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On the iterates of asymptotically k-strict pseudocontractive mappings in Hilbert spaces with convergence analysis

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Abstract
In this paper, we consider the approximations of an iterative process for asymptotically k-strict pseudocontractive mappings in Hilbert spaces. Finally, we present some examples to study the rate of convergence.
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1 Preliminaries

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] in 1972. They proved that if C is a nonempty bounded, closed and convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T of C has a fixed point. Further, the set $F(T)$ of fixed points of T is closed and convex. In [2], Zegeye *et al.* introduced a class of Lipschitz pseudocontractive mappings in a Banach space. Since then, the weak and strong convergence problems of the iterative algorithms for such a class of mappings have been studied by several researchers under suitable conditions (see Yao *et al.* [3] and [4]; Thakur *et al.* [5, 6]; Dewangan *et al.* [7, 8]; Zegeye and Shahzad [9]; Jung [10]). Also, the class of nonexpansive mappings via iteration methods was extensively studied (see Tan and Xu [11]; Thakur *et al.* [12]).

In 2011, Ceng *et al.* [13] considered the following concept of asymptotically k-strict pseudocontractive type mapping in the intermediate sense in a Hilbert space $\mathcal{H}$. For an asymptotically k-strict pseudocontractive type mapping T with sequence $\{\gamma_n\}$, Ceng *et al.* proved that the Mann iteration sequence converges weakly to a fixed point of T.

In this paper, based on [13], the convergence of the iteration approximation of asymptotically k-strict pseudocontractive type mappings in a Hilbert space is studied. Finally, we study the rate of convergence of the iteration. Also, some illustrative numerical examples (using Matlab software) are presented.

We need the following definitions and lemmas for the main results.

Definition 1.1 [13] Let B be a nonempty subset of a Hilbert space $\mathcal{H}$. A mapping $T : B \to B$ is called an asymptotically k-strict pseudocontractive type mapping in the intermediate sense with sequence $\{\gamma_n\}$ if there exists a constant $k \in [0,1)$ and a sequence $\{\gamma_n\}$ in $[0,\infty)$

with $\lim_{n\to\infty}\gamma_n = 0$ such that

$$\limsup_{n\to\infty}\sup_{x,y\in B}\Bigl(\|T^n x - T^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - k\max\bigl\{\|x - T^n x - (y - T^n y)\|, \|x - T^n x + (y - T^n y)\|\bigr\}^2\Bigr) \le 0. \tag{1.1}$$

Throughout this paper we assume

$$\Theta_n := \max\Bigl\{0, \sup_{x,y\in B}\Bigl(\|T^n x - T^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - k\max\bigl\{\|x - T^n x - (y - T^n y)\|, \|x - T^n x + (y - T^n y)\|\bigr\}^2\Bigr)\Bigr\}.$$

Then $\Theta_n \ge 0$ ($\forall n \ge 1$), $\Theta_n \to 0$ ($n\to\infty$), and (1.1) reduces to the relation

$$\|T^n x - T^n y\|^2 \le (1+\gamma_n)\|x-y\|^2 + k\max\bigl\{\|x - T^n x - (y - T^n y)\|, \|x - T^n x + (y - T^n y)\|\bigr\}^2 + \Theta_n$$

for all $x, y \in B$ and $n \ge 1$.

Lemma 1.2 [13] *Suppose that $\{\delta_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences of nonnegative numbers satisfying the recursive inequality*

$$\delta_{n+1} \le \beta_n\delta_n + \gamma_n, \quad \forall n \ge 1,$$

if $\beta_n \ge 1$, $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$, then $\lim_{n\to\infty}\delta_n$ exists.

Lemma 1.3 [14] *Assume that $\{a_n\}$ is a sequence of nonnegative numbers such that*

$$a_{n+1} \le (1-\alpha_n)a_n + \delta_n, \quad n \ge 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence of real numbers such that
- (I) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;
- (II) $\limsup_{n\to\infty}\frac{\delta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty}a_n = 0$.

Lemma 1.4 [15] *Let X be a uniformly convex Banach space, $\{t_n\}$ be a sequence of real numbers in $(0,1)$ bounded away from 0 and 1, and $\{x_n\}$ and $\{y_n\}$ be sequences of X such that $\limsup_{n\to\infty}\|x_n\| \le a$, $\limsup_{n\to\infty}\|y_n\| \le a$ and $\limsup_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = a$ for some $a \ge 0$, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 1.5 [13] *Let $\{x_n\}$ be a bounded sequence on a reflexive Banach space X. If $w_w(\{x_n\}) = \{x\}$, then $x_n \rightharpoonup x$.*

Lemma 1.6 [13] *Let $\mathcal{H}$ be a real Hilbert space.*
- (i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all $x, y \in \mathcal{H}$;*
- (ii) $\|(1-t)x + ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - t(1-t)\|x-y\|^2$ *for all $t \in [0,1]$ and for all $x, y \in \mathcal{H}$;*

(iii) *If* $\{x_n\}$ *is a sequence in* $\mathcal{H}$ *such that* $x_n \rightharpoonup x$, *it follows that*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in \mathcal{H}.$$

Lemma 1.7 [13] *Let* B *be a nonempty subset of a Hilbert space* $\mathcal{H}$ *and* $T : B \to B$ *be an asymptotically* k*-strict pseudocontractive type mapping in the intermediate sense with sequence* $\{\gamma_n\}$. *Then*

$$\|T^n x - T^n y\| \leq \frac{1}{1-k}\Big(k\|x-y\| + \sqrt{\big(1+(1-k)\gamma_n\big)\|x-y\|^2 + (1-k)h_n(x,y)}\Big)$$

for all $x, y \in B$ *and* $n \geq 1$, *where* $h_n(x,y) = 4k\|y - T^n y\|\|x - T^n x + y - T^n y\| + \Theta_n$. *In particular, if* $F(T) \neq \phi$, *then the above inequality reduces to the following*

$$\|T^n x - q\| \leq \frac{1}{1-k}\Big(k\|x-q\| + \sqrt{\big(1+(1-k)\gamma_n\big)\|x-q\|^2 + (1-k)\Theta_n}\Big)$$

for all $x \in B$, $q \in F(T)$ *and* $n \geq 1$.

Lemma 1.8 [13] *Let* B *be a nonempty subset of a Hilbert space* $\mathcal{H}$ *and* $T : B \to B$ *be a uniformly continuous asymptotically* k*-strict pseudocontractive type mapping in the intermediate sense with sequence* $\{\gamma_n\}$. *Let* $\{x_n\}$ *be a bounded sequence in* B *such that* $\|x_n - x_{n+1}\| \to 0$ *and* $\|x_n - T^n x_n\| \to 0$ *as* $n \to \infty$. *If* $F(T) \neq \phi$, *then* $\|x_n - Tx_n\| \to 0$ *as* $n \to \infty$.

Proposition 1.9 [13] *Let* B *be a nonempty, closed and convex subset of a Hilbert space* $\mathcal{H}$ *and* $T : B \to B$ *be a continuous asymptotically* k*-strict pseudocontractive type mapping in the intermediate sense with sequence* $\{\gamma_n\}$ *such that* $F(T) \neq \phi$. *Then* $I - T$ *is demiclosed at zero in the sense that if* $\{x_n\}$ *is a sequence in* B *such that* $x_n \rightharpoonup x \in B$ *and* $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| = 0$, *then* $(I - T)x = 0$.

2 Semigroup

Let B be a nonempty and closed subset of a Hilbert space $\mathcal{H}$ and $I : B \to B$ be an identity mapping.

