Liu et al. *Fixed Point Theory and Applications* (2015) 2015:20
DOI 10.1186/s13663-015-0271-z

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Common fixed point theorems for four mappings satisfying ψ-weakly contractive conditions

Zeqing Liu[1], Xiaoping Zhang[2], Jeong Sheok Ume[3*] and Shin Min Kang[4]

*Correspondence: jsume@changwon.ac.kr
[3]Department of Mathematics, Changwon National University, Changwon, 641-773, Korea
Full list of author information is available at the end of the article

Abstract

The existence and uniqueness of common fixed points for four mappings satisfying ψ- and (ψ,φ)-weakly contractive conditions in metric spaces are proved. Four examples are given to demonstrate that the results presented in this paper generalize indeed some well-known results in the literature.

MSC: 54H25

Keywords: common fixed point; ψ-weakly contractive conditions; (ψ,φ)-weakly contractive conditions; weakly compatible mappings

1 Introduction and preliminaries

In 2001, Rhoades [1] introduced the concept of φ-weakly contractive mappings and proved the following fixed point theorem, which is a generalization of the Banach fixed point theorem.

Theorem 1.1 ([1]) *Let (X,d) be a complete metric space, and let $T:X\to X$ be a mapping such that*

$$d(Tx,Ty)\le d(x,y)-\varphi\bigl(d(x,y)\bigr),\quad \forall x,y\in X,$$

where $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is continuous and nondecreasing, and $\varphi(t)=0$ if and only if $t=0$. Then T has a unique fixed point.

Afterwards, the researchers [2–8] continued the study of Rhoades by introducing a few φ- and (ψ,φ)-weakly contractive conditions relative to one, two or three mappings and discussed the existence of fixed and common fixed point for these mappings. In particular, Abbas and Dorić [2], Abbas and Khan [3], and Dutta and Choudhury [5] proved the following fixed and common fixed point theorems for the φ- and (ψ,φ)-weakly contractive mappings.

Theorem 1.2 ([5]) *Let (X,d) be a complete metric space, and let $T:X\to X$ be a mapping satisfying the inequality*

$$\psi\bigl(d(Tx,Ty)\bigr)\le\psi\bigl(d(x,y)\bigr)-\varphi\bigl(d(x,y)\bigr),\quad \forall x,y\in X,$$

where $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ are both continuous and monotone nondecreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Theorem 1.3 ([3]) *Let T, S be two self mappings in a metric space (X, d) satisfying*

$$\psi\bigl(d(Tx, Ty)\bigr) \le \psi\bigl(d(Sx, Sy)\bigr) - \varphi\bigl(d(Sx, Sy)\bigr), \quad \forall x, y \in X,$$

where $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ are both continuous and monotone nondecreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. If range of S contains the range of T and $S(X)$ is a complete subspace of X, then T and S have a unique point of coincidence in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point.

Theorem 1.4 ([2]) *Suppose that A, B, S, and T are selfmaps of a complete metric space (X, d), $T(X) \subseteq B(X)$, $S(X) \subseteq A(X)$ and the pairs $\{A, T\}$ and $\{B, S\}$ are weakly compatible. If*

$$\psi\bigl(d(Tx, Sy)\bigr) \le \psi\bigl(M(x, y)\bigr) - \varphi\bigl(M(x, y)\bigr), \quad \forall x, y \in X,$$

where

$$M(x, y) = \max\left\{d(Ax, By), d(Ax, Tx), d(By, Sy), \frac{1}{2}\bigl[d(Ax, Sy) + d(Tx, By)\bigr]\right\}, \quad \forall x, y \in X,$$

$\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is lower semi-continuous, $\varphi(0) = 0$, $\varphi(t) > 0$ for all $t > 0$, $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing with $\psi(t) = 0$ if and only if $t = 0$, then A, B, S and T have a unique common fixed point in X provided one of the ranges of $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is closed.

Motivated by the results in [1–9], in this paper, we introduce the concepts of ψ- and (ψ, φ)-weakly contractive conditions relative to four mappings A, B, S and T:

$$d(Tx, Sy) \le \psi\bigl(M_i(x, y)\bigr), \quad \forall x, y \in X, \tag{1.1}$$

$$\psi\bigl(d(Tx, Sy)\bigr) \le \psi\bigl(M_i(x, y)\bigr) - \varphi\bigl(M_i(x, y)\bigr), \quad \forall x, y \in X, \tag{1.2}$$

where $i \in \{1, 2, 3\}$, $\psi \in \Phi_3$, $(\psi, \varphi) \in \Phi_1 \times \Phi_2$, respectively,

$$\begin{aligned} M_1(x, y) = \max\biggl\{ & d(Ax, By), d(Ax, Tx), d(By, Sy), \\ & \frac{1}{2}\bigl[d(Ax, Sy) + d(Tx, By)\bigr], \frac{d(Ax, Sy)d(Tx, By)}{1 + d(Ax, By)}, \\ & \frac{d(Ax, Tx)d(By, Sy)}{1 + d(Ax, By)}, \frac{1 + d(Ax, Sy) + d(Tx, By)}{1 + d(Ax, Tx) + d(By, Sy)} d(Ax, Tx)\biggr\}, \quad \forall x, y \in X, \end{aligned} \tag{1.3}$$

$$\begin{aligned} M_2(x, y) = \max\biggl\{ & d(Ax, By), d(Ax, Tx), d(By, Sy), \frac{1}{2}\bigl[d(Ax, Sy) + d(Tx, By)\bigr], \\ & \frac{1 + d(Ax, Tx)}{1 + d(Ax, By)} d(By, Sy), \frac{1 + d(By, Sy)}{1 + d(Ax, By)} d(Ax, Tx), \\ & \frac{1 + d(Ax, Sy) + d(Tx, By)}{1 + d(Ax, Tx) + d(By, Sy)} d(By, Sy)\biggr\}, \quad \forall x, y \in X \end{aligned} \tag{1.4}$$

and

$$M_3(x,y) = \max\left\{d(Ax,By), d(Ax,Tx), d(By,Sy), \frac{1}{2}\left[d(Ax,Sy) + d(Tx,By)\right]\right\},$$
$$\forall x,y \in X \tag{1.5}$$

and establish sufficient conditions which ensure the existence and uniqueness of common fixed points for the four mappings A, B, S and T satisfying ψ- and (ψ,φ)-weakly contractive conditions, respectively, in metric spaces. Our results extend, improve and unify the corresponding results in [1–5]. Four nontrivial examples are included.

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{R}^+ = [0,+\infty)$ and

$$\Phi_1 = \big\{\psi : \psi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is continuous and nondecreasing,}$$
$$\text{and } \psi(t) = 0 \text{ if and only if } t = 0\big\},$$
$$\Phi_2 = \big\{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is lower semi-continuous, and } \varphi(t) = 0 \text{ if and only if } t = 0\big\},$$
$$\Phi_3 = \Big\{\psi : \psi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is upper semi-continuous,}$$
$$\text{and } \lim_{n\to\infty} a_n = 0 \text{ for each sequence } \{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+ \text{ with } a_{n+1} \le \psi(a_n), \forall n \in \mathbb{N}\Big\}.$$

Definition 1.1 ([10]) A pair of self mappings f and g in a metric space (X,d) are said to be *weakly compatible* if for all $t \in X$ the equality $ft = gt$ implies $fgt = gft$.

