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A Nemytskii-Edelstein type fixed point theorem for partial metric spaces

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Abstract

In 1994, Matthews obtained an extension of the celebrated Banach fixed point theorem to the partial metric framework (Ann. N.Y. Acad. Sci. 728:183-197, 1994). Motivated by the Matthews extension of the Banach theorem, we present a Nemytskii-Edelstein type fixed point theorem for self-mappings in partial metric spaces in such a way that the classical one can be retrieved as a particular case of our new result. We give examples which show that the assumed hypothesis in our new result cannot be weakened. Moreover, we show that our new fixed point theorem allows one to find fixed points of mappings in some cases in which the Matthews result and the classical Nemytskii-Edelstein one cannot be applied. Furthermore, we provide a negative answer to the question about whether our new result can be retrieved as a particular case of the classical Nemytskii-Edelstein one whenever the metrization technique, developed by Hitzler and Seda (Mathematical Aspects of Logic Programming Semantics, 2011), is applied to partial metric spaces.

1 Introduction

In 1922, S Banach proved in the context of metric spaces his celebrated fixed point result. Let us recall his result below. To this end, we denote by $\mathbb{R}^+$ the set of nonnegative real numbers.

Theorem 1 *Let f be a mapping of a complete metric space (X,d) into itself such that there is $s \in \mathbb{R}^+$ with $0 \leq s < 1$, satisfying*

$$d\big(f(x),f(y)\big) \leq sd(x,y) \tag{1}$$

for all $x,y \in X$. Then f has a unique fixed point.

The origins of the preceding theorem lies in the methods for solving differential equations via successive approximations. However, since Banach proved the above fixed point result, a wide range of applications has been given in very different frameworks. A class of such applications are obtained mainly through two extensions of Theorem 1. One of them to the context of quasi-metric spaces and the other one to the framework of partial metric spaces (for a detailed discussion see, for instance, [1]).

In order to recall the aforesaid extension to the quasi-metric framework, let us fix a few pertinent concepts about quasi-metric spaces.

Following [2], a quasi-metric on a nonempty set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x,y,z \in X$:

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(i) $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$.
(ii) $d(x,y) \leq d(x,z) + d(z,y)$.

If, in addition, a quasi-metric d satisfies (iii) $d(x,y) = 0 \Leftrightarrow x = y$, then d is called a T_1 quasi-metric.

Obviously a metric on a set X is a quasi-metric d on X satisfying (iv) $d(x,y) = d(y,x)$ for all $x,y \in X$.

Of course, a (T_1) quasi-metric space is a pair (X,d) such that X is a nonempty set and d is a (T_1) quasi-metric on X.

Each quasi-metric d on X generates a T_0-topology $\mathcal{T}(d)$ on X which has as a base the family of open d-balls $\{B_d(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Clearly, if the quasi-metric space (X,d) is T_1, then the topology $\mathcal{T}(d)$ is T_1.

Given a quasi-metric d on X, then the function $d^{-1} : X \times X \to \mathbb{R}^+$ defined by $d^{-1}(x,y) = d(y,x)$ for all $x,y \in X$ is also a quasi-metric on X that is called the conjugate quasi-metric of d. Furthermore the function $d^s : X \times X \to \mathbb{R}^+$ defined by $d^s(x,y) = \max\{d(x,y), d^{-1}(x,y)\}$ for all $x,y \in X$ is a metric on X.

A quasi-metric space (X,d) is called bicomplete if the metric space (X,d^s) is complete.

The next example of quasi-metric space will play a central role in the sequel.

Example 2 Consider the function $d_{\boldsymbol{u}} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ given by $d_{\boldsymbol{u}}(x,y) = \max\{y - x, 0\}$ for all $x,y \in \mathbb{R}^+$. It is not hard to see that $(\mathbb{R}^+, d_{\boldsymbol{u}})$ is a quasi-metric space which is called the upper quasi-metric space. Note that $\mathcal{T}(d_{\boldsymbol{u}})$ is the so-called upper topology on $\mathbb{R}^+$ (see [2] and [3]). Moreover, the quasi-metric $d_{\boldsymbol{u}}^{-1}$ is defined on $\mathbb{R}^+$ by $d_{\boldsymbol{u}}^{-1}(x,y) = \max\{x - y, 0\}$ for all $x,y \in \mathbb{R}^+$. The quasi-metric space $(\mathbb{R}^+, d_{\boldsymbol{u}}^{-1})$ is called the lower quasi-metric space. Furthermore, the upper quasi-metric induces the Euclidean metric on $\mathbb{R}^+$, *i.e.*, $d_{\boldsymbol{u}}^s(x,y) = |y - x|$ for all $x,y \in \mathbb{R}^+$. Thus the upper quasi-metric space $(\mathbb{R}^+, d_{\boldsymbol{u}})$ is bicomplete.

As we have announced before, Theorem 1 can be generalized in the following easy way.