A one-parameter family $\zeta = \{T(t) : 0 \leq t < \infty\}$ from self-mappings of a nonempty closed convex subset B of a Hilbert space $\mathcal{H}$ is said to be a nonexpansive semigroup on B if the following conditions are satisfied:

(I) $T(0)x = x$ for all $x \in B$;

(II) $T(s+t)x = T(s)T(t)x$ for all $x \in B$ and $s, t \geq 0$;

(III) For each $x \in B$, the mapping $t \to T(t)x$ is continuous on $[0, \infty)$;

(IV) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in B$.

We denote by $\mathrm{Fix}(\zeta)$ the set of all common fixed points of ζ; that is, $\mathrm{Fix}(\zeta) = \{x \in B : T(s)x = x, \forall s > 0\}$. $\mathrm{Fix}(\zeta)$ is nonempty if B is bounded.

Lemma 2.1 [16] *Let* B *be a nonempty, bounded, closed and convex subset of* $\mathcal{H}$ *and* $\zeta = \{T(t) : 0 \leq t < \infty\}$ *be a nonexpansive semigroup on* B. *Then*

$$\limsup_{s\to\infty} \limsup_{t\to\infty} \sup_{x\in B} \left\| \frac{1}{t}\int_0^t T(u)x\,du - T(s)\left(\frac{1}{t}\int_0^t T(u)x\,du\right)\right\| = 0.$$

In this section, we study a new modified iteration process. This process is defined by

$$\begin{cases} x_1 \in B, \\ u_n = \frac{1}{n+1}\sum_{j=0}^{n} T_0^j x_n, \\ x_{n,1} = (1-\beta_{n,0})x_n + \beta_{n,0}\frac{1}{t_n}\int_0^{t_n}\phi(s)u_n\,ds, \\ x_{n,2} = (1-\beta_{n,1})x_n + \beta_{n,1}T_1^n x_{n,1}, \\ \vdots \\ x_{n,m} = (1-\beta_{n,m-1})x_n + \beta_{n,m-1}T_{m-1}^n x_{n,m-1}, \\ x_{n+1} = (1-\beta_{n,m})\psi(x_n) + \beta_{n,m}T_m^n x_{n,m}, \end{cases} \tag{2.1}$$

where $\{\beta_{n,i}\}$, $i = 0,\dots,m$, are the sequences in $(0,1)$ such that the following conditions are satisfied:

(1) $0 < \liminf_{n\to\infty}\beta_{n,i} \le \limsup_{n\to\infty}\beta_{n,i} < 1$.

(2) $T_i : B \to B$ are uniformly continuous asymptotically k_i-strict pseudocontractive type mappings in the intermediate sense with sequences $\{\gamma_{n,i}\}$ if there exist constants $k_i \in [0,1)$ and sequences $\{\gamma_{n,i}\}$ in $[0,\infty)$ with $\lim_{n\to\infty}\gamma_{n,i} = 0$ for $i = 0,\dots,m$ such that

$$\begin{aligned} &\left\|T_i^n x - T_i^n y\right\|^2 \\ &\quad\le (1+\gamma_{ni})\|x-y\|^2 \\ &\qquad + k_i \max\left\{\left\|x - T_i^n x - \left(y - T_i^n y\right)\right\|, \left\|x - T_i^n x + \left(y - T_i^n y\right)\right\|\right\}^2 + \Theta_{ni}, \end{aligned}$$

for $i = 0,\dots,m$ and for all $x, y \in B$, $n \ge 1$.

(3) ψ is a contractive mapping on B with coefficient λ.

(4) $\zeta = \{\phi(t) : 1 \le t < \infty\}$ is a nonexpansive semigroup on B.

(5) $\{t_n\} \subset [1,+\infty)$ is a positive real divergent sequence.

Now, we prove that the sequence $\{x_n\}$ generated by (2.1) is weakly convergent in a Hilbert space $\mathcal{H}$.

Theorem 2.2 *Let B be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$. Suppose that $T_i : B \to B$ are uniformly continuous asymptotically k_i-strict pseudocontractive type mappings in the intermediate sense for $i = 0,\dots,m$, for all $x \in B$, $\prec x - T_i x, T_i x \succ \ge 0$. Assume that $\sum_{n=1}^{\infty}\gamma_{ni} < \infty$ and $\{x_n\}$ is a sequence defined by* (2.1). *If $F = \bigcap_{i=0}^{m} F(T_i) \cap F(\psi) \cap F(\xi) \ne \phi$, then*

(1) $\lim_{n\to\infty}\|x_n - q\|$ *exists for all* $q \in F$;

(2) $\lim_{n\to\infty}\|x_n - u_n\| = 0$ *and* $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ $(i = 0,\dots,m)$;

(3) *The sequence $\{x_n\}$ is weakly convergent to $q \in F$.*

Proof By the Hölder inequality,

$$\|u_n - q\|^2 = \left\|\frac{1}{n+1}\sum_{j=0}^{n} T_0^j x_n - q\right\|^2 \le \frac{n}{(n+1)^2}\sum_{j=0}^{n}\left\|T_0^j x_n - q\right\|^2$$

for $1 \le j \le n$,

$$\left\|T_0^j x_n - q\right\|^2 \le (1+\gamma_{j0})\|x_n - q\|^2 + k_0\left\|x_n - T_0^j x_n\right\|^2 + \Theta_{j0}.$$

Using Lemma 1.6(i), we have

$$\begin{aligned}\|x_n - T_0^j x_n\|^2 &= \|(x_n - q) - (T_0^j x_n - q)\|^2 \\ &= \|x_n - q\|^2 - \|T_0^j x_n - q\|^2 - 2\prec x_n - T_0^j x_n, T_0^j x_n\succ,\end{aligned}$$

since $\prec x_n - T_0^j x_n, T_0^j x_n\succ \geq 0$,

$$\|T_0^j x_n - q\|^2 \leq (1+\gamma_{j0})\|x_n - q\|^2 + k_0\|x_n - q\|^2 - k_0\|T_0^j x_n - q\|^2 + \Theta_{j0},$$

therefore

$$\|T_0^j x_n - q\|^2 \leq \left(1 + \frac{\gamma_{j0}}{1+k_0}\right)\|x_n - q\|^2 + \frac{\Theta_{j0}}{1+k_0}.$$

By process (2.1), we get

$$\begin{aligned}\|u_n - q\|^2 &\leq \frac{n}{(n+1)^2}\left(\|x_n - q\|^2 + \sum_{j=1}^n \|T_0^j x_n - q\|^2\right) \\ &\leq \frac{n}{(n+1)^2}\left(\|x_n - q\|^2 + \sum_{j=1}^n \left(\left(1 + \frac{\gamma_{j0}}{1+k_0}\right)\|x_n - q\|^2 + \frac{\Theta_{j0}}{1+k_0}\right)\right) \\ &\leq \frac{n(n+1)(1+\frac{\varsigma_n}{1+k_0})}{(n+1)^2}\|x_n - q\|^2 + \frac{n^2}{(n+1)^2}\frac{\upsilon_n}{1+k_0} \\ &\leq \left(1 + \frac{\varsigma_n}{1+k_0}\right)\|x_n - q\|^2 + \frac{\upsilon_n}{1+k_0}, \end{aligned} \tag{2.2}$$

where $\varsigma_n = \max\{\gamma_{j0}, 1 \leq j \leq n\}$ and $\upsilon_n = \max\{\Theta_{j0}, 1 \leq j \leq n\}$. By process (2.1) and inequality (2.2),