Lemma 1.1 ([9]) *Let $\psi \in \Psi_3$. Then $\psi(0) = 0$ and $\psi(t) < t$ for all $t > 0$.*

Lemma 1.2 *Let A, B, S and T be self mappings in a metric space (X,d) satisfying* (1.2), *where $(\psi,\varphi) \in \Phi_1 \times \Phi_2$ and $i \in \{1,2,3\}$. Assume that $I : \mathbb{R}^+ \to \mathbb{R}^+$ is the identity mapping and*

$$\psi_1(t) = (\psi + I)^{-1}(\psi + I - \varphi)(t), \quad \forall t \in \mathbb{R}^+. \tag{1.6}$$

Then $\psi_1 \in \Phi_3$ and

$$d(Tx,Sy) \le \psi_1\big(M_i(x,y)\big), \quad \forall x,y \in X. \tag{1.7}$$

Proof It follows from $\psi \in \Phi_1$ that $\psi + I : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and increasing and $(\psi + I)(t) = 0$ if and only if $t = 0$. So does $(\psi + I)^{-1}$. Obviously, $(\psi,\varphi) \in \Phi_1 \times \Phi_2$ and (1.6) guarantee

$$\psi_1 \text{ is upper semi-continuous and } \psi_1(0) = 0. \tag{1.8}$$

Assume that $\{a_n\}_{n\in\mathbb{N}}$ is an arbitrary sequence in $\mathbb{R}^+$ with

$$a_{n+1} \le \psi_1(a_n), \quad \forall n \in \mathbb{N}. \tag{1.9}$$

Suppose that $a_{n_0} = 0$ for some $n_0 \in \mathbb{N}$. It follows from (1.6), (1.8) and (1.9) that

$$0 \le a_{n_0+1} \le \psi_1(a_{n_0}) = \psi_1(0) = 0,$$

that is, $a_{n_0+1} = 0$. Similarly we have $a_n = a_{n-1} = \cdots = a_{n_0} = 0$ for each $n > n_0$, that is, $\lim_{n\to\infty} a_n = 0$. Suppose that $a_n > 0$ for all $n \in \mathbb{N}$. If $a_{k+1} \geq a_k$ for some $k \in \mathbb{N}$, it follows from (1.6), (1.9) and $(\psi, \varphi) \in \Phi_1 \times \Phi_2$ that

$$\begin{aligned}\psi(a_k) + a_k &\leq \psi(a_{k+1}) + a_{k+1} = (\psi + I)(a_{k+1}) \leq (\psi + I)\psi_1(a_k) = (\psi + I - \varphi)(a_k) \\ &= \psi(a_k) + a_k - \varphi(a_k) < \psi(a_k) + a_k,\end{aligned}$$

which is a contradiction. Consequently, $\{a_n\}_{n\in\mathbb{N}}$ is positive and decreasing, which implies that $\{a_n\}_{n\in\mathbb{N}}$ converges to some $a \geq 0$. Suppose that $a > 0$. By means of (1.8) and (1.9), we find

$$0 < a = \limsup_{n\to\infty} a_{n+1} \leq \limsup_{n\to\infty} \psi_1(a_n) \leq \psi_1(a),$$

which together with (1.6) and $(\psi, \varphi) \in \Phi_1 \times \Phi_2$ means

$$\psi(a) + a \leq \psi(a) + a - \varphi(a) < \psi(a) + a,$$

which is a contradiction. Hence $a = 0$. Consequently, $\psi_1 \in \Phi_3$.

In order to prove (1.7), we have to consider two possible cases as follows:

Case 1. $M_i(x_0, y_0) = 0$ for some $x_0, y_0 \in X$. It is easy to verify

$$d(Ax_0, By_0) = d(Ax_0, Tx_0) = d(By_0, Sy_0) = 0,$$

which yields

$$Tx_0 = Ax_0 = By_0 = Sy_0,$$

and

$$d(Tx_0, Sy_0) = 0 = \psi_1\big(M_i(x_0, y_0)\big);$$

Case 2. $M_i(x, y) > 0$ for all $x, y \in X$. It follows from (1.2), (1.6) and $(\psi, \varphi) \in \Phi_1 \times \Phi_2$ that

$$\psi\big(d(Tx, Sy)\big) \leq \psi\big(M_i(x, y)\big) - \varphi\big(M_i(x, y)\big) < \psi\big(M_i(x, y)\big), \quad \forall x, y \in X,$$

which yields

$$d(Tx, Sy) < M_i(x, y), \quad \forall x, y \in X$$

and

$$\begin{aligned}(\psi + I)\big(d(Tx, Sy)\big) &= \psi\big(d(Tx, Sy)\big) + d(Tx, Sy) < \psi\big(M_i(x, y)\big) - \varphi\big(M_i(x, y)\big) + M_i(x, y) \\ &= (\psi + I - \varphi)\big(M_i(x, y)\big), \quad \forall x, y \in X,\end{aligned}$$

which together with (1.6) gives (1.7). This completes the proof. □

Remark 1.1 It follows from Lemma 1.2 that the (ψ, φ)-weakly contractive conditions (1.2) relative to four mappings A, B, S and T implies the ψ_1-weakly contractive conditions (1.1) relative to four mappings A, B, S and T.

2 Common fixed point theorems

Our main results are as follows.

Theorem 2.1 *Let A, B, S, and T be self mappings in a metric space (X,d) such that*

$$\{A,T\} \text{ and } \{B,S\} \text{ are weakly compatible;} \tag{2.1}$$

$$T(X) \subseteq B(X) \text{ and } S(X) \subseteq A(X); \tag{2.2}$$

$$\text{one of } A(X), B(X), S(X), \text{ and } T(X) \text{ is complete;} \tag{2.3}$$

$$d(Tx,Sy) \le \psi\big(M_1(x,y)\big), \quad \forall x,y \in X, \tag{2.4}$$

where ψ_1 is in Φ_3 and M_1 is defined by (1.3). *Then A, B, S, and T have a unique common fixed point in X.*

Proof Let $x_0 \in X$. It follows from (2.2) that there exist two sequences $\{y_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}_0}$ in X such that

$$y_{2n+1} := Bx_{2n+1} = Tx_{2n}, \qquad y_{2n+2} := Ax_{2n+2} = Sx_{2n+1}, \quad \forall n \in \mathbb{N}_0. \tag{2.5}$$

Put $d_n = d(y_n, y_{n+1})$ for all $n \in \mathbb{N}$.

Now we prove

$$\lim_{n\to\infty} d_n = 0. \tag{2.6}$$

Using (2.4) and (2.5), we derive

$$d_{2n} = d(Tx_{2n}, Sx_{2n-1}) \le \psi\big(M_1(x_{2n}, x_{2n-1})\big), \quad \forall n \in \mathbb{N} \tag{2.7}$$

and

$$\begin{aligned}
&M_1(x_{2n}, x_{2n-1}) \\
&\quad = \max\bigg\{ d(Ax_{2n}, Bx_{2n-1}), d(Ax_{2n}, Tx_{2n}), d(Bx_{2n-1}, Sx_{2n-1}), \\
&\qquad \frac{1}{2}\big[d(Ax_{2n}, Sx_{2n-1}) + d(Tx_{2n}, Bx_{2n-1})\big], \\
&\qquad \frac{d(Ax_{2n}, Sx_{2n-1})d(Tx_{2n}, Bx_{2n-1})}{1 + d(Ax_{2n}, Bx_{2n-1})}, \frac{d(Ax_{2n}, Tx_{2n})d(Bx_{2n-1}, Sx_{2n-1})}{1 + d(Ax_{2n}, Bx_{2n-1})}, \\
&\qquad \frac{1 + d(Ax_{2n}, Sx_{2n-1}) + d(Tx_{2n}, Bx_{2n-1})}{1 + d(Ax_{2n}, Tx_{2n}) + d(Bx_{2n-1}, Sx_{2n-1})} d(Ax_{2n}, Tx_{2n}) \bigg\} \\
&\quad = \max\bigg\{ d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), \frac{1}{2}\big[d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})\big], \\
&\qquad \frac{d(y_{2n}, y_{2n})d(y_{2n+1}, y_{2n-1})}{1 + d(y_{2n}, y_{2n-1})}, \frac{d(y_{2n}, y_{2n+1})d(y_{2n-1}, y_{2n})}{1 + d(y_{2n}, y_{2n-1})}, \\
&\qquad \frac{1 + d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})}{1 + d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n})} d(y_{2n}, y_{2n+1}) \bigg\}
\end{aligned}$$

$$= \max\left\{d_{2n-1}, d_{2n}, d_{2n-1}, \frac{1}{2}d(y_{2n+1}, y_{2n-1}), 0, \frac{d_{2n}d_{2n-1}}{1+d_{2n-1}}, \frac{1+d(y_{2n+1}, y_{2n-1})}{1+d_{2n}+d_{2n-1}}d_{2n}\right\}$$
$$= \max\{d_{2n-1}, d_{2n}\}, \quad \forall n \in \mathbb{N}. \tag{2.8}$$