Theorem 3 *Let f be a mapping from a bicomplete quasi-metric space (X,d) into itself such that there is $s \in \mathbb{R}^+$ with $0 \leq s < 1$, satisfying*

$$d\bigl(f(x), f(y)\bigr) \leq s d(x,y) \tag{2}$$

for all $x,y \in X$. Then f has a unique fixed point.

Applications of the above quasi-metric version of the Banach fixed point theorem to asymptotic complexity of algorithms can be found in [4] and [5].

With the aim of introducing the second aforementioned extension of the Banach fixed point theorem, let us recall a few notions about partial metric spaces.

According to [6], a partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x,y,z \in X$:

(i) $p(x,x) = p(x,y) = p(y,y) \Leftrightarrow x = y$.
(ii) $p(x,x) \leq p(x,y)$.
(iii) $p(x,y) = p(y,x)$.
(iv) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. Clearly, a metric on a set X is a partial metric p on X such that $p(x,x) = 0$ for all $x \in X$.

From now on, we will denote by $\mathbb{N}$ the set of positive integer numbers.

Each partial metric p on X generates a T_0 topology $\mathcal{T}(p)$ on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. From this fact it immediately follows that a sequence $(x_n)_n$ in a partial metric space (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a partial metric space (X,p) is called a Cauchy sequence if there exists $\lim_{n,m\to\infty} p(x_n,x_m)$.

A partial metric space (X,p) is said to be complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X converges, with respect to $\mathcal{T}(p)$, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Given a partial metric p on X, then the function $p^s : X \times X \to \mathbb{R}^+$ defined by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ for all $x,y \in X$ is a metric on X. In addition, on account on [6], a partial metric space (X,p) is complete if and only if its associated metric space (X,p^s) is complete.

The next example will be needed in the sequel.

Example 4 Consider the function $p_{\max} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ given by $p_{\max}(x,y) = \max\{x,y\}$ for all $x,y \in \mathbb{R}^+$. It is not hard to see that $(\mathbb{R}^+, p_{\max})$ is a partial metric space. Moreover, it is clear that $p^s_{\max}(x,y) = |y-x|$ and, thus, $(\mathbb{R}^+, p_{\max})$ is complete.

The aforementioned partial metric version of Banach fixed point theorem can be stated as follows.

Theorem 5 *Let f be a mapping from a complete partial metric space (X,p) into itself such that there is $s \in \mathbb{R}^+$ with $0 \leq s < 1$, satisfying*

$$p\bigl(f(x),f(y)\bigr) \leq sp(x,y) \tag{3}$$

for all $x,y \in X$. Then f has a unique fixed point.

An application of the preceding partial metric version of the Banach fixed point theorem to program correctness can be found in [7] (see also [6]).

In 1962, M Edelstein put forth a natural question about the possibility of obtaining a fixed point theorem by keeping the completeness of the metric space and replacing the contractive condition (1) in Theorem 1 by another slightly modified condition. In particular he proposed the following contractive condition for a mapping f from a metric space (X,d) into itself:

$$d\bigl(f(x),f(y)\bigr) < d(x,y) \tag{4}$$

for all $x,y \in X$ with $x \neq y$.

A negative answer to this query was given when he showed that completeness of the metric space is not sufficient to guarantee even the existence of a fixed point whenever the Banach contractive condition (1) is replaced by (4). However, Edelstein himself proved a positive result, Theorem 6 below, for self-mappings satisfying the new contractive condition (4), although the class of spaces to which it applies is much more restrictive (see [8] and [9]). This result was proved independently by Nemytskii [10].

Theorem 6 *Let f be a mapping from a compact metric space (X,d) into itself satisfying*

$$d\big(f(x),f(y)\big) < d(x,y) \tag{5}$$

for all $x,y \in X$ with $x \neq y$. Then f has a unique fixed point.

The preceding theorem, just like Theorem 1, is a powerful tool to obtain relevant applications of fixed point theory to different fields of science. Some of the aforementioned applications can be found in [11] (and references therein), [12, 13], and [14].

Inspired by the Matthews extension of the Banach theorem (Theorem 5 above), many fixed point theorems for self-mappings in metric spaces have been extended to the partial metric framework. A few recent works in this direction can be found, for example, in [1, 15–32] and [33]. For details, we refer to [34]. Motivated by this intense research activity in fixed point theory in partial metric spaces, we present a Nemytskii-Edelstein type fixed point theorem for self-mappings in partial metric spaces in such a way that the classical one can be retrieved as a particular case of our new result. Moreover, we show that our new fixed point theorem allows one to find fixed points of mappings in some cases in which the Matthews result and the classical Nemytskii-Edelstein one cannot be applied. Furthermore, a discussion about whether our new results can be retrieved as a particular case of the Nemytskii-Edelstein classical fixed point result through the application of the metrization technique introduced in [35] (see also [36]) is performed.