$$\begin{aligned}\|x_{n,1} - q\|^2 &= \left\|(1-\beta_{n,0})x_n + \beta_{n,0}\frac{1}{t_n}\int_0^{t_n}\phi(s)u_n\,ds - q\right\|^2 \\ &= \left\|(1-\beta_{n,0})(x_n - q) + \beta_{n,0}\frac{1}{t_n}\left(\int_0^{t_n}(\phi(s)u_n - q)\,ds\right)\right\|^2 \\ &= (1-\beta_{n,0})\|x_n - q\|^2 + \beta_{n,0}\left\|\frac{1}{t_n}\int_0^{t_n}(\phi(s)u_n - q)\,ds\right\|^2 \\ &\quad - \beta_{n,0}(1-\beta_{n,0})\left\|x_n - \frac{1}{t_n}\int_0^{t_n}\phi(s)u_n\,ds\right\|^2 \\ &\leq (1-\beta_{n,0})\|x_n - q\|^2 + \beta_{n,0}\|u_n - q\|^2 \\ &\quad - \beta_{n,0}(1-\beta_{n,0})\left\|x_n - \frac{1}{t_n}\int_0^{t_n}\phi(s)u_n\,ds\right\|^2 \\ &\leq (1-\beta_{n,0})\|x_n - q\|^2 + \beta_{n,0}\left[\rho_0\|x_n - q\|^2 + \sigma_0\right] \\ &\quad - \beta_{n,0}(1-\beta_{n,0})\left\|x_n - \frac{1}{t_n}\int_0^{t_n}\phi(s)u_n\,ds\right\|^2, \end{aligned} \tag{2.3}$$

where $\rho_0 = 1 + \frac{\varsigma_n}{k_0+1}$ and $\sigma_0 = \frac{\upsilon_n}{k_0+1}$. For $2 \le i \le m$,

$$\begin{aligned}
\|x_{n,i} - q\|^2 &= \left\|(1 - \beta_{n,i-1})x_n + \beta_{n,i-1}T_{i-1}^n x_{n,i-1} - q\right\|^2 \\
&= (1 - \beta_{n,i-1})\|x_n - q\|^2 + \beta_{n,i-1}\left\|T_{i-1}^n x_{n,i-1} - q\right\|^2 \\
&\quad - \beta_{n,i-1}(1 - \beta_{n,i-1})\left\|x_n - T_{i-1}^n x_{n,i-1}\right\|^2,
\end{aligned}$$

using Lemma 1.6(i), for $2 \le i \le m$,

$$\left\|T_{i-1}^n x_{n,i-1} - q\right\|^2 \le \left(1 + \frac{\gamma_{ni-1}}{1 + k_{i-1}}\right)\|x_{n,i-1} - q\|^2 + \frac{\Theta_{ni-1}}{1 + k_{i-1}}. \tag{2.4}$$

Let $\rho_i = 1 + \frac{\gamma_{ni}}{1+k_i}$ and $\sigma_i = \frac{\Theta_{ni}}{1+k_i}$ for $i = 1, \dots, m$,

$$\begin{aligned}
\|x_{n,i} - q\|^2 &\le (1 - \beta_{n,i-1})\|x_n - q\|^2 + \beta_{n,i-1}\left[\rho_{i-1}\|x_{n,i-1} - q\|^2 + \sigma_{i-1}\right] \\
&\quad - \beta_{n,i-1}(1 - \beta_{n,i-1})\left\|x_n - T_{i-1}^n x_{n,i-1}\right\|^2.
\end{aligned}$$

Also

$$\begin{aligned}
\|x_{n,m} - q\|^2 &\le (1 - \beta_{n,m-1})\|x_n - q\|^2 + \beta_{n,m-1}\left[\rho_{m-1}\|x_{n,m-1} - q\|^2 + \sigma_{m-1}\right] \\
&\quad - \beta_{n,m-1}(1 - \beta_{n,m-1})\left\|x_n - T_{m-1}^n x_{n,m-1}\right\|^2 \\
&\le (1 - \beta_{n,m-1})\|x_n - q\|^2 + \beta_{n,m-1}\left[\rho_{m-1}\left[(1 - \beta_{n,m-2})\|x_n - q\|^2\right.\right. \\
&\quad + \beta_{n,m-2}\left[\rho_{m-2}\|x_{n,m-2} - q\|^2 + \sigma_{m-2}\right] \\
&\quad \left.\left. - \beta_{n,m-2}(1 - \beta_{n,m-2})\left\|x_n - T_{m-2}^n x_{n,m-2}\right\|^2\right] + \sigma_{m-1}\right] \\
&\quad - \beta_{n,m-1}(1 - \beta_{n,m-1})\left\|x_n - T_{m-1}^n x_{n,m-1}\right\|^2,
\end{aligned}$$

continuing this process

$$\begin{aligned}
\|x_{n,m} - q\|^2 &\le \left\{(1 - \beta_{n,m-1}) + \beta_{n,m-1}\rho_{m-1}\left[(1 - \beta_{n,m-2})\right.\right. \\
&\quad + \beta_{n,m-2}\rho_{m-2}\left[(1 - \beta_{n,m-3}) + \beta_{n,m-3}\rho_{m-3}\left[\cdots\left[\cdots\left[(1 - \beta_{n,1})\right.\right.\right.\right. \\
&\quad \left.\left.\left.\left.\left.\left. + \beta_{n,1}\rho_1\left[(1 - \beta_{n,0}) + \beta_{n,0}\rho_0\right]\right]\cdots\right]\right\}\|x_n - q\|^2\right.\right.\right. \\
&\quad + \beta_{n,m-1}\sigma_{m-1} + \beta_{n,m-1}\beta_{n,m-2}\rho_{m-1}\sigma_{m-2} \\
&\quad + \beta_{n,m-1}\beta_{n,m-2}\beta_{n,m-3}\rho_{m-1}\rho_{m-2}\sigma_{m-3} \\
&\quad + \cdots \\
&\quad + \beta_{n,m-1}\beta_{n,m-2}\cdots\beta_{n,0}\rho_{m-1}\rho_{m-2}\cdots\rho_1\sigma_0,
\end{aligned} \tag{2.5}$$

so

$$\|x_{n,m} - q\|^2 \le \mu_n\|x_n - q\|^2 + \eta_n, \tag{2.6}$$

where

$$\begin{aligned}\mu_n &= (1-\beta_{n,m-1}) + \beta_{n,m-1}\rho_{m-1}\big[(1-\beta_{n,m-2})\\ &\quad + \beta_{n,m-2}\rho_{m-2}\big[(1-\beta_{n,m-3}) + \beta_{n,m-3}\rho_{m-3}\big[\cdots\big[\cdots\big[(1-\beta_{n,1})\\ &\quad + \beta_{n,1}\rho_1\big[(1-\beta_{n,0}) + \beta_{n,0}\rho_0\big]\big]\cdots\big],\end{aligned}$$

and

$$\begin{aligned}\eta_n &= \beta_{n,m-1}\sigma_{m-1} + \beta_{n,m-1}\beta_{n,m-2}\rho_{m-1}\sigma_{m-2}\\ &\quad + \beta_{n,m-1}\beta_{n,m-2}\beta_{n,m-3}\rho_{m-1}\rho_{m-2}\sigma_{m-3}\\ &\quad + \cdots\\ &\quad + \beta_{n,m-1}\cdots\beta_{n,0}\rho_{m-1}\rho_{m-2}\cdots\rho_1\sigma_0.\end{aligned}$$