Suppose that $d_{2n_0-1} < d_{2n_0}$ for some $n_0 \in \mathbb{N}$. It follows from (2.7), (2.8), $\psi \in \Phi_3$, and Lemma 1.1 that

$$d_{2n_0} \le \psi\bigl(M_1(x_{2n_0}, x_{2n_0-1})\bigr) = \psi\bigl(\max\{d_{2n_0-1}, d_{2n_0}\}\bigr) = \psi(d_{2n_0}) < d_{2n_0},$$

which is a contradiction. Hence

$$d_{2n} \le d_{2n-1} = M_1(x_{2n}, x_{2n-1}), \quad \forall n \in \mathbb{N}. \tag{2.9}$$

Similarly we infer

$$d_{2n+1} \le d_{2n} = M_1(x_{2n}, x_{2n+1}), \quad \forall n \in \mathbb{N},$$

which together with (2.9) ensures

$$d_{n+1} \le d_n, \quad \forall n \in \mathbb{N},$$

which means that the sequence $\{d_n\}_{n\in\mathbb{N}}$ is nonincreasing and bounded. Consequently there exists $r \ge 0$ with $\lim_{n\to\infty} d_n = r$. Suppose that $r > 0$. It follows from (2.7), (2.9), $\psi \in \Phi_3$, and Lemma 1.1 that

$$r = \limsup_{n\to\infty} d_{2n} \le \limsup_{n\to\infty} \psi\bigl(M_1(x_{2n}, x_{2n-1})\bigr)$$
$$= \limsup_{n\to\infty} \psi(d_{2n-1}) \le \psi(r) < r,$$

which is a contradiction. Hence $r = 0$, that is, (2.6) holds.

Next we prove that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Because of (2.6) it is sufficient to verify that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. It follows that there exist $\varepsilon > 0$ and two subsequences $\{y_{2m(k)}\}_{k\in\mathbb{N}}$ and $\{y_{2n(k)}\}_{k\in\mathbb{N}}$ of $\{y_{2n}\}_{n\in\mathbb{N}}$ such that

$$2n(k) > 2m(k) > 2k, \qquad d(y_{2m(k)}, y_{2n(k)}) \ge \varepsilon, \quad \forall k \in \mathbb{N}, \tag{2.10}$$

where $2n(k)$ is the smallest index satisfying (2.10). It follows that

$$d(y_{2m(k)}, y_{2n(k)-2}) < \varepsilon, \quad \forall k \in \mathbb{N}. \tag{2.11}$$

Taking advantage of (2.10), (2.11), and the triangle inequality, we get

$$\varepsilon \le d(y_{2m(k)}, y_{2n(k)})$$
$$\le d(y_{2m(k)}, y_{2n(k)-2}) + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)})$$
$$< \varepsilon + d_{2n(k)-2} + d_{2n(k)-1}, \quad \forall k \in \mathbb{N} \tag{2.12}$$

and

$$\begin{aligned} &\left|d(y_{2m(k)}, y_{2n(k)-1}) - d(y_{2m(k)}, y_{2n(k)})\right| \le d_{2n(k)-1}, \quad \forall k \in \mathbb{N}; \\ &\left|d(y_{2m(k)+1}, y_{2n(k)}) - d(y_{2m(k)}, y_{2n(k)})\right| \le d_{2m(k)}, \quad \forall k \in \mathbb{N}; \\ &\left|d(y_{2m(k)+1}, y_{2n(k)-1}) - d(y_{2m(k)}, y_{2n(k)-1})\right| \le d_{2m(k)}, \quad \forall k \in \mathbb{N}. \end{aligned} \tag{2.13}$$

Letting $k \to \infty$ in (2.12) and (2.13) and using (2.6), we deduce

$$\begin{aligned} \lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) &= \lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)-1}) = \lim_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)}) \\ &= \lim_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)-1}) = \varepsilon. \end{aligned} \tag{2.14}$$

Note that (1.3) and (2.14) yield

$$\begin{aligned} &M_1(x_{2m(k)}, x_{2n(k)-1}) \\ &\quad = \max\Bigg\{ d(Ax_{2m(k)}, Bx_{2n(k)-1}), d(Ax_{2m(k)}, Tx_{2m(k)}), d(Bx_{2n(k)-1}, Sx_{2n(k)-1}), \\ &\qquad \frac{1}{2}\left[d(Ax_{2m(k)}, Sx_{2n(k)-1}) + d(Tx_{2m(k)}, Bx_{2n(k)-1})\right], \\ &\qquad \frac{d(Ax_{2m(k)}, Sx_{2n(k)-1}) d(Tx_{2m(k)}, Bx_{2n(k)-1})}{1 + d(Ax_{2m(k)}, Bx_{2n(k)-1})}, \\ &\qquad \frac{d(Ax_{2m(k)}, Tx_{2m(k)}) d(Bx_{2n(k)-1}, Sx_{2n(k)-1})}{1 + d(Ax_{2m(k)}, Bx_{2n(k)-1})}, \\ &\qquad \frac{1 + d(Ax_{2m(k)}, Sx_{2n(k)-1}) + d(Tx_{2m(k)}, Bx_{2n(k)-1})}{1 + d(Ax_{2m(k)}, Tx_{2m(k)}) + d(Bx_{2n(k)-1}, Sx_{2n(k)-1})} d(Ax_{2m(k)}, Tx_{2m(k)}) \Bigg\} \\ &\quad = \max\Bigg\{ d(y_{2m(k)}, y_{2n(k)-1}), d(y_{2m(k)}, y_{2m(k)+1}), d(y_{2n(k)-1}, y_{2n(k)}), \\ &\qquad \frac{1}{2}\left[d(y_{2m(k)}, y_{2n(k)}) + d(y_{2m(k)+1}, y_{2n(k)-1})\right], \\ &\qquad \frac{d(y_{2m(k)}, y_{2n(k)}) d(y_{2m(k)+1}, y_{2n(k)-1})}{1 + d(y_{2m(k)}, y_{2n(k)-1})}, \frac{d(y_{2m(k)}, y_{2m(k)+1}) d(y_{2n(k)-1}, y_{2n(k)})}{1 + d(y_{2m(k)}, y_{2n(k)-1})}, \\ &\qquad \frac{1 + d(y_{2m(k)}, y_{2n(k)}) + d(y_{2m(k)+1}, y_{2n(k)-1})}{1 + d(y_{2m(k)}, y_{2m(k)+1}) + d(y_{2n(k)-1}, y_{2n(k)})} d(y_{2m(k)}, y_{2m(k)+1}) \Bigg\} \\ &\quad \to \max\left\{ \varepsilon, 0, 0, \frac{1}{2}(\varepsilon + \varepsilon), \frac{\varepsilon^2}{1+\varepsilon}, 0, 0 \right\} \\ &\quad = \varepsilon \quad \text{as } k \to \infty. \end{aligned} \tag{2.15}$$

In view of (2.4), (2.14), (2.15), $\psi \in \Phi_3$, and Lemma 1.1, we gain

$$\begin{aligned} \varepsilon &= \limsup_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)}) = \limsup_{k\to\infty} d(Tx_{2m(k)}, Sx_{2n(k)-1}) \\ &\le \limsup_{k\to\infty} \psi\big(M_1(x_{2m(k)}, x_{2n(k)-1})\big) \le \psi(\varepsilon) < \varepsilon, \end{aligned}$$

which is a contradiction. Hence $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Assume that $A(X)$ is complete. Observe that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $A(X)$. Consequently there exists $(z,v)\in A(X)\times X$ with $\lim_{n\to\infty} y_{2n} = z = Av$. It is easy to see

$$z = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n-1} = \lim_{n\to\infty} Ax_{2n}. \tag{2.16}$$