2 Statement of the problem

The natural attempt to extend the classical Nemytskii-Edelstein fixed point theorem to the context of partial metric spaces would consist of replacing in Theorem 5 the contractive condition (3) by a strict one and the completeness of the partial metric space by a suitable notion of compactness. Thus one may conjecture that the next result would be desirable.

Conjecture 1 *Let (X,p) be a compact partial metric space and let f be a mapping from (X,p) into itself satisfying*

$$p\big(f(x),f(y)\big) < p(x,y) \tag{6}$$

for all $x,y \in X$ with $x \neq y$. Then f has a unique fixed point.

In [6] (see also [37]), it was showed that, given a partial metric space (X,p), the function $d_p : X \times X \to \mathbb{R}^+$, defined by $d_p(x,y) = p(x,y) - p(x,x)$ for all $x,y \in X$, is a quasi-metric on X such that $\mathcal{T}(p) = \mathcal{T}(d_p)$.

In the light of the preceding relationship between partial metric spaces and quasi-metrics, it also seems natural to wonder whether the following quasi-metric version of the Nemytskii-Edelstein fixed point theorem holds.

Conjecture 2 *Let (X,p) be a compact partial metric space and let f be a mapping from (X,p) into itself satisfying*

$$d_p\bigl(f(x),f(y)\bigr) < d_p(x,y) \tag{7}$$

for all $x,y \in X$ with $d_p(x,y) \neq 0$ and $x \neq y$. Then f has a unique fixed point.

In the remainder of the paper we provide a negative answer to the posed question. Concretely, in Section 3 we provide counterexamples that show that such conjectures are not true when we consider a few possible notions of compactness in partial metric spaces. In spite of this handicap, we prove that an additional assumption, which is not too much restrictive, on the self-mapping is enough to provide Nemytskii-Edelstein type fixed point theorems in the spirit of the above conjectures. In addition, we give examples that show that the additional hypothesis on the self-mapping in our new results cannot be weakened. Furthermore, in Section 4, we provide a negative answer to the question about whether our new result can be retrieved as a particular case of the classical Nemytskii-Edelstein one whenever the metrization technique, given in [35], is applied to partial metric spaces.

3 The new fixed point theorem

In order to achieve our aim we need to introduce an appropriate notion of compactness in the context of partial metric spaces. Thus, from now on, we will say that a partial metric space (X,p) is compact provided that the quasi-metric space (X,d_p^{-1}) is compact, *i.e.*, the topological space $(X,\mathcal{T}(d_p^{-1}))$ is compact. As we will see later on, Theorem 11, our main result, justifies the definition of compactness introduced.

Next we provide an example of compact partial metric space.

Example 7 Let $p_{\max}$ be the partial metric introduced in Example 4. Consider the partial metric space $([0,1[,p_{\max})$ where we have also denoted by $p_{\max}$ the restriction of the partial metric $p_{\max}$ to $[0,1[$. It is clear that $([0,1[,p_{\max})$ is compact, since $B_{d_{p_{\max}}^{-1}}(0,\varepsilon) = [0,1[$ for each $\varepsilon > 0$.

In [38], O'Neill defined a different notion of compactness in partial metric spaces. Concretely, a partial metric space (X,p) is said to be compact in the O'Neill sense whenever the metric space (X,p^s) is compact (or equivalently (X,d_p^s) is compact). In the sequel, compact partial metric spaces in the O'Neill sense will be called O-compact partial metric spaces.

Next we give an example of O-compact partial metric spaces.

Example 8 Consider the partial metric space $([0,1],p_{\max})$, where $p_{\max}$ is defined as in Example 7. It is obvious that $p_{\max}^s(x,y) = |y-x|$ for all $x,y \in [0,1]$. Therefore $([0,1],p_{\max})$ is an O-compact partial metric space.

Clearly O-compact partial metric spaces are compact but the converse is not true. Example 7 provides an instance of compact partial metric space which is not O-compact, since the partial metric space $([0,1[,p_{\max}^s)$ is not compact.

In the light of the different notions of compactness introduced, we are able to discuss whether the conjectures given in Section 2, 'Conjecture 1' and 'Conjecture 2', are foolproof.

Regarding 'Conjecture 1', the next example shows that it does not hold.

Example 9 Consider the complete partial metric space $(\mathcal{C}, p_{\mathcal{C}})$ introduced in [37], where

$$\mathcal{C} = \left\{ f : \mathbb{N} \to]0, \infty] : \sum_{n=1}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty \right\}$$

and

$$p_{\mathcal{C}}(f, g) = \sum_{n=1}^{\infty} 2^{-n} \max\left\{ \frac{1}{f(n)}, \frac{1}{g(n)} \right\}$$