By process (2.1),

$$\begin{aligned}\|x_{n+1}-q\|^2 &= (1-\beta_{n,m})\big\|\psi(x_n)-q\big\|^2 + \beta_{n,m}\big\|T_m^n x_{n,m}-q\big\|^2\\ &\quad - \beta_{n,m}(1-\beta_{n,m})\big\|\psi(x_n)-T_m^n x_{n,m}\big\|^2\\ &\le (1-\beta_{n,m})\lambda^2\|x_n-q\|^2 + \beta_{n,m}\big[\rho_m\|x_{n,m}-q\|^2+\sigma_m\big]\\ &\quad - \beta_{n,m}(1-\beta_{n,m})\big\|\psi(x_n)-T_m^n x_{n,m}\big\|^2,\end{aligned} \tag{2.7}$$

by inequalities (2.6) and (2.7)

$$\|x_{n+1}-q\|^2 \le \big[(1-\beta_{n,m})\lambda^2 + \beta_{n,m}\rho_m\mu_n\big]\|x_n-q\|^2 + \beta_{n,m}(\rho_m\eta_n+\sigma_m). \tag{2.8}$$

Since $\lim_{n\to\infty}[(1-\beta_{n,m})\lambda^2 + \beta_{n,m}\rho_m\mu_n] \le 1$ and $\lim_{n\to\infty}\beta_{n,m}(\rho_m\eta_n+\sigma_m) = 0$, by Lemma 1.2 we deduce that $\lim_{n\to\infty}\|x_n-q\| = h$ exists for some $h>0$.

By inequality (2.6), $\limsup_{n\to\infty}\|x_{n,m}-q\| \le h$, also by inequality (2.7), since

$$\frac{\|x_{n+1}-q\|^2 - \|x_n-q\|^2 + \beta_{n,m}\|x_n-q\|^2}{\beta_{n,m}\rho_m} \le \|x_{n,m}-q\|^2 + \frac{\sigma_m}{\rho_m},$$

then $h \le \liminf_{n\to\infty}\|x_{n,m}-q\|$ and $\lim_{n\to\infty}\|x_{n,m}-q\| = h$.

By the same argument, $\lim_{n\to\infty}\|x_{n,i}-q\| = h$ for $i = 2,\ldots,m-1$. By process (2.1),

$$\lim_{n\to\infty}\|x_{n,i}-q\| = \lim_{n\to\infty}\big\|(1-\beta_{n,i-1})(x_n-q) + \beta_{n,i-1}\big(T_{i-1}^n x_{n,i-1}-q\big)\big\| = h,$$

and by inequality (2.4), $\lim_{n\to\infty}\|T_i^n x_{n,i}-q\| \le h$, then by Lemma 1.4, $\lim_{n\to\infty}\|x_n - T_{i-1}^n x_{n,i-1}\| = 0$ for $i = 2,\ldots,m$.

Now, we show $\|x_n - T_{i-1}^n x_n\| \to 0$ for $i = 2,\ldots,m$, by process (2.1),

$$\|x_{n,i}-x_n\| = \beta_{n,i-1}\big\|x_n - T_{i-1}^n x_{n,i-1}\big\| \to 0 \quad \text{as } n\to\infty,$$

also

$$\big\|x_n - T_{i-1}^n x_n\big\| \le \big\|x_n - T_{i-1}^n x_{n,i-1}\big\| + \big\|T_{i-1}^n x_{n,i-1} - T_{i-1}^n x_n\big\|,$$

and by Lemma 1.7,

$$\begin{aligned}
&\left\| T_{i-1}^n x_n - T_{i-1}^n x_{n,i-1} \right\| \\
&\quad \leq \frac{1}{1-k_{i-1}} \\
&\qquad \times \Bigl(k_{i-1}\|x_n - x_{n,i-1}\| + \sqrt{\bigl(1+(1-k_{i-1})\gamma_{ni-1}\bigr)\|x_n - x_{n,i-1}\|^2 + (1-k_{i-1})h_n(x_n, x_{n,i-1})} \Bigr),
\end{aligned}$$

where $h_n(x_n, x_{n,i-1}) = 4k_{i-1}\|x_{n,i-1} - T_{i-1}^n x_{n,i-1}\| \|x_n - T_{i-1}^n x_n + x_{n,i-1} - T_{i-1}^n x_{n,i-1}\| + \Theta_{ni-1}$. We have

$$\begin{aligned}
&\left\| x_{n,i-1} - T_{i-1}^n x_{n,i-1} \right\|^2 \\
&\quad = \left\| (1-\beta_{n,i-2})\bigl(x_n - T_{i-1}^n x_{n,i-1}\bigr) + \beta_{n,i-2}\bigl(T_{i-2}^n x_{n,i-2} - T_{i-1}^n x_{n,i-1}\bigr) \right\|^2 \\
&\quad = (1-\beta_{n,i-2})\left\| x_n - T_{i-1}^n x_{n,i-1} \right\|^2 + \beta_{n,i-2}\left\| T_{i-2}^n x_{n,i-2} - T_{i-1}^n x_{n,i-1} \right\|^2 \\
&\qquad - \beta_{n,i-2}(1-\beta_{n,i-2})\left\| x_n - T_{i-2}^n x_{n,i-2} \right\|^2 \\
&\quad \leq (1-\beta_{n,i-2})\bigl[(1+\gamma_{ni-1})\|x_n - x_{n,i-1}\|^2 \\
&\qquad + k_{i-1}\left\| x_{n,i-1} - T_{i-1}^n x_{n,i-1} \right\|^2 + \Theta_{ni-1}\bigr] \\
&\qquad + \beta_{n,i-2}\bigl[(1+\gamma_{ni-1})\left\| T_{i-2}^n x_{n,i-2} - x_{n,i-1} \right\|^2 \\
&\qquad + k_{i-1}\left\| x_{n,i-1} - T_{i-1}^n x_{n,i-1} \right\|^2 + \Theta_{ni-1}\bigr] \\
&\qquad - \beta_{n,i-2}(1-\beta_{n,i-2})\left\| x_n - T_{i-2}^n x_{n,i-2} \right\|^2 \\
&\quad = (1-\beta_{n,i-2})\bigl[(1+\gamma_{ni-1})\beta_{n,i-2}^2\left\| x_n - T_{i-2}^n x_{n,i-2} \right\|^2 \\
&\qquad + k_{i-1}\left\| x_{n,i-1} - T_{i-1}^n x_{n,i-1} \right\|^2 + \Theta_{ni-1}\bigr] \\
&\qquad + \beta_{n,i-2}\bigl[(1+\gamma_{ni-1})(1-\beta_{n,i-2})^2\left\| x_n - T_{i-2}^n x_{n,i-2} \right\|^2 \\
&\qquad + k_{i-1}\left\| x_{n,i-1} - T_{i-1}^n x_{n,i-1} \right\|^2 + \Theta_{ni-1}\bigr] - \beta_{n,i-2}(1-\beta_{n,i-2})\left\| x_n - T_{i-2}^n x_{n,i-2} \right\|^2 \\
&\quad = k_{i-1}\left\| x_{n,i-1} - T_{i-1}^n x_{n,i-1} \right\|^2 + \beta_{n,i-2}(1-\beta_{n,i-2})\gamma_{ni-1}\left\| x_n - T_{i-2}^n x_{n,i-2} \right\|^2 + \Theta_{ni-1},
\end{aligned}$$