Suppose that $Tv \neq z$. Note that (1.3) and (2.16) imply

$$\begin{aligned}
&M_1(v,x_{2n+1})\\
&\quad= \max\biggl\{d(Av,Bx_{2n+1}), d(Av,Tv), d(Bx_{2n+1},Sx_{2n+1}),\\
&\qquad \frac{1}{2}\bigl[d(Av,Sx_{2n+1})+d(Tv,Bx_{2n+1})\bigr],\\
&\qquad \frac{d(Av,Sx_{2n+1})d(Tv,Bx_{2n+1})}{1+d(Av,Bx_{2n+1})}, \frac{d(Av,Tv)d(Bx_{2n+1},Sx_{2n+1})}{1+d(Av,Bx_{2n+1})},\\
&\qquad \frac{1+d(Av,Sx_{2n+1})+d(Tv,Bx_{2n+1})}{1+d(Av,Tv)+d(Bx_{2n+1},Sx_{2n+1})}d(Av,Tv)\biggr\}\\
&\quad\to \max\biggl\{d(Av,z), d(Av,Tv), d(z,z), \frac{1}{2}\bigl[d(Av,z)+d(Tv,z)\bigr],\\
&\qquad \frac{d(Av,z)d(Tv,z)}{1+d(Av,z)}, \frac{d(Av,Tv)d(z,z)}{1+d(Av,z)}, \frac{1+d(Av,z)+d(Tv,z)}{1+d(Av,Tv)+d(z,z)}d(Av,Tv)\biggr\}\\
&\quad= \max\biggl\{0, d(z,Tv), 0, \frac{1}{2}d(Tv,z), 0, 0, d(z,Tv)\biggr\}\\
&\quad= d(Tv,z) \quad \text{as } n\to\infty,
\end{aligned}$$

which together with (2.4), $\psi\in\Phi_3$, and Lemma 1.1 gives

$$\begin{aligned}
d(Tv,z) &= \limsup_{n\to\infty} d(Tv,y_{2n+2}) = \limsup_{n\to\infty} d(Tv,Sx_{2n+1})\\
&\le \limsup_{n\to\infty} \psi\bigl(M_1(v,x_{2n+1})\bigr) \le \psi\bigl(d(Tv,z)\bigr) < d(Tv,z),
\end{aligned}$$

which is a contradiction. Hence $Tv = z$. It follows from (2.2) that there exists a point $w\in X$ with $z = Bw = Tv$. Suppose that $Sw \neq z$. In light of (1.3) and (2.16), we deduce

$$\begin{aligned}
&M_1(x_{2n},w)\\
&\quad= \max\biggl\{d(Ax_{2n},Bw), d(Ax_{2n},Tx_{2n}), d(Bw,Sw), \frac{1}{2}\bigl[d(Ax_{2n},Sw)+d(Tx_{2n},Bw)\bigr],\\
&\qquad \frac{d(Ax_{2n},Sw)d(Tx_{2n},Bw)}{1+d(Ax_{2n},Bw)}, \frac{d(Ax_{2n},Tx_{2n})d(Bw,Sw)}{1+d(Ax_{2n},Bw)},\\
&\qquad \frac{1+d(Ax_{2n},Sw)+d(Tx_{2n},Bw)}{1+d(Ax_{2n},Tx_{2n})+d(Bw,Sw)}d(Ax_{2n},Tx_{2n})\biggr\}\\
&\quad\to \max\biggl\{d(z,Bw), d(z,z), d(Bw,Sw), \frac{1}{2}\bigl[d(z,Sw)+d(z,Bw)\bigr],\\
&\qquad \frac{d(z,Sw)d(z,Bw)}{1+d(z,Bw)}, \frac{d(z,z)d(Bw,Sw)}{1+d(z,Bw)}, \frac{1+d(z,Sw)+d(z,Bw)}{1+d(z,z)+d(Bw,Sw)}d(z,z)\biggr\}
\end{aligned}$$

$$= \max\left\{0, 0, d(z, Sw), \frac{1}{2}d(z, Sw), 0, 0, 0\right\}$$
$$= d(z, Sw) \quad \text{as } n \to \infty,$$

which together with (2.4), $\psi \in \Phi_3$, and Lemma 1.1 yields

$$\begin{aligned} d(z, Sw) &= \limsup_{n\to\infty} d(y_{2n+1}, Sw) = \limsup_{n\to\infty} d(Tx_{2n}, Sw) \\ &\leq \limsup_{n\to\infty} \psi\big(M_1(x_{2n}, w)\big) \leq \psi\big(d(z, Sw)\big) < (d(z, Sw), \end{aligned}$$

which is impossible, and hence $Sw = z$. Thus (2.1) means $Az = ATv = TAv = Tz$ and $Bz = BSw = SBw = Sz$. Suppose that $Tz \neq Sz$. It follows from (1.3), (2.4), $\psi \in \Phi_3$, and Lemma 1.1 that

$$\begin{aligned} &M_1(z, z) \\ &\quad = \max\left\{d(Az, Bz), d(Az, Tz), d(Bz, Sz), \frac{1}{2}\big[d(Az, Sz) + d(Tz, Bz)\big], \right. \\ &\qquad \frac{d(Az, Sz)d(Tz, Bz)}{1 + d(Az, Bz)}, \frac{d(Az, Tz)d(Bz, Sz)}{1 + d(Az, Bz)}, \\ &\qquad \left. \frac{1 + d(Az, Sz) + d(Tz, Bz)}{1 + d(Az, Tz) + d(Bz, Sz)} d(Az, Tz)\right\} \\ &\quad = \max\left\{d(Tz, Sz), 0, 0, \frac{1}{2}\big[d(Tz, Sz) + d(Tz, Sz)\big], \frac{d^2(Tz, Sz)}{1 + d(Tz, Sz)}, 0, 0\right\} \\ &\quad = d(Tz, Sz) \end{aligned}$$

and

$$d(Tz, Sz) \leq \psi\big(M_1(z, z)\big) = \psi\big(d(Tz, Sz)\big) < d(Tz, Sz),$$

which is a contradiction, and hence $Tz = Sz$.

Suppose that $Tz \neq z$. It follows from (1.3) that

$$\begin{aligned} &M_1(z, w) \\ &\quad = \max\left\{d(Az, Bw), d(Az, Tz), d(Bw, Sw), \frac{1}{2}\big[d(Az, Sw) + d(Tz, Bw)\big], \right. \\ &\qquad \frac{d(Az, Sw)d(Tz, Bw)}{1 + d(Az, Bw)}, \frac{d(Az, Tz)d(Bw, Sw)}{1 + d(Az, Bw)}, \\ &\qquad \left. \frac{1 + d(Az, Sw) + d(Tz, Bw)}{1 + d(Az, Tz) + d(Bw, Sw)} d(Az, Tz)\right\} \\ &\quad = \max\left\{d(Tz, z), 0, 0, \frac{1}{2}\big[d(Tz, z) + d(Tz, z)\big], \frac{d^2(Tz, z)}{1 + d(Tz, z)}, 0, 0\right\} \\ &\quad = d(Tz, z), \end{aligned}$$

which together with (2.4), $\psi \in \Phi_3$, and Lemma 1.1 implies

$$d(Tz,z) = d(Tz,Sw) \leq \psi\big(M_1(z,w)\big) = \psi\big(d(Tz,z)\big) < d(Tz,z),$$

which is impossible and hence $Tz = z$, that is, z is a common fixed point of A, B, S, and T.