for all $f, g \in \mathcal{C}$. Of course, it is adopted the convention that $\frac{1}{\infty} = 0$. Next consider the subset $\mathcal{C}_1 \subseteq \mathcal{C}$ given by $\mathcal{C}_1 = \{f \in \mathcal{C} : 1 \leq f(1) < \frac{4}{3}$ and $f(n) = \infty$ for all $n \in \mathbb{N}_2\}$ where $\mathbb{N}_2 = \{n \in \mathbb{N} : n \geq 2\}$. According to [39, 40] and [37], the pair $(\mathcal{C}_1, p^s_{\mathcal{C}})$ is totally bounded (note that we have also denoted by $p^s_{\mathcal{C}}$ the restriction of $p^s_{\mathcal{C}}$ to $\mathcal{C}_1$). In addition, it is not hard to see that the subset $\mathcal{C}_1$ is closed in $(\mathcal{C}_1, p^s_{\mathcal{C}})$, and, thus, complete. Whence we find that the metric space $(\mathcal{C}_1, p^s_{\mathcal{C}})$ is compact. So the partial metric space $(\mathcal{C}_1, p_{\mathcal{C}})$ is O-compact.

Now define the mapping $F : \mathcal{C}_1 \to \mathcal{C}_1$ by

$$F(f)(n) = \begin{cases} 2 - \frac{f(1)}{2} & \text{if } n = 1, \\ \infty & \text{if } n > 1. \end{cases}$$

Then is clear that F satisfies the required contractive condition, *i.e.*,

$$p_{\mathcal{C}}\big(F(f), F(g)\big) < p_{\mathcal{C}}(f, g)$$

for all $f, g \in \mathcal{C}_1$ with $f \neq g$. However, F has no fixed point.

In the light of the preceding example 'Conjecture 1' is not true even when we consider O-compactness instead of compactness.

Unfortunately 'Conjecture 2' does not hold as the next example shows.

Example 10 Consider the compact partial metric space $([0,1], p_{\max})$ introduced in Example 8. Define the mapping $f : [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

It is clear that

$$d_{p_{\max}}\big(f(x), f(y)\big) < d_{p_{\max}}(x, y)$$

for all $x, y \in [0,1]$ with $x \neq y$ and $d_{p_{\max}}(x, y) \neq 0$. But the mapping f is fixed point free.

Of course, the preceding example shows that 'Conjecture 2' is not also satisfied even when one considers O-compactness.

In the light of the preceding discussion we yield sufficient conditions in order to guarantee that the Nemytskii-Edelstein contractive type conditions (6) and (7) provide a unique fixed point.

To this end let us recall that, given a partial metric space (X,p), a mapping from (X,p) into itself is said to be continuous provided that it is continuous from $(X,\mathcal{T}(p))$ to $(X,\mathcal{T}(p))$. Moreover, we will say that f is conjugate continuous if f is continuous from $(X,\mathcal{T}(d_p^{-1}))$ to $(X,\mathcal{T}(d_p^{-1}))$. Besides, let us recall that a function f from a topological space $(X,\mathcal{T})$ into $(\mathbb{R}^+,\mathcal{T}(|\cdot|))$ is lower semicontinuous on $(X,\mathcal{T})$ if and only if f is continuous from $(X,\mathcal{T})$ to $(\mathbb{R}^+,\mathcal{T}(d_u^{-1}))$ (see [3]), where d_u is the upper quasi-metric introduced in Example 2. Furthermore, it is well known that every lower semicontinuos function on a compact topological space attains a minimum value (see Theorem 2.40 in [41]).

Next we prove our announced fixed point result.

Theorem 11 *Let (X,p) be a compact partial metric space. If f is a mapping from (X,p) into itself which is conjugate continuous and satisfies*

$$p\bigl(f(x),f(y)\bigr) < p(x,y) \tag{8}$$

for all $x,y \in X$ with $x \neq y$, then f has a unique fixed point.

Proof First we show the existence of fixed point. Define the function $F: X \to \mathbb{R}^+$ by $F(x) = p(x,f(x))$. Next we show that F is lower semicontinuous on $(X,\mathcal{T}(d_p^{-1}))$. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $\lim_{n\to+\infty} d_p^{-1}(x,x_n) = 0$. Then, given $\varepsilon > 0$, since f is a conjugate continuous mapping there exists $n_0 \in \mathbb{N}$ such that $d_p(x_n,x) < \varepsilon$ and $d_p^{-1}(f(x),f(x_n)) < \varepsilon$ for all $n \geq n_0$. Whence

$$\begin{aligned}
F(x) - F(x_n) &= p\bigl(x,f(x)\bigr) - p\bigl(x_n,f(x_n)\bigr) \\
&\leq p(x,x_n) + p\bigl(x_n,f(x)\bigr) - p(x_n,x_n) - p\bigl(x_n,f(x_n)\bigr) \\
&= d_p^{-1}(x,x_n) + p\bigl(x_n,f(x)\bigr) - p\bigl(x_n,f(x_n)\bigr) \\
&< \varepsilon + p\bigl(x_n,f(x)\bigr) - p\bigl(x_n,f(x_n)\bigr) \\
&\leq \varepsilon + p\bigl(x_n,f(x_n)\bigr) + p\bigl(f(x_n),f(x)\bigr) - p\bigl(f(x_n),f(x_n)\bigr) - p\bigl(x_n,f(x_n)\bigr) \\
&= \varepsilon + d_p^{-1}\bigl(f(x),f(x_n)\bigr) \\
&< 2\varepsilon
\end{aligned}$$