therefore

$$(1-k_{i-1})\left\| x_{n,i-1} - T_{i-1}^n x_{n,i-1} \right\|^2 \leq \beta_{n,i-2}(1-\beta_{n,i-2})\gamma_{ni-1}\left\| x_n - T_{i-2}^n x_{n,i-2} \right\|^2 + \Theta_{ni-1},$$

it means that $\|x_{n,i-1} - T_{i-1}^n x_{n,i-1}\| \to 0$, $h_n(x_n, x_{n,i-1}) \to 0$, $\|T_{i-1}^n x_n - T_{i-1}^n x_{n,i-1}\| \to 0$ and $\|x_n - T_{i-1}^n x_n\| \to 0$ for $i = 2, \ldots, m$. Since

$$\lim_{n\to\infty} \|x_{n+1} - q\| = \lim_{n\to\infty} \left\| (1-\beta_{n,m})\bigl(\psi(x_n) - q\bigr) + \beta_{n,m}\bigl(T_m^n x_{n,m} - q\bigr) \right\| = h,$$

and

$$\lim_{n\to\infty} \left\| \psi(x_n) - q \right\| \leq \lim_{n\to\infty} \lambda\|x_n - q\| \leq h,$$

by inequality (2.4), $\lim_{n\to\infty}\|T_m^n x_{n,m}-q\|\leq h$, by Lemma 1.4, $\lim_{n\to\infty}\|\psi(x_n)-T_m^n x_{n,m}\|=0$. Since

$$\left\|x_{n+1}-T_m^n x_{n,m}\right\|=(1-\beta_{n,m})\left\|\psi(x_n)-T_m^n x_{n,m}\right\|,$$

then $\lim_{n\to\infty}\|x_{n+1}-T_m^n x_{n,m}\|=0$.

We show $\|x_n-T_m^n x_n\|\to 0$ and $\|x_{n+1}-x_n\|\to 0$. We have

$$\|x_{n+1}-x_n\|\leq\left\|x_{n+1}-T_m^n x_{n,m}\right\|+\left\|T_m^n x_{n,m}-x_{n,m}\right\|+\|x_{n,m}-x_n\|,$$

by the same argument

$$(1-k_m)\left\|x_{n,m}-T_m^n x_{n,m}\right\|^2\leq\beta_{n,m}(1-\beta_{n,m})\gamma_{nm}\left\|x_n-T_{m-1}^n x_{n,m-1}\right\|^2+\Theta_{nm},\tag{2.9}$$

it means that $\|x_{n,m}-T_m^n x_{n,m}\|\to 0$ and $\|x_{n+1}-x_n\|\to 0$. Since

$$\begin{aligned}\left\|x_n-T_m^n x_n\right\|&\leq\left\|x_n-T_m^n x_{n,m}\right\|+\left\|T_m^n x_{n,m}-T_m^n x_n\right\|\\&\leq\|x_n-x_{n+1}\|+\left\|x_{n+1}-T_m^n x_{n,m}\right\|+\left\|T_m^n x_{n,m}-T_m^n x_n\right\|,\end{aligned}$$

by Lemma 1.7 and inequality (2.9), $\|x_n-T_m^n x_n\|\to 0$.

We show $\lim_{n\to\infty}\|u_n-x_n\|=0$ and $\lim_{n\to\infty}\|T_0^n x_n-x_n\|=0$. Let $\Omega_n=\frac{1}{t_n}\int_0^{t_n}\phi(s)u_n\,ds$, by inequality (2.3) and process (2.1), $\lim_{n\to\infty}\|x_{n,1}-q\|=h$. Also

$$\lim_{n\to\infty}\|x_{n,1}-q\|=\lim_{n\to\infty}\left\|(1-\beta_{n,0})(x_n-q)+\beta_{n,0}(\Omega_n-q)\right\|=h.$$

By inequality (2.3),

$$\|\Omega_n-q\|^2=\left\|\frac{1}{t_n}\int_0^{t_n}\phi(s)u_n\,ds-q\right\|^2\leq\rho_0\|x_n-q\|^2+\sigma_0,$$

and $\lim_{n\to\infty}\|\Omega_n-q\|\leq h$ and by Lemma 1.4, $\lim_{n\to\infty}\|\Omega_n-x_n\|=0$. Since $\|x_{n,1}-x_n\|=\beta_{n,0}\|\Omega_n-x_n\|$, then $\lim_{n\to\infty}\|x_{n,1}-x_n\|=0$. Also $\|x_n-q\|\leq\|x_n-\Omega_n\|+\|\Omega_n-q\|$ and $\lim_{n\to\infty}\|\Omega_n-q\|=h$. By inequality (2.2), $\|u_n-q\|^2\leq\rho_0\|x_n-q\|^2+\sigma_0$, so $\limsup_{n\to\infty}\|u_n-q\|\leq\limsup_{n\to\infty}\|x_n-q\|=h$. On the other hand, by $\|\Omega_n-q\|^2=\|\frac{1}{t_n}\int_0^{t_n}K(s)u_n\,ds-q\|^2\leq\|u_n-q\|^2$, we have

$$\lim_{n\to\infty}\|x_n-q\|=\lim_{n\to\infty}\|\Omega_n-q\|=\lim_{n\to\infty}\|u_n-q\|=h,$$

and by the same argument $\lim_{n\to\infty}\|u_n-x_n\|=0$, $\lim_{n\to\infty}\|T_0^n x_n-x_n\|=0$. For all $0\leq r<\infty$, we note that

$$\begin{aligned}\left\|\phi(r)x_n-x_n\right\|&\leq\left\|\phi(r)x_n-\phi(r)\Omega_n\right\|+\left\|\phi(r)\Omega_n-\Omega_n\right\|+\|\Omega_n-x_n\|\\&\leq 2\|x_n-\Omega_n\|+\left\|\phi(r)\Omega_n-\Omega_n\right\|,\end{aligned}$$

by Lemma 2.1, $\|\phi(r)x_n-x_n\|\to 0$. Also by Lemma 1.8 we have $\|x_n-T_ix_n\|\to 0$ as $n\to\infty$ for $i=0,\dots,m$. Assume that $x_{ni}\to u$ weakly and $x_{nj}\to v$ weakly as $n\to\infty$. Then $u,v\in F$.

We prove that $u = v$. If $u \neq v$, by Opial's condition,

$$\begin{aligned}\lim_{n\to\infty} \|x_n - u\| &= \lim_{i\to\infty} \|x_{ni} - u\| \\ &< \lim_{i\to\infty} \|x_{ni} - v\| \\ &= \lim_{n\to\infty} \|x_n - v\| \\ &< \lim_{j\to\infty} \|x_{nj} - u\| \\ &= \lim_{n\to\infty} \|x_n - u\|,\end{aligned}$$

which is a contradiction. Therefore, we have the conclusion. □

Theorem 2.3 *Suppose that all of the conditions of Theorem* 2.2 *hold. If* $0 < \liminf_{n\to\infty} \beta_{n,i} \leq \limsup_{n\to\infty} \beta_{n,i} < 1$ *for all* $i = 0,\ldots,m-1$, $\lim_{n\to\infty} \beta_{n,m} = 0$ *and* $\sum_{n=1}^{\infty} \beta_{n,m} = \infty$, *then the sequence* $\{x_n\}_{n\geq 0}$ *generated by* (2.1) *is strongly convergent to* $q \in F$ *in* B.