Suppose that A, B, S, and T have another common fixed point $u \in X \setminus \{z\}$. It follows from (1.3), (2.4), $\psi \in \Phi_3$, and Lemma 1.1 that

$$\begin{aligned}
&M_1(u,z)\\
&\quad= \max\biggl\{d(Au,Bz), d(Au,Tu), d(Bz,Sz), \frac{1}{2}\bigl[d(Au,Sz)+d(Tu,Bz)\bigr],\\
&\qquad\frac{d(Au,Sz)d(Tu,Bz)}{1+d(Au,Bz)}, \frac{d(Au,Tu)d(Bz,Sz)}{1+d(Au,Bz)},\\
&\qquad\frac{1+d(Au,Sz)+d(Tu,Bz)}{1+d(Au,Tu)+d(Bz,Sz)}d(Au,Tu)\biggr\}\\
&\quad= \max\biggl\{d(u,z), 0, 0, \frac{1}{2}\bigl[d(u,z)+d(u,z)\bigr], \frac{d^2(u,z)}{1+d(u,z)}, 0, 0\biggr\}\\
&\quad= d(u,z)
\end{aligned}$$

and

$$d(u,z) = d(Tu,Sz) \leq \psi\big(M_1(u,z)\big) = \psi\big(d(u,z)\big) < d(u,z),$$

which is a contradiction and hence z is a unique common fixed point of A, B, S, and T in X.

Similarly we conclude that A, B, S, and T have a unique common fixed point in X if one of $B(X)$, $S(X)$, and $T(X)$ is complete. This completes the proof. □

Theorem 2.2 *Let A, B, S, and T be self mappings in a metric space (X,d) satisfying* (2.1)-(2.3) *and*

$$d(Tx,Sy) \leq \psi\big(M_2(x,y)\big), \quad \forall x,y \in X, \tag{2.17}$$

where ψ is in Φ_3 and M_2 is defined by (1.4). *Then A, B, S, and T have a unique common fixed point in X.*

Proof Let $x_0 \in X$. It follows from (2.2) that there exist two sequences $\{y_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}_0}$ in X satisfying (2.5). Put $d_n = d(y_n, y_{n+1})$ for all $n \in \mathbb{N}$.

Now we prove that (2.6) holds. In view of (1.4) and (2.17), we deduce

$$d_{2n} = d(Tx_{2n}, Sx_{2n-1}) \leq \psi\big(M_2(x_{2n}, x_{2n-1})\big), \quad \forall n \in \mathbb{N} \tag{2.18}$$

and

$$\begin{aligned}
&M_2(x_{2n}, x_{2n-1})\\
&\quad= \max\biggl\{d(Ax_{2n}, Bx_{2n-1}), d(Ax_{2n}, Tx_{2n}), d(Bx_{2n-1}, Sx_{2n-1}),
\end{aligned}$$

$$
\begin{aligned}
&\frac{1}{2}\big[d(Ax_{2n},Sx_{2n-1})+d(Tx_{2n},Bx_{2n-1}\big],\\
&\frac{1+d(Ax_{2n},Tx_{2n})}{1+d(Ax_{2n},Bx_{2n-1})}d(Bx_{2n-1},Sx_{2n-1}),\frac{1+d(Bx_{2n-1},Sx_{2n-1})}{1+d(Ax_{2n},Bx_{2n-1})}d(Ax_{2n},Tx_{2n}),\\
&\frac{1+d(Ax_{2n},Sx_{2n-1})+d(Tx_{2n},Bx_{2n-1})}{1+d(Ax_{2n},Tx_{2n})+d(Bx_{2n-1},Sx_{2n-1})}d(Bx_{2n-1},Sx_{2n-1})\bigg\}\\
={}&\max\bigg\{d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n}),\\
&\frac{1}{2}\big[d(y_{2n},y_{2n})+d(y_{2n+1},y_{2n-1})\big],\frac{1+d(y_{2n},y_{2n+1})}{1+d(y_{2n},y_{2n-1})}d(y_{2n-1},y_{2n}),\\
&\frac{1+d(y_{2n-1},y_{2n})}{1+d(y_{2n},y_{2n-1})}d(y_{2n},y_{2n+1}),\frac{1+d(y_{2n},y_{2n})+d(y_{2n+1},y_{2n-1})}{1+d(y_{2n},y_{2n+1})+d(y_{2n-1},y_{2n})}d(y_{2n-1},y_{2n})\bigg\}\\
={}&\max\bigg\{d_{2n-1},d_{2n},d_{2n-1},\frac{1}{2}d(y_{2n+1},y_{2n-1}),\frac{1+d_{2n}}{1+d_{2n-1}}d_{2n-1},d_{2n},\\
&\frac{1+d(y_{2n+1},y_{2n-1})}{1+d_{2n}+d_{2n-1}}d_{2n-1}\bigg\}\\
={}&\max\bigg\{d_{2n-1},d_{2n},\frac{1+d_{2n}}{1+d_{2n-1}}d_{2n-1}\bigg\},\quad \forall n\in\mathbb{N}.
\end{aligned}
$$

Suppose that $d_{2n_0-1} < d_{2n_0}$ for some $n_0 \in \mathbb{N}$. It follows that

$$d_{2n_0}(1+d_{2n_0-1}) = d_{2n_0} + d_{2n_0}d_{2n_0-1} > d_{2n_0-1} + d_{2n_0}d_{2n_0-1} = d_{2n_0-1}(1+d_{2n_0}),$$

that is,

$$d_{2n_0} > \frac{1+d_{2n_0}}{1+d_{2n_0-1}}d_{2n_0-1},$$

which implies $M_2(x_{2n_0},x_{2n_0-1}) = d_{2n_0}$. By means of (2.18), $\psi \in \Phi_3$, and Lemma 1.1, we conclude

$$d_{2n_0} \le \psi\big(M_2(x_{2n_0},x_{2n_0-1})\big) = \psi(d_{2n_0}) < d_{2n_0},$$

which is a contradiction. Consequently, we deduce

$$d_{2n} \le d_{2n-1} = M_2(x_{2n},x_{2n-1}),\quad \forall n \in \mathbb{N}. \tag{2.19}$$

Similarly we have

$$d_{2n+1} \le d_{2n} = M_2(x_{2n},x_{2n+1}),\quad \forall n \in \mathbb{N}. \tag{2.20}$$

It follows from (2.19) and (2.20) that

$$d_{n+1} \le d_n,\quad \forall n \in \mathbb{N},$$

which means that the sequence $\{d_n\}_{n\in\mathbb{N}}$ is nonincreasing and bounded. Consequently there exists $r \ge 0$ with $\lim_{n\to\infty} d_n = r$. Suppose that $r > 0$. It follows from (2.18), (2.19),

$\psi \in \Phi_3$, and Lemma 1.1 that

$$\begin{aligned} r &= \limsup_{n\to\infty} d_{2n} \le \limsup_{n\to\infty} \psi\big(M_2(x_{2n}, x_{2n-1})\big) \\ &= \limsup_{n\to\infty} \psi(d_{2n-1}) \le \psi(r) < r, \end{aligned}$$

which is a contradiction. Hence $r = 0$, that is, (2.6) holds.

In order to prove that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, we need only to show that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. It follows that there exist $\varepsilon > 0$ and two subsequences $\{y_{2m(k)}\}_{k\in\mathbb{N}}$ and $\{y_{2n(k)}\}_{k\in\mathbb{N}}$ of $\{y_{2n}\}_{n\in\mathbb{N}}$ satisfying (2.10)-(2.14) and