whenever $n \geq n_0$. So $d_u^{-1}(F(x),F(x_n)) < \varepsilon$ for all $n \geq n_0$ and, thus, F is lower semicontinuous on $(X,\mathcal{T}(d_p^{-1}))$. Since (X,d_p^{-1}) is a compact quasi-metric space, F attains its minimum, say at $x_0 \in X$. Consequently, we find that $x_0 = f(x_0)$, because otherwise $p(x_0,f(x_0)) \neq 0$ and, thus by (8),

$$F\bigl(f(x_0)\bigr) = p\bigl(f(x_0),f\bigl(f(x_0)\bigr)\bigr) < p\bigl(x_0,f(x_0)\bigr) = F(x_0).$$

This contradicts the fact that F attains its minimum at x_0.

Now we prove the uniqueness. Suppose that there exists $y_0 \in X$ such that $f(y_0) = y_0$ and $y_0 \neq x_0$. Then $p(x_0,y_0) \neq 0$ and, by (8),

$$p(x_0,y_0) = p\bigl(f(x_0),f(y_0)\bigr) < p(x_0,y_0).$$

Therefore the fixed point of f is unique. □

Example 9 shows that the conjugate continuity of the mapping cannot be deleted from the hypotheses of Theorem 11 in order to guarantee the existence of a fixed point.

Remark 12 Note that as a particular case of Theorem 11 we obtain the Nemytskii-Edelstein fixed point theorem (Theorem 6 in Section 1) when the partial metric is, in fact, a metric.

The next example shows that Theorem 11 cannot be deduced, in general, from Theorem 6 when an O-compact partial metric space and its associated metric space are considered.

Example 13 Consider the O-compact partial metric space $([0,1], p_{\max})$ introduced in Example 8. Define the mapping $f : [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{2}, \\ \frac{1}{2} & \text{if } x > \frac{1}{2}. \end{cases}$$

It is clear that

$$p_{\max}\bigl(f(x), f(y)\bigr) < p_{\max}(x, y)$$

for all $x, y \in [0,1]$ such that $x \neq y$. Nevertheless,

$$\frac{1}{2} = p^s_{\max}\left(f(1), f\left(\frac{1}{2}\right)\right) \nless p^s_{\max}\left(1, \frac{1}{2}\right) = \frac{1}{2}.$$

From Theorem 11 we obtain the following results.

Corollary 14 *Let (X, p) be a compact partial metric space and let f be a mapping from (X, p) into itself satisfying*

$$p\bigl(f(x), f(y)\bigr) < p(x, y) \tag{9}$$

for all $x, y \in X$ with $x \neq y$ and

$$p\bigl(f(x), f(x)\bigr) \geq p(x, x) \tag{10}$$

for all $x \in X$, then f has a unique fixed point.

Proof We only have to prove that the mapping f is conjugate continuous. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $\lim_{n\to+\infty} d_p^{-1}(x, x_n) = 0$. Next we show that $\lim_{n\to+\infty} d_p^{-1}(f(x), f(x_n)) = 0$. Obviously we can assume that $f(x) \neq f(x_n)$ for all $n \geq n_0$, because otherwise we immediately obtain $d_p^{-1}(f(x), f(x_n)) = 0$ for all $n \in \mathbb{N}$ such that $n \geq n_0$ and $f(x) = f(x_n)$. Since $\lim_{n\to+\infty} d_p^{-1}(x, x_n) = 0$ there exists $n_0 \in \mathbb{N}$ such that $p(x, x_n) - p(x_n, x_n) < \varepsilon$ for all $n \geq n_0$. Whence we find that

$$\begin{aligned} d_p^{-1}\bigl(f(x), f(x_n)\bigr) &= p\bigl(f(x), f(x_n)\bigr) - p\bigl(f(x_n), f(x_n)\bigr) \\ &\leq p\bigl(f(x), f(x_n)\bigr) - p(x_n, x_n) < p(x, x_n) - p(x_n, x_n) < \varepsilon \end{aligned}$$

for all $n \geq n_0$, which is the desired conclusion. By Theorem 11 we obtain the existence and uniqueness of a fixed point. □

Example 9 proves that the contractive condition (10) cannot be deleted in statement of Corollary 14.

Motivated by the fact that Example 10 shows that the proposed quasi-metric version of the Nemytskii-Edelstein fixed point theorem given in 'Conjecture 2' does not hold, we end the section focusing our attention on the study of the aforesaid version.