Proof By the same argument of Theorem 2.2,

$$\|x_{n+1} - q\|^2 \leq \bigl[(1-\beta_{n,m})\lambda^2 + \beta_{n,m}\rho_m\mu_n\bigr]\|x_n - q\|^2 + \beta_{n,m}(\rho_m\eta_n + \sigma_m). \tag{2.10}$$

Since $\lim_{n\to\infty} \beta_{n,m} = 0$ and $\lim_{n\to\infty} \beta_{n,m}(\rho_m\eta_n + \sigma_m) = 0$, by Lemma 1.3 we deduce that $\lim_{n\to\infty} \|x_n - q\| = 0$. Also $\lim_{n\to\infty} \|x_n - T_{n,m}x_n\| = 0$ for $i = 0,\ldots,m$ and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. □

Now, we study a new modified iteration process. This process is defined by

$$\begin{cases} x_1 \in B, \\ u_n = \frac{1}{n+1}\sum_{j=0}^{n} T_0^j x_n, \\ x_{n,1} = (1-\beta_{n,0})x_n + \beta_{n,0}\frac{1}{t_n}\int_0^{t_n} \phi(u_n, s)\,ds, \\ x_{n,2} = (1-\beta_{n,1})x_n + \beta_{n,1}T_1^n x_{n,1}, \\ \vdots \\ x_{n,m} = (1-\beta_{n,m-1})x_n + \beta_{n,m-1}T_{m-1}^n x_{n,m-1}, \\ x_{n+1} = (1-\beta_{n,m})\psi(x_n) + \beta_{n,m}T_m^n x_{n,m}, \end{cases} \tag{2.11}$$

where $\{\beta_{n,i}\}$, $i = 0,\ldots,m$, are the sequences in $(0,1)$ and $\phi : B \times I \to B$ is an integrable function and $I = [0,b] \subset \mathbb{R}$. We have the following theorem.

Theorem 2.4 *Suppose that all of the conditions of Theorem* 2.2 *hold. If* $\phi : B \times I \to B$ *is a contraction mapping with constant* $l < 1 \in \mathbb{R}^+$ *such that for all* $x, y \in B$ *and* $t \in I$,

$$\bigl\|\phi(x,t) - \phi(y,t)\bigr\| \leq l\|x - y\|.$$

Also $0 < \liminf_{n\to\infty} \beta_{n,i} \leq \limsup_{n\to\infty} \beta_{n,i} < 1$ *for all* $i = 0,\ldots,m-1$, $\lim_{n\to\infty} \beta_{n,m} = 0$ *and* $\sum_{n=1}^{\infty} \beta_{n,m} = \infty$, *then the sequence* $\{x_n\}_{n\geq 0}$ *generated by* (2.11) *is strongly convergent to* $q \in F = \bigcap_{i=0}^{m} F(T_i) \cap F(\psi) \cap F(\phi) \neq \phi$ *in* B.

Proof By the same argument of Theorem 2.2,

$$
\begin{aligned}
\|x_{n,1} - q\|^2 &= \left\|(1 - \beta_{n,0})x_n + \beta_{n,0}\frac{1}{t_n}\int_0^{t_n} \phi(u_n, s)\,ds - q\right\|^2 \\
&= \left\|(1 - \beta_{n,0})(x_n - q) + \beta_{n,0}\frac{1}{t_n}\left(\int_0^{t_n} \big(\phi(u_n, s) - q\big)\,ds\right)\right\|^2 \\
&= (1 - \beta_{n,0})\|x_n - q\|^2 + \beta_{n,0}\left\|\frac{1}{t_n}\int_0^{t_n} \big(\phi(u_n, s) - q\big)\,ds\right\|^2 \\
&\quad - \beta_{n,0}(1 - \beta_{n,0})\left\|x_n - \frac{1}{t_n}\int_0^{t_n} \phi(u_n, s)\,ds\right\|^2 \\
&\leq (1 - \beta_{n,0})\|x_n - q\|^2 + \beta_{n,0}l^2\|u_n - q\|^2 \\
&\quad - \beta_{n,0}(1 - \beta_{n,0})\left\|x_n - \frac{1}{t_n}\int_0^{t_n} \phi(u_n, s)\,ds\right\|^2 \\
&\leq (1 - \beta_{n,0})\|x_n - q\|^2 + \beta_{n,0}l^2\left[\left(1 + \frac{\varsigma_n}{k_0 + 1}\right)\|x_n - q\|^2 + \frac{\upsilon_n}{k_0 + 1}\right] \\
&\quad - \beta_{n,0}(1 - \beta_{n,0})\left\|x_n - \frac{1}{t_n}\int_0^{t_n} \phi(u_n, s)\,ds\right\|^2.
\end{aligned}
\tag{2.12}
$$

Then

$$
\|x_{n+1} - q\|^2 \leq \big[(1 - \beta_{n,m}) + \beta_{n,m}\rho_m\mu_n\big]\|x_n - q\|^2 + \beta_{n,m}(\rho_m\eta_n + \sigma_m). \tag{2.13}
$$

Since $\lim_{n\to\infty} \beta_{n,m} = 0$ and $\lim_{n\to\infty} \beta_{n,m}(\rho_m\eta_n + \sigma_m) = 0$, by Lemma 1.3 we deduce that $\lim_{n\to\infty} \|x_n - q\| = 0$. □

3 Some examples

In this section, we consider the following examples to illustrate the theoretical results.

Example 3.1 Let $\mathcal{H} = \mathbb{R}$ be the set of real numbers and $B = [0, \infty)$. For each $x \in B$, we define

$$
T(x) = \begin{cases} \frac{k\sin x}{1+x} & \text{if } x \in [0, \frac{1}{2}], \\ 0 & \text{if } x \in (\frac{1}{2}, \infty), \end{cases} \tag{3.1}
$$

where $0 < k < \frac{1}{2}$. Set $B_1 = [0, \frac{1}{2}]$ and $B_2 = (\frac{1}{2}, \infty)$. Then, for all $x, y \in B_1$ and $n \geq 1$,

$$
|Tx - Ty| = \left|\frac{k\sin x}{1 + x} - \frac{k\sin y}{1 + y}\right| \leq k|\sin x - \sin y| \leq k|x - y|,
$$

and

$$
\left|T^2x - T^2y\right| = \left|\frac{kTx}{1 + Tx} - \frac{kTy}{1 + Ty}\right| \leq k|Tx - Ty| \leq k^2|x - y|,
$$

then for all $n \geq 1$, $|T^nx - T^ny| \leq k^n|x - y|$.

For all $x, y \in B_2$ and $n \geq 1$, $|T^n x - T^n y| = 0 \leq |x - y|$.
For all $x \in B_1$ and $y \in B_2$, $|Tx - Ty| = |\frac{k \sin x}{1+x} - 0| \leq |kx - 0|$,

$$\begin{aligned}
|T^n x - T^n y|^2 &\leq |k^n x - 0|^2 \\
&= |k^n(x - y) + k^n y|^2 \\
&\leq \left(\frac{k^{n-1}|x - y| + k^{n-1}|y|}{2}\right)^2 \\
&\leq \frac{1}{4}k^{2(n-1)}|x - y| + \frac{1}{4}k^{2(n-1)}\left|\left(y + x - T^n x\right) - \left(x - T^n y\right)\right|^2 \\
&\leq |x - y|^2 + \frac{1}{2}\max\left\{\left|x - T^n x - \left(y - T^n y\right)\right|,\right. \\
&\qquad \left.\left|x - T^n x + y - T^n y\right|\right\}^2 + \frac{1}{2}k^{2(n-1)}.
\end{aligned}$$

Therefore $T : B \to B$ is an asymptotically $\frac{1}{2}$-strict pseudocontractive type mapping in the intermediate sense.