$$\begin{aligned} &M_2(x_{2m(k)}, x_{2n(k)-1}) \\ &\quad= \max\bigg\{ d(Ax_{2m(k)}, Bx_{2n(k)-1}), d(Ax_{2m(k)}, Tx_{2m(k)}), d(Bx_{2n(k)-1}, Sx_{2n(k)-1}), \\ &\qquad \frac{1}{2}\big[d(Ax_{2m(k)}, Sx_{2n(k)-1}) + d(Tx_{2m(k)}, Bx_{2n(k)-1})\big], \\ &\qquad \frac{1 + d(Ax_{2m(k)}, Tx_{2m(k)})}{1 + d(Ax_{2m(k)}, Bx_{2n(k)-1})} d(Bx_{2n(k)-1}, Sx_{2n(k)-1}), \\ &\qquad \frac{1 + d(Bx_{2n(k)-1}, Sx_{2n(k)-1})}{1 + d(Ax_{2m(k)}, Bx_{2n(k)-1})} d(Ax_{2m(k)}, Tx_{2m(k)}), \\ &\qquad \frac{1 + d(Ax_{2m(k)}, Sx_{2n(k)-1}) + d(Tx_{2m(k)}, Bx_{2n(k)-1})}{1 + d(Ax_{2m(k)}, Tx_{2m(k)}) + d(Bx_{2n(k)-1}, Sx_{2n(k)-1})} d(Bx_{2n(k)-1}, Sx_{2n(k)-1}) \bigg\} \\ &\quad= \max\bigg\{ d(y_{2m(k)}, y_{2n(k)-1}), d(y_{2m(k)}, y_{2m(k)+1}), d(y_{2n(k)-1}, y_{2n(k)}), \\ &\qquad \frac{1}{2}\big[d(y_{2m(k)}, y_{2n(k)}) + d(y_{2m(k)+1}, y_{2n(k)-1})\big], \\ &\qquad \frac{1 + d(y_{2m(k)}, y_{2m(k)+1})}{1 + d(y_{2m(k)}, y_{2n(k)-1})} d(y_{2n(k)-1}, y_{2n(k)}), \\ &\qquad \frac{1 + d(y_{2n(k)-1}, y_{2n(k)})}{1 + d(y_{2m(k)}, y_{2n(k)-1})} d(y_{2m(k)}, y_{2m(k)+1}), \\ &\qquad \frac{1 + d(y_{2m(k)}, y_{2n(k)}) + d(y_{2m(k)+1}, y_{2n(k)-1})}{1 + d(y_{2m(k)}, y_{2m(k)+1}) + d(y_{2n(k)-1}, y_{2n(k)})} d(y_{2n(k)-1}, y_{2n(k)}) \bigg\} \\ &\quad\to \max\bigg\{ \varepsilon, 0, 0, \frac{1}{2}(\varepsilon + \varepsilon), 0, 0, 0 \bigg\} \\ &\quad= \varepsilon \quad \text{as } k \to \infty. \end{aligned} \tag{2.21}$$

By virtue of (2.14), (2.17), (2.21), $\psi \in \Phi_3$, and Lemma 1.1, we infer

$$\begin{aligned} \varepsilon &= \limsup_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)}) = \limsup_{k\to\infty} d(Tx_{2m(k)}, Sx_{2n(k)-1}) \\ &\le \limsup_{k\to\infty} \psi\big(M_2(x_{2m(k)}, x_{2n(k)-1})\big) \le \psi(\varepsilon) < \varepsilon, \end{aligned}$$

which is impossible. Hence $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Assume that $A(X)$ is complete. Observe that $\{y_{2n}\}_{n\in\mathbb{N}} \subseteq A(X)$ is a Cauchy sequence. It follows that there exists $(z,v) \in A(X) \times X$ with $\lim_{n\to\infty} y_{2n} = z = Av$. It is easy to show that (2.16) holds.

Suppose that $Tv \neq z$. Note that (1.4), (2.16), (2.17), and $\psi \in \Phi_3$ imply

$$\begin{aligned}
&M_2(v,x_{2n+1})\\
&\quad= \max\biggl\{d(Av,Bx_{2n+1}), d(Av,Tv), d(Bx_{2n+1},Sx_{2n+1}),\\
&\qquad\frac{1}{2}\bigl[d(Av,Sx_{2n+1}) + d(Tv,Bx_{2n+1})\bigr],\\
&\qquad\frac{1+d(Av,Tv)}{1+d(Av,Bx_{2n+1})}d(Bx_{2n+1},Sx_{2n+1}), \frac{1+d(Bx_{2n+1},Sx_{2n+1})}{1+d(Av,Bx_{2n+1})}d(Av,Tv),\\
&\qquad\frac{1+d(Av,Sx_{2n+1})+d(Tv,Bx_{2n+1})}{1+d(Av,Tv)+d(Bx_{2n+1},Sx_{2n+1})}d(Bx_{2n+1},Sx_{2n+1})\biggr\}\\
&\quad\to \max\biggl\{d(Av,z), d(Av,Tv), d(z,z), \frac{1}{2}\bigl[d(Av,z)+d(Tv,z)\bigr],\\
&\qquad\frac{1+d(Av,Tv)}{1+d(Av,z)}d(z,z), \frac{1+d(z,z)}{1+d(Av,z)}d(Av,Tv), \frac{1+d(Av,z)+d(Tv,z)}{1+d(Av,Tv)+d(z,z)}d(z,z)\biggr\}\\
&\quad= \max\biggl\{0, d(z,Tv), 0, \frac{1}{2}d(Tv,z), 0, d(z,Tv), 0\biggr\}\\
&\quad= d(Tv,z) \quad \text{as } n\to\infty,
\end{aligned}$$

which together with (2.17), $\psi \in \Phi_3$, and Lemma 1.1 gives

$$\begin{aligned}
d(Tv,z) &= \limsup_{n\to\infty} d(Tv,y_{2n+2}) = \limsup_{n\to\infty} d(Tv,Sx_{2n+1})\\
&\leq \limsup_{n\to\infty} \psi\bigl(M_2(v,x_{2n+1})\bigr) \leq \psi\bigl(d(Tv,z)\bigr) < d(Tv,z),
\end{aligned}$$

which is a contradiction. Hence $Tv = z$.

Since $T(X) \subseteq B(X)$, it follows that there exists a point $w \in X$ such that $z = Bw = Tv$. Suppose that $Sw \neq z$. In light of (1.4) and (2.16), we obtain

$$\begin{aligned}
&M_2(x_{2n},w)\\
&\quad= \max\biggl\{d(Ax_{2n},Bw), d(Ax_{2n},Tx_{2n}), d(Bw,Sw), \frac{1}{2}\bigl[d(Ax_{2n},Sw)+d(Tx_{2n},Bw)\bigr],\\
&\qquad\frac{1+d(Ax_{2n},Tx_{2n})}{1+d(Ax_{2n},Bw)}d(Bw,Sw), \frac{1+d(Bw,Sw)}{1+d(Ax_{2n},Bw)}d(Ax_{2n},Tx_{2n}),\\
&\qquad\frac{1+d(Ax_{2n},Sw)+d(Tx_{2n},Bw)}{1+d(Ax_{2n},Tx_{2n})+d(Bw,Sw)}d(Bw,Sw)\biggr\}\\
&\quad\to \max\biggl\{d(z,z), d(z,z), d(z,Sw), \frac{1}{2}\bigl[d(z,Sw)+d(z,Bw)\bigr],\\
&\qquad\frac{1+d(z,z)}{1+d(z,z)}d(z,Sw), \frac{1+d(z,Sw)}{1+d(z,z)}d(z,z),\\
&\qquad\frac{1+d(z,Sw)+d(z,z)}{1+d(z,z)+d(z,Sw)}d(z,Sw)\biggr\}
\end{aligned}$$

$$
\begin{aligned}
&= \max\left\{0, 0, d(z, Sw), \frac{1}{2}d(z, Sw), d(z, Sw), 0, d(z, Sw)\right\} \\
&= d(z, Sw) \quad \text{as } n \to \infty,
\end{aligned}
$$

which together with (2.17), $\psi \in \Phi_3$, and Lemma 1.1 yields

$$
\begin{aligned}
d(z, Sw) &= \limsup_{n\to\infty} d(y_{2n+1}, Sw) = \limsup_{n\to\infty} d(Tx_{2n}, Sw) \\
&\leq \limsup_{n\to\infty} \psi\big(M_2(x_{2n}, w)\big) \leq \psi\big(d(z, Sw)\big) < d(z, Sw),
\end{aligned}
$$

which is impossible, and hence $Sw = z$. Clearly, (2.1) yields $Az = ATv = TAv = Tz$ and $Bz = BSw = SBw = Sz$. Suppose that $Tz \neq Sz$. It follows from (1.4) that

$$
\begin{aligned}
M_2(z, z) &= \max\bigg\{d(Az, Bz), d(Az, Tz), d(Bz, Sz), \frac{1}{2}\big[d(Az, Sz) + d(Tz, Bz)\big], \\
&\quad \frac{1 + d(Az, Tz)}{1 + d(Az, Bz)} d(Bz, Sz), \frac{1 + d(Bz, Sz)}{1 + d(Az, Bz)} d(Az, Tz), \\
&\quad \frac{1 + d(Az, Sz) + d(Tz, Bz)}{1 + d(Az, Tz) + d(Bz, Sz)} d(Bz, Sz)\bigg\} \\
&= \max\left\{d(Tz, Sz), 0, 0, \frac{1}{2}\big[d(Tz, Sz) + d(Tz, Sz)\big], 0, 0, 0\right\} \\
&= d(Tz, Sz).
\end{aligned}
$$

Taking account of (2.17), $\psi \in \Phi_3$, and Lemma 1.1, we conclude

$$
d(Tz, Sz) \leq \psi\big(M_2(z, z)\big) = \psi\big(d(Tz, Sz)\big) < d(Tz, Sz),
$$

which is a contradiction, and hence $Tz = Sz$.