Corollary 15 *Let (X,p) be a compact partial metric space and let f be a mapping from (X,p) into itself satisfying*

$$d_p\big(f(x),f(y)\big) \leq d_p(x,y) \tag{11}$$

for all $x,y \in X$ and

$$p\big(f(x),f(y)\big) < p(x,y) \tag{12}$$

for all $x,y \in X$ with $x \neq y$, then f has a unique fixed point.

Proof Consider a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\lim_{n\to+\infty} d_p^{-1}(x,x_n) = 0$. By condition (11) we deduce that $d_p(f(x_n),f(x)) \leq d_p(x_n,x)$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}\lim_{n\to+\infty} d_p^{-1}\big(f(x),f(x_n)\big) &= \lim_{n\to+\infty} d_p\big(f(x_n),f(x)\big)\\ &\leq \lim_{n\to+\infty} d_p(x_n,x) = \lim_{n\to+\infty} d_p^{-1}(x,x_n) = 0.\end{aligned}$$

It follows that f is conjugate continuous and, hence, by Theorem 11, we obtain the existence and uniqueness of a fixed point. □

Example 10 shows that the contractive condition (11) cannot be deleted in statement of the preceding result.

Corollary 16 *Let (X,p) be a compact partial metric space. If f is a mapping from (X,p) into itself which satisfies*

$$d_p(x,y) = 0 \quad \Rightarrow \quad d_p\big(f(x),f(y)\big) = 0 \tag{13}$$

for all $x,y \in X$ and

$$d_p\big(f(x),f(y)\big) < d_p(x,y) \tag{14}$$

for all $x,y \in X$ such that $x \neq y$ with $d_p(x,y) \neq 0$ and, in addition,

$$p\big(f(x),f(x)\big) \leq p(x,x) \tag{15}$$

for all $x \in X$, then f has a unique fixed point in X.

Proof It is a simple matter to check that the mapping f is conjugate continuous. Next we show that f holds the contractive condition (8). To this end, consider $x, y \in X$ such that $x \neq y$. Then

$$p\bigl(f(x),f(y)\bigr) - p(x,x) \leq d_p\bigl(f(x),f(y)\bigr) < d_p(x,y) = p(x,y) - p(x,x).$$

It follows that

$$p\bigl(f(x),f(y)\bigr) < p(x,y).$$

Whence, by Theorem 11, we deduce that f has a unique fixed point. □

The next example shows that the contractive condition (15) cannot be deleted in the statement of Corollary 16.

Example 17 Let $(\mathcal{C}, p_{\mathcal{C}})$ be the partial metric space introduced in Example 9. Next consider the subset $\mathcal{C}_2 \subseteq \mathcal{C}$ given by $\mathcal{C}_2 = \mathcal{C} \setminus \{f_2\}$, where $f_2(n) = 2$ for all $n \in \mathbb{N}$. It is clear that $(\mathcal{C}_2, p_{\mathcal{C}})$ is compact, where the restriction of $p_{\mathcal{C}}$ to $\mathcal{C}_2$ has also been denoted by $p_{\mathcal{C}}$. Note that $B_{d_{p_{\mathcal{C}}}^{-1}}(f_\infty, \varepsilon) = \mathcal{C}_2$ for each $\varepsilon > 0$, where f_∞ is the element of $\mathcal{C}_2$ given by $f_\infty(n) = \infty$ for all $n \in \mathbb{N}$.

Now define the mapping $F : \mathcal{C}_2 \to \mathcal{C}_2$ by

$$F(f)(n) = \begin{cases} 3 & \text{if } n = 1, \\ f(n-1) & \text{if } n > 1. \end{cases}$$

Then it is clear that $d_{p_{\mathcal{C}}}(F(f), F(g)) = 0$ whenever $d_{p_{\mathcal{C}}}(f, g) = 0$. Moreover,

$$d_{p_{\mathcal{C}}}\bigl(F(f), F(g)\bigr) < d_{p_{\mathcal{C}}}(f, g)$$

for all $f, g \in \mathcal{C}_2$ with $f \neq g$ and $d_{p_{\mathcal{C}}}(f, g) \neq 0$. Nevertheless,

$$p_{\mathcal{C}}\bigl(F(f), F(f)\bigr) \not< p_{\mathcal{C}}(f, f)$$

for all $f \in \mathcal{C}_2$. Of course, F has not fixed point.

The next result will be needed in order to provide another quasi-metric version of the Nemytskii-Edelstein fixed point theorem given in 'Conjecture 2'.

Theorem 18 *Let (X, d) be a quasi-metric space such that the metric space (X, d^s) is compact. If f is a mapping from (X, d) into itself which satisfies*

$$d(x,y) = 0 \quad \Rightarrow \quad d\bigl(f(x),f(y)\bigr) = 0 \tag{16}$$

for all $x, y \in X$ and

$$d\bigl(f(x),f(y)\bigr) < d(x,y) \tag{17}$$

for all $x, y \in X$ such that $x \neq y$ with $d(x, y) \neq 0$, then f has a unique fixed point in X.