Example 3.2 Let $\mathcal{H} = \mathbb{R}$ be the set of real numbers and $B = [0, \infty)$. For each $x \in B$, we define

$$T(x) = \begin{cases} \frac{kx}{1+x} & \text{if } x \in [0, \frac{1}{8}], \\ 0 & \text{if } x \in (\frac{1}{8}, \infty), \end{cases} \tag{3.2}$$

where $0 < k < \frac{1}{8}$. Therefore $T : B \to B$ is an asymptotically $\frac{1}{8}$-strict pseudocontractive type mapping in the intermediate sense.

Example 3.3 [13] Let $\mathcal{H} = \mathbb{R}$ be the set of real numbers and $B = [0, \infty)$. Suppose that $T : B \to B$ is defined by [13]

$$T(x) = \begin{cases} kx & \text{if } x \in [0, 1], \\ 0 & \text{if } x \in (1, \infty), \end{cases} \tag{3.3}$$

where $0 < k < \frac{1}{4}$. Then $T : B \to B$ is an asymptotically $\frac{1}{4}$-strict pseudocontractive type mapping in the intermediate sense.

Example 3.4 Let $\mathcal{H} = \mathbb{R}$ be the set of real numbers and $B = [0, \infty)$. Consider the following conditions:

(1) $T_0 x = \frac{x}{100}$, $T_1 x = \frac{x}{200}$, $T_2 x = \frac{\sin x}{100(1+x)}$ and $T_3 x = \frac{x}{100(1+x)}$;
(2) $\psi(x) = \frac{x}{100}$, therefore ψ is a contraction mapping with constant $\lambda = \frac{1}{100}$;
(3) $\beta_{n,0} = \frac{n}{10n+1}$, $\beta_{n,1} = \frac{n}{20n+1}$, $\beta_{n,2} = \frac{n}{30n+1}$ and $\beta_{n,3} = \frac{n}{40n+1}$;
(4) $\phi(s) = e^{-s}$ and $t_n = n$.

Let $\{x_n\}$ be the sequence defined by (2.1). So

n	x_n	n	x_n
1	1	11	7.875×10^{-21}
2	0.009876	12	7.678×10^{-23}
3	9.634×10^{-5}	13	7.478×10^{-25}
4	9.395×10^{-7}	14	7.3×10^{-27}
5	9.162×10^{-9}	15	7.118×10^{-29}
6	8.934×10^{-11}	16	6.94×10^{-31}
7	8.711×10^{-13}	17	6.767×10^{-33}
8	8.494×10^{-15}	18	6.598×10^{-35}
9	8.283×10^{-17}	19	6.433×10^{-37}
10	8.076×10^{-19}	20	6.273×10^{-39}

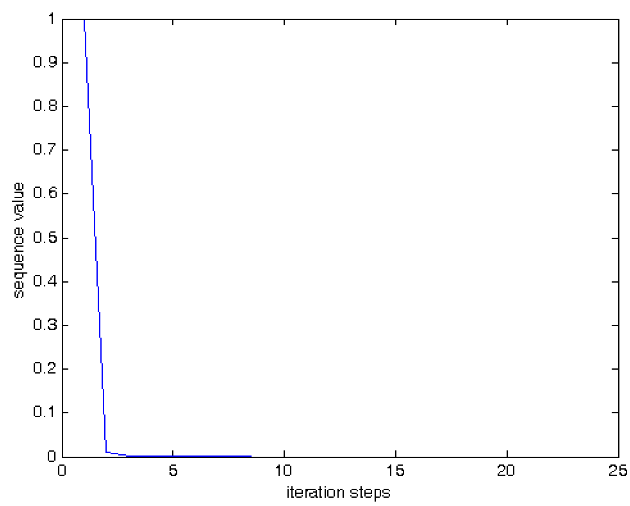

Figure 1 The iteration chart with initial value $x_1 = 1$.

$$\begin{cases} u_n = \frac{1}{n+1} \sum_{j=0}^{n} \frac{x_n}{100^j}, \\ x_{n,1} = \frac{9n+1}{10n+1} x_n + \frac{n}{10n+1} (\frac{1-e^{-n}}{n}) u_n, \\ x_{n,2} = \frac{19n+1}{20n+1} x_n + \frac{n}{20n+1} \frac{1}{200^n} x_{n,1}, \\ x_{n,3} = \frac{29n+1}{30n+1} x_n + \frac{n}{30n+1} T_2^n x_{n,2}, \\ x_{n+1} = \frac{39n+1}{40n+1} \frac{1}{100} x_n + \frac{n}{40n+1} T_3^n x_{n,3}, \end{cases}$$

and $\mathrm{Fix}(\zeta) \cap \mathrm{Fix}(\psi) \bigcap_{i=0}^{3} \mathrm{Fix}(T_i) = \{0\}$. Set $x_1 = 1$ (see Figure 1).

Example 3.5 Suppose that all of the conditions of Example 3.4 hold. Suppose that $\{x_n\}$ is defined by the process

$$\begin{cases} x_{n,1} = \frac{9n+1}{10n+1} x_n + \frac{n}{10n+1} \frac{1}{100^n} x_n, \\ x_{n,2} = \frac{19n+1}{20n+1} x_n + \frac{n}{20n+1} \frac{1}{200^n} x_{n,1}, \\ x_{n,3} = \frac{29n+1}{30n+1} x_n + \frac{n}{30n+1} T_2^n x_{n,2}, \\ x_{n+1} = \frac{39n+1}{40n+1} x_n + \frac{n}{40n+1} T_3^n x_{n,3}, \end{cases}$$

and set $x_1 = 1$ (see Figure 2).

n	x_n	n	x_n
1	1	11	0.77786
2	0.97572	12	0.75846
3	0.95163	13	0.73954
4	0.92804	14	0.72108
5	0.90498	15	0.70309
6	0.88247	16	0.68554
7	0.8605	17	0.66843
8	0.83906	18	0.65174
9	0.81815	19	0.63547
10	0.79775	20	0.6196

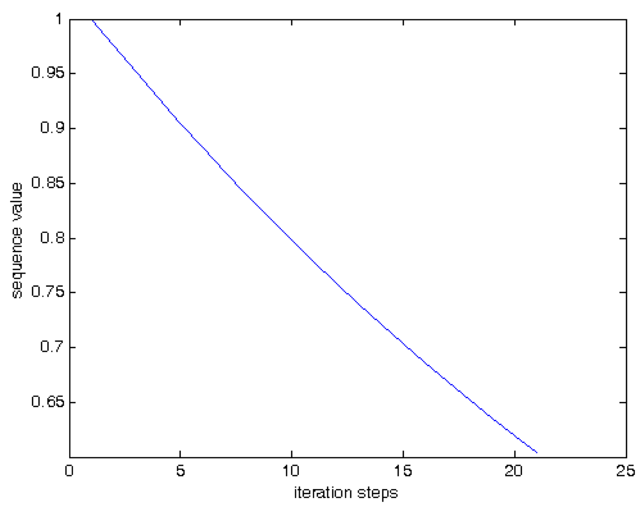

Figure 2 The iteration chart with initial value $x_1 = 1$.

The rate of convergence of our approximation is faster than the corresponding one in [13].