Suppose that $Tz \neq z$. It follows from (1.4) that

$$
\begin{aligned}
M_2(z, w) &= \max\bigg\{d(Az, Bw), d(Az, Tz), d(Bw, Sw), \frac{1}{2}\big[d(Az, Sw) + d(Tz, Bw)\big], \\
&\quad \frac{1 + d(Az, Tz)}{1 + d(Az, Bw)} d(Bw, Sw), \frac{1 + d(Bw, Sw)}{1 + d(Az, Bw)} d(Az, Tz), \\
&\quad \frac{1 + d(Az, Sw) + d(Tz, Bw)}{1 + d(Az, Tz) + d(Bw, Sw)} d(Bw, Sw)\bigg\} \\
&= \max\left\{d(Tz, z), 0, 0, \frac{1}{2}\big[d(Tz, z) + d(Tz, z)\big], 0, 0, 0\right\} \\
&= d(Tz, z),
\end{aligned}
$$

which together with (2.17), $\psi \in \Phi_3$, and Lemma 1.1 means

$$
d(Tz, z) = d(Tz, Sw) \leq \psi\big(M_2(z, w)\big) = \psi\big(d(Tz, z)\big) < d(Tz, z),
$$

which is impossible, and hence $Tz = z$, that is, z is a common fixed point of A, B, S, and T.

Suppose that A, B, S, and T have another common fixed point $u \in X \setminus \{z\}$. It follows from (1.4) that

$$\begin{aligned} M_2(u,z) &= \max\biggl\{d(Au,Bz), d(Au,Tu), d(Bz,Sz), \frac{1}{2}\bigl[d(Au,Sz)+d(Tu,Bz)\bigr], \\ &\quad \frac{1+d(Au,Tu)}{1+d(Au,Bz)}d(Bz,Sz), \frac{1+d(Bz,Sz)}{1+d(Au,Bz)}d(Au,Tu), \\ &\quad \frac{1+d(Au,Sz)+d(Tu,Bz)}{1+d(Au,Tu)+d(Bz,Sz)}d(Bz,Sz)\biggr\} \\ &= \max\biggl\{d(u,z), 0, 0, \frac{1}{2}\bigl[d(u,z)+d(u,z)\bigr], 0, 0, 0\biggr\} \\ &= d(u,z), \end{aligned}$$

which together with (2.17), $\psi \in \Phi_3$, and Lemma 1.1 ensures

$$d(u,z) = d(Tu,Sz) \le \psi\bigl(M_2(u,z)\bigr) = \psi\bigl(d(u,z)\bigr) < d(u,z),$$

which is a contradiction, and hence z is a unique common fixed point of A, B, S, and T in X.

Similarly we conclude that A, B, S, and T have a unique common fixed point in X if one of $B(X)$, $S(X)$, and $T(X)$ is complete. This completes the proof. □

Similar to the proofs of Theorems 2.1 and 2.2, we have the following result and omit its proof.

Theorem 2.3 *Let A, B, S, and T be self mappings in a metric space (X,d) satisfying* (2.1)-(2.3) *and*

$$d(Tx,Sy) \le \psi\bigl(M_3(x,y)\bigr), \quad \forall x,y \in X, \tag{2.22}$$

where ψ is in Φ_3 and M_3 is defined by (1.5). *Then A, B, S, and T have a unique common fixed point in X.*

Utilizing Theorems 2.1-2.3, Lemma 1.2, and Remark 1.1, we get the following results.

Theorem 2.4 *Let A, B, S, and T be self mappings in a metric space (X,d) satisfying* (2.1)-(2.3) *and*

$$\psi\bigl(d(Tx,Sy)\bigr) \le \psi\bigl(M_1(x,y)\bigr) - \varphi\bigl(M_1(x,y)\bigr), \quad \forall x,y \in X, \tag{2.23}$$

where (ψ,φ) is in $\Phi_1 \times \Phi_2$ and M_1 is defined by (1.3). *Then A, B, S, and T have a unique common fixed point in X.*

Theorem 2.5 *Let A, B, S, and T be self mappings in a metric space (X,d) satisfying* (2.1)-(2.3) *and*

$$\psi\bigl(d(Tx,Sy)\bigr) \le \psi\bigl(M_2(x,y)\bigr) - \varphi\bigl(M_2(x,y)\bigr), \quad \forall x,y \in X, \tag{2.24}$$

where (ψ,φ) *is in* $\Phi_1 \times \Phi_2$ *and* M_2 *is defined by* (1.4). *Then* A, B, S, *and* T *have a unique common fixed point in* X.

Theorem 2.6 *Let* A, B, S, *and* T *be self mappings in a metric space* (X,d) *satisfying* (2.1)-(2.3) *and* (2.22), *where* (ψ,φ) *is in* $\Phi_1 \times \Phi_2$ *and* M_3 *is defined by* (1.5). *Then* A, B, S, *and* T *have a unique common fixed point in* X.

Remark 2.1 Condition (2.3) in Theorem 2.6 is weaker than the conditions of (X,d) is complete and one of the ranges of the four mappings A, B, S, and T is closed in Theorem 2.1 in [2]. Hence Theorem 2.6 is a slight generalizations of Theorem 2.1 in [2]. Note that Theorem 2.4 generalizes Theorems 2.1 and 2.2 in [4]. Example 2.1 below shows that Theorem 2.6 is a substantial generalization of Theorem 2.1 in [2] and Theorems 2.1 and 2.2 in [4].

Example 2.1 Let $X = (-1,1)$ be endowed with the Euclidean metric $d(x,y) = |x-y|$ for all $x,y \in X$. Let $A,B,S,T : X \to X$ be defined by

$$Ax = x^2, \qquad Bx = x, \qquad Sx = 0, \quad \forall x \in X, \qquad Tx = \begin{cases} 0, & \forall x \in X \setminus \{\frac{1}{2}\}, \\ -\frac{1}{4}, & x = \frac{1}{2}. \end{cases}$$

Since the metric space (X,d) is not complete, it follows that Theorem 2.1 in [2] is useless in proving the existence of common fixed points of A, B, S, and T in X and Theorems 2.1 and 2.2 in [4] are unapplicable in proving the existence of common fixed points of S and T and fixed points of T, respectively.