Proof Define the function $F : X \to \mathbb{R}^+$ by $F(x) = d(x,f(x))$. Next we show that F is lower semicontinuous on $(X, \mathcal{T}(d^s))$. Indeed, let $(x_n)_{n\in\mathbb{N}} \subset X$ such that $\lim_{n\to+\infty} d^s(x,x_n) = 0$. Then, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\max\{d(x,x_n), d(x_n,x)\} < \varepsilon$$

for all $n \geq n_0$. Moreover, for each $n \in \mathbb{N}$ we can distinguish two cases:

Case 1. $d(x_n,x) = 0$. Then, by (16), $d(f(x_n),f(x)) = 0$.

Case 2. $d(x_n,x) \neq 0$. Then, by (17),

$$d(f(x_n),f(x)) < d(x_n,x) < \varepsilon.$$

From the preceding assertions we have

$$\begin{aligned}
F(x) - F(x_n) &= d(x,f(x)) - d(x_n,f(x_n)) \\
&\leq d(x,x_n) + d(x_n,f(x)) - d(x_n,f(x_n)) \\
&< \varepsilon + d(x_n,f(x)) - d(x_n,f(x_n)) \\
&\leq \varepsilon + d(x_n,f(x_n)) + d(f(x_n),f(x)) - d(x_n,f(x_n)) \\
&\leq \varepsilon + d(f(x_n),f(x)) \\
&< 2\varepsilon
\end{aligned}$$

whenever $n \geq n_0$. So $d_u^{-1}(F(x),F(x_n)) < \varepsilon$ for all $n \geq n_0$ and, thus, F is lower semicontinuous on $(X, \mathcal{T}(d^s))$. Since (X,d^s) is compact we deduce that F attains its minimum, say at $x_0 \in X$. Hence we deduce that $x_0 = f(x_0)$ because otherwise

$$F(f(x_0)) = d(f(x_0),f(f(x_0))) < d(x_0,f(x_0)) = F(x_0),$$

which contradicts the fact that F attains its minimum at x_0.

Finally, we suppose that there exists $y_0 \in X$ such that $f(y_0) = y_0$ and $y_0 \neq x_0$. Then $d^s(x_0,y_0) \neq 0$. Suppose that $d^s(x_0,y_0) = d(x_0,y_0)$. By (17) we obtain

$$d(x_0,y_0) = d(f(x_0),f(y_0)) < d(x_0,y_0).$$

Therefore the fixed point of f is unique. Of course, the same conclusion is concluded if $d^s(x_0,y_0) = d(y_0,x_0)$. □

Example 10 shows that the separation condition (16) cannot be deleted in the statement of Theorem 18.

Remark 19 Observe that, in general, the contractive conditions (16) and (17) do not imply the contractive condition (5) for the metric case. Indeed, consider the O-compact partial metric space $([0,1], p_{\max})$ given in Example 10. Define the mapping $f : [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x < \frac{1}{2}, \\ \frac{1}{2} & \text{if } x \geq \frac{1}{2}. \end{cases}$$

It is clear that

$$d_{p_{\max}}\big(f(x),f(y)\big) < d_{p_{\max}}(x,y)$$

for all $x, y \in [0,1]$ with $x \neq y$ and $d_{p\max}(x,y) \neq 0$. Moreover,

$$d_{p_{\max}}(x,y) = 0 \quad \Rightarrow \quad d_{p_{\max}}\big(f(x),f(y)\big) = 0.$$

Nevertheless

$$\frac{1}{2} = d^s_{p_{\max}}\left(f(0),f\left(\frac{1}{2}\right)\right) \not< d^s_{p_{\max}}\left(0,\frac{1}{2}\right) = \frac{1}{2}.$$

As a direct consequence of Theorem 18 we obtain the following one for partial metric spaces in the spirit of 'Conjecture 2'.

Corollary 20 *Let (X,p) be an O-compact partial metric space. If f is a mapping from (X,p) into itself which satisfies*

$$d_p(x,y) = 0 \quad \Rightarrow \quad d_p\big(f(x),f(y)\big) = 0 \tag{18}$$

for all $x, y \in X$ and

$$d_p\big(f(x),f(y)\big) < d_p(x,y) \tag{19}$$

for all $x, y \in X$ such that $x \neq y$ with $d_p(x,y) \neq 0$, then f has a unique fixed point in X.

We end the section giving an instance in which our new fixed point theorem allows one to find the fixed point of a mapping where, however, the Matthews fixed point result cannot be applied. Moreover, the classical Nemytskii-Edelstein fixed point theorem also cannot be applied when the metric induced by the partial metric is considered.

Example 21 Consider the O-compact partial metric space $([0,1], p_{\max})$ and the mapping f introduced in Example 13. It is not hard to check that f is conjugate continuous. Moreover, as we have pointed out in the aforesaid example, it is easy to see that

$$p_{\max}\big(f(x),f(y)\big) < p_{\max}(x,y)$$

for all $x, y \in [0,1]$ such that $x \neq y$. Therefore all conditions in Theorem 11 are satisfied and, thus, f has a unique fixed point which is obviously $x_0 = 0$.