Example 3.6 Let $\mathcal{H} = \mathbb{R}$ be the set of real numbers and $B = [0, \infty)$. Consider the following conditions:

(1) $T_0 x = \frac{x}{100}$, $T_1 x = \frac{x}{200}$, $T_2 x = \frac{\sin x}{100(1+x)}$ and $T_3 x = \frac{x}{100(1+x)}$;
(2) $\psi(x) = \frac{x}{100}$, therefore ψ is a contraction mapping with constant $l = \frac{1}{100}$;
(3) $\beta_{n,0} = \frac{n}{10n+1}$, $\beta_{n,1} = \frac{n}{20n+1}$, $\beta_{n,2} = \frac{n}{30n+1}$ and $\beta_{n,3} = \frac{n}{40n+1}$;
(4) $\phi(x,t) = \frac{\sin x}{e^t}$ and $t_n = n$.

Let $\{x_n\}$ be the sequence defined by the process

$$\begin{cases} u_n = \frac{1}{n+1} \sum_{j=0}^{n} \frac{x_n}{100^j}, \\ x_{n,1} = \frac{9n+1}{10n+1} x_n + \frac{n}{10n+1} (\frac{1-e^{-n}}{n}) \sin u_n, \\ x_{n,2} = \frac{19n+1}{20n+1} x_n + \frac{n}{20n+1} \frac{1}{200^n} x_{n,1}, \\ x_{n,3} = \frac{29n+1}{30n+1} x_n + \frac{n}{30n+1} T_2^n x_{n,2} \\ x_{n+1} = \frac{39n+1}{40n+1} \frac{1}{100^n} x_n + \frac{n}{40n+1} T_3^n x_{n,3}, \end{cases}$$

and $\mathrm{Fix}(\phi) \cap \mathrm{Fix}(\psi) \bigcap_{i=0}^{3} \mathrm{Fix}(T_i) = \{0\}$. Set $x_1 = 1$ (see Figure 3).

n	x_n	n	x_n
1	1	11	9.798×10^{-111}
2	0.0098	12	9.79×10^{-133}
3	9.863×10^{-7}	13	9.782×10^{-157}
4	9.855×10^{-13}	14	9.774×10^{-183}
5	9.184×10^{-21}	15	9.766×10^{-211}
6	9.839×10^{-31}	16	9.758×10^{-241}
7	9.831×10^{-43}	17	9.75×10^{-273}
8	9.823×10^{-57}	18	9.741×10^{-307}
9	9.815×10^{-73}	19	0.0000
10	9.806×10^{-91}	20	0.0000

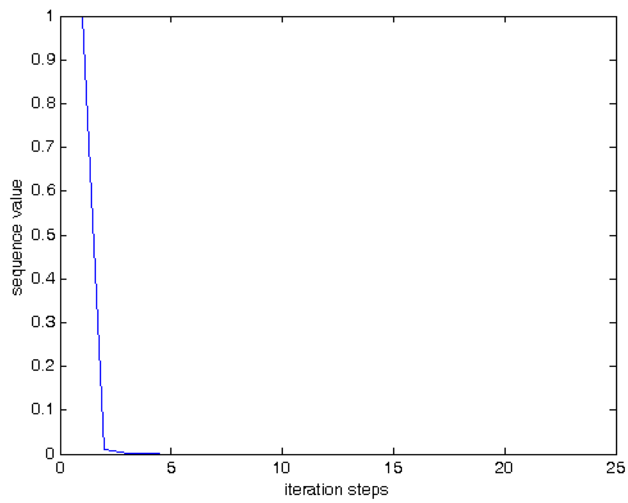

Figure 3 The iteration chart with initial value $x_1 = 1$.

Example 3.7 Suppose that all of the conditions of Example 3.4 hold. Set $\beta_{n,3} = \frac{1}{10n}$ and let $\{x_n\}$ be the sequence defined by the process

$$\begin{cases} u_n = \frac{1}{n+1}\sum_{j=0}^{n} \frac{x_n}{100^j}, \\ x_{n,1} = \frac{9n+1}{10n+1}x_n + \frac{n}{10n+1}(\frac{1-e^{-n}}{n})u_n, \\ x_{n,2} = \frac{19n+1}{20n+1}x_n + \frac{n}{20n+1}\frac{1}{200^n}x_{n,1}, \\ x_{n,3} = \frac{29n+1}{30n+1}x_n + \frac{n}{30n+1}T_2^n x_{n,2}, \\ x_{n+1} = \frac{10n-1}{10n}\frac{1}{100^n}x_n + \frac{1}{10n}T_3^n x_{n,3}, \end{cases}$$

and $\mathrm{Fix}(\zeta) \cap \mathrm{Fix}(\psi) \bigcap_{i=0}^{3} \mathrm{Fix}(T_i) = \{0\}$. Suppose $x_1 = 1$ (see Figure 4).

4 Conclusion

The stability of a fixed point iterative procedure was first studied by Ostrowski [17] in the case of Banach contraction mappings, and this subject was later developed for certain contractive definitions by several authors.

Let (X, d) be a complete metric space, $T : X \to X$ be a map and $x_{n+1} = f(T, x_n)$ be an iteration procedure. Suppose that T has at least one fixed point and that the sequence

n	x_n	n	x_n
1	1	11	9.4233×10^{-111}
2	0.00949	12	9.4204×10^{-133}
3	9.468×10^{-7}	13	9.4178×10^{-157}
4	9.4575×10^{-13}	14	9.4154×10^{-183}
5	9.4497×10^{-21}	15	9.413×10^{-211}
6	9.4435×10^{-31}	16	9.4111×10^{-241}
7	9.4383×10^{-43}	17	9.4091×10^{-273}
8	9.4338×10^{-57}	18	9.4073×10^{-307}
9	9.4299×10^{-73}	19	0.0000
10	9.4264×10^{-91}	20	0.0000

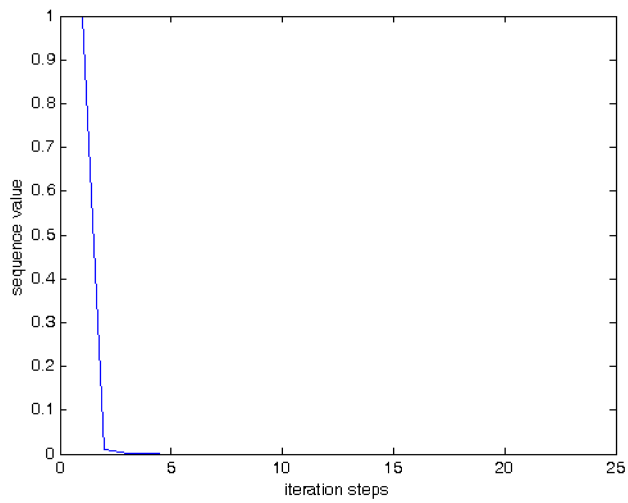

Figure 4 The iteration chart with initial value $x_1 = 1$.

$\{x_n\}$ converges to a fixed point $x^* \in X$. We denote the set of fixed points of the mapping T by $F(T)$. Let $\{y_n\}$ be an arbitrary sequence in X and $\varepsilon_n = d(y_{n+1}, f(T, y_n))$. If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = x^*$, then the iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T-stable. If $\{y_n\}$ is a bounded sequence and $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = x^*$, then the iteration procedure $x_{n+1} = f(T, x_n)$ is said to be boundedly T-stable.

According to the above definition, Haghi *et al.* [18] studied the T-stability of Picard's iteration for generalized ϕ-contraction mappings on a metric space. Also Olatinwo and Postolache [19] studied the stability for Jungck-type iterative processes in convex metric spaces.

Now, consider the modified iteration process (2.1), one interesting problem is studying the stability of the iteration scheme $\{x_n\}$ generated by (2.1).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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