Next we show that Theorem 5 also cannot be applied to show the existence and uniqueness of the fixed point of f. Assume that there exists $c \in [0,1[$ such that

$$p_{\max}\big(f(x),f(y)\big) \leq c p_{\max}(x,y)$$

for all $x, y \in [0,1]$. Then

$$\frac{1}{2} = p_{\max}\left(f\left(\frac{1}{2}\right),f\left(\frac{1}{2}+\frac{1}{n}\right)\right) \leq c p_{\max}\left(\frac{1}{2},\frac{1}{2}+\frac{1}{n}\right) = c\left(\frac{1}{2}+\frac{1}{n}\right)$$

for all $n \in \mathbb{N}$ with $n \geq 2$. Hence

$$c \geq \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{n}}$$

for all $n \geq 2$. Consequently $c \geq \lim_{n\to\infty} \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{n}} = 1$, which contradicts the fact that $c \in [0,1[$.

Finally we prove that Theorem 6 cannot be applied to show the existence and uniqueness of the fixed point of f when the metric space $([0,1], p^s_{\max})$ is considered. Observe that $p^s_{\max}(x,y) = |y-x|$ for all $x,y \in [0,1]$. In addition,

$$p^s_{\max}\left(f\left(\frac{1}{2}\right), f(1)\right) = p^s_{\max}\left(\frac{1}{2}, 1\right) = \frac{1}{2}.$$

Whence we conclude that

$$p^s_{\max}\left(f\left(\frac{1}{2}\right), f(1)\right) \not< p^s_{\max}\left(\frac{1}{2}, 1\right).$$

4 A reflection

This section is devoted to a discussion of the possibility of retrieving our new result, Theorem 11, from the classical Nemytskii-Edelstein fixed point result whenever a special metrization technique of partial metrics is considered.

In [35], Hitzler and Seda have proved that each partial metric p on a nonempty set X induces a metric m_p on X such that $\mathcal{T}(d^s_p) \subseteq \mathcal{T}(m_p)$, where

$$m_p(x,y) = \begin{cases} 0 & \text{if } x = y, \\ p(x,y) & \text{if } x \neq y. \end{cases} \tag{20}$$

In [36], Haghi *et al.*, inspired by the work of Hiztler and Seda, stressed that the preceding technique to generate a metric from a partial metric allows one to retrieve many fixed point results for self-mappings in the partial metric context from the known counterpart fixed point results for self-mappings in the metric framework. Taking into account the preceding comment, we show that this is not the situation regarding the Nemytskii-Edelstein fixed point result.

Let (X,p) be a compact partial metric space and let f be a mapping from (X,p) into itself which holds the contractive condition introduced in Theorem 11, *i.e.*,

$$p\big(f(x), f(y)\big) < p(x,y) \tag{21}$$

for all $x,y \in X$ such that $x \neq y$. Then we immediately find that f satisfies in addition the next contractive condition

$$m_p\big(f(x), f(y)\big) < m_p(x,y) \tag{22}$$

for all $x,y \in X$ with $x \neq y$. However, the existence and uniqueness of fixed point of f cannot be deduced in general from Theorem 6 (the classical Nemytskii-Edelstein result) through the use of the metric m_p. This is due to the fact that given a compact partial metric space

(X, p), the associated metric space (X, m_p) is not compact in general. The following examples illustrate this fact.

First we give an example of a compact partial metric space whose induced metric space, obtained following (20), is not compact.

Example 22 Consider the compact partial metric space $([0,1[, p_{\max})$ introduced in Example 7. The associated metric space $([0,1[, m_{p_{\max}})$ is not compact. Indeed, let $x \in [0,1[$ and consider the sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n = \frac{1}{2} - \frac{1}{2n}$ for all $n \in \mathbb{N}$. Then we have, on the one hand, $m_p(x, x_n) = x$ provided that $x \geq \frac{1}{2}$. On the other hand, if $x < \frac{1}{2}$ then there exists $n_0 \in \mathbb{N}$ such that $m_p(x, x_n) = \frac{1}{2} - \frac{1}{2n}$ for all $n \geq n_0$. Therefore every subsequence of the sequence $(x_n)_{n\in\mathbb{N}}$ is not convergent in $([0,1[, m_{p_{\max}})$. Consequently $([0,1[, m_{p_{\max}})$ is not compact.

Next we give an example of an O-compact partial metric space whose induced metric space, obtained following (20), is not compact.

Example 23 Consider the O-compact partial metric space $([0,1], p_{\max})$ introduced in Example 8. Clearly $([0,1], m_{p_{\max}})$ is not compact, since every subsequence of the sequence $(x_n)_{n\in\mathbb{N}}$ introduced in Example 22 is not convergent.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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