Now we use Theorem 2.6 to prove the existence of common fixed points of A, B, S, and T in X. Define $\psi,\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\psi(t) = \begin{cases} \sqrt{t}, & \forall t \in [0,\frac{1}{2}), \\ \frac{\sqrt{2}}{2}, & \forall t \in [\frac{1}{2},+\infty) \end{cases}$$

and

$$\varphi(t) = \begin{cases} t^3, & \forall t \in [0,\frac{1}{2}), \\ \frac{1}{16}, & \forall t \in [\frac{1}{2},+\infty). \end{cases}$$

It is easy to verify that (2.1)-(2.3) holds, $(\psi,\varphi) \in \Phi_1 \times \Phi_2$, $\psi(t) \geq \varphi(t)$ for each $t \in \mathbb{R}^+$. Put $x,y \in X$. In order to verify (2.22), we consider two cases as follows:

Case 1. $x \in X \setminus \{\frac{1}{2}\}$. It is clear that

$$\psi\bigl(d(Tx,Sy)\bigr) = \psi(0) = 0 \leq \psi\bigl(M_3(x,y)\bigr) - \varphi\bigl(M_3(x,y)\bigr);$$

Case 2. $x = \frac{1}{2}$. Clearly we have

$$\begin{aligned} M_3(x,y) &= \max\left\{d(Ax,By), d(Ax,Tx), d(By,Sy), \frac{1}{2}\bigl[d(Ax,Sy) + d(Tx,By)\bigr]\right\} \\ &\geq d(Ax,Tx) = d\left(\frac{1}{4},-\frac{1}{4}\right) = \frac{1}{2}. \end{aligned}$$

It follows that

$$\psi\big(d(Tx,Sy)\big) = \psi\left(d\left(-\frac{1}{4},0\right)\right) = \psi\left(\frac{1}{4}\right) = \frac{1}{2}$$
$$\leq \frac{\sqrt{2}}{2} - \frac{1}{16} = \psi\big(M_3(x,y)\big) - \varphi\big(M_3(x,y)\big).$$

That is, (2.22) holds. Hence the conditions of Theorem 2.6 are satisfied. It follows from Theorem 2.6 that A, B, S, and T in X possess a unique common fixed point $0 \in X$.

Remark 2.2 Theorems 2.4-2.6 extend, improve and unify Theorem 2.1 in [3], Theorem 2.1 in [5] and Theorem 1 in [1]. Note that Examples 2.2-2.4 below deal with the existence of common fixed points of four mappings A, B, S, and T, but Theorem 2.1 in [3], Theorem 2.1 in [5] and Theorem 1 in [1] deal with the existence of fixed and common fixed points of at most three mappings, therefore the results in [1, 3, 5] are useless in proving the existence of common fixed points of four mappings A, B, S, and T. That is, Theorems 2.4-2.6 extend indeed Theorem 2.1 in [3], Theorem 2.1 in [5] and Theorem 1 in [1].

Example 2.2 Let $X = \mathbb{R}^+$ be endowed with the Euclidean metric $d(x,y) = |x-y|$ for all $x, y \in X$. Let $B, T : X \to X$ be defined by

$$Bx = x^2, \quad \forall x \in X \quad \text{and} \quad Tx = \begin{cases} 1, & \forall x \in \mathbb{R}^+ - \{\frac{1}{32}\}, \\ \frac{15}{16}, & x = \{\frac{1}{32}\}. \end{cases}$$

Firstly we claim that Theorem 2.1 in [5] and Theorem 1 in [1] and Theorem 2.1 in [3] cannot be used to prove the existence of fixed and common fixed points for the mapping T and the mappings B and T, respectively, in the complete metric space X.

Suppose that there exist $\varphi \in \Phi_1$ satisfying

$$d(Tx,Ty) \leq d(x,y) - \varphi\big(d(x,y)\big), \quad \forall x, y \in X,$$

which implies

$$\frac{1}{16} = d\left(1,\frac{15}{16}\right) = d\left(T0,T\frac{1}{32}\right) \leq d\left(0,\frac{1}{32}\right) - \varphi\left(d\left(0,\frac{1}{32}\right)\right)$$
$$= \frac{1}{32} - \varphi\left(\frac{1}{32}\right),$$

that is,

$$0 < \varphi\left(\frac{1}{32}\right) \leq \frac{1}{32} - \frac{1}{16} = -\frac{1}{32},$$

which is a contradiction.

Suppose that there exists $\psi, \varphi \in \Phi_1$ satisfying

$$\psi\big(d(Tx,Ty)\big) \leq \psi\big(d(x,y)\big) - \varphi\big(d(x,y)\big), \quad \forall x, y \in X,$$

which yields

$$\begin{aligned}\psi\left(\frac{1}{16}\right) &= \psi\left(1-\frac{15}{16}\right) = \psi\left(d\left(T\frac{3}{32}, T\frac{1}{32}\right)\right) \\ &\le \psi\left(d\left(\frac{3}{32}, \frac{1}{32}\right)\right) - \varphi\left(d\left(\frac{3}{32}, \frac{1}{32}\right)\right) = \psi\left(\frac{1}{16}\right) - \varphi\left(\frac{1}{16}\right),\end{aligned}$$

that is,

$$0 < \varphi\left(\frac{1}{16}\right) \le \psi\left(\frac{1}{16}\right) - \psi\left(\frac{1}{16}\right) = 0,$$

which is impossible.

Suppose that there exists $(\psi, \varphi) \in \Phi_1 \times \Phi_2$ satisfying

$$\psi\big(d(Tx, Ty)\big) \le \psi\big(d(Bx, By)\big) - \varphi\big(d(Bx, By)\big), \quad \forall x, y \in X,$$

which gives

$$\begin{aligned}\psi\left(\frac{1}{16}\right) &= \psi\left(1-\frac{15}{16}\right) = \psi\left(d\left(T\frac{3}{64}, T\frac{1}{32}\right)\right) \\ &\le \psi\left(d\left(B\frac{3}{64}, B\frac{1}{32}\right)\right) - \varphi\left(d\left(B\frac{3}{64}, B\frac{1}{32}\right)\right) \\ &= \psi\left(\left(\frac{3}{64}\right)^2 - \left(\frac{1}{32}\right)^2\right) - \varphi\left(\left(\frac{3}{64}\right)^2 - \left(\frac{1}{32}\right)^2\right) \\ &= \psi\left(\frac{5}{4{,}096}\right) - \varphi\left(\frac{5}{4{,}096}\right) \\ &< \psi\left(\frac{5}{4{,}096}\right) \le \psi\left(\frac{5}{4{,}095}\right) = \psi\left(\frac{1}{819}\right) \\ &\le \psi\left(\frac{1}{16}\right),\end{aligned}$$

which is impossible.

Secondly we claim that the mappings A, B, S, and T satisfy the conditions of Theorem 2.6, where $A, S : X \to X$ and $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ are defined by

$$Ax = x^3, \qquad Sx = 1, \quad \forall x \in X$$

and

$$\psi(t) = \begin{cases} 16t, & \forall t \in [0, \frac{1}{16}), \\ 512t^2 - 1, & \forall t \in [\frac{1}{16}, +\infty), \end{cases} \qquad \varphi(t) = \begin{cases} t^2, & \forall t \in [0, \frac{1}{16}), \\ \frac{t^2}{1+8t^2}, & \forall t \in [\frac{1}{16}, +\infty). \end{cases}$$

Clearly, (2.1)-(2.3) hold, $(\psi, \varphi) \in \Phi_1 \times \Phi_2$, $\psi(t) \ge \varphi(t)$ for any $t \in \mathbb{R}^+$, and $\varphi(\mathbb{R}^+) \subset [0, \frac{1}{8})$. Put $x, y \in X$. In order to verify (2.22), we have to consider the following two possible cases:

Case 1. $x \in X \setminus \{\frac{1}{32}\}$. It follows that

$$\psi\big(d(Tx, Sy)\big) = \psi(0) = 0 \le \psi\big(M_3(x, y)\big) - \varphi\big(M_3(x, y)\big);$$

Case 2. $x = \frac{1}{32}$. It follows that

$$\begin{aligned} M_3\left(\frac{1}{32},y\right) &= \max\left\{\left|\frac{1}{32^3}-y^2\right|,\left|\frac{1}{32^3}-\frac{15}{16}\right|,\left|1-y^2\right|,\frac{1}{2}\left(\left|\frac{1}{32^3}-1\right|+\left|\frac{15}{16}-y^2\right|\right)\right\} \\ &\geq \frac{15}{16}-\frac{1}{32^3}=\frac{30{,}719}{32{,}768}>\frac{1}{16} \end{aligned}$$

and

$$\begin{aligned} \psi\left(d\left(T\frac{1}{32},Sy\right)\right) &= \psi\left(d\left(\frac{15}{16},1\right)\right)=\psi\left(\frac{1}{16}\right)=1 \\ &< 400<\frac{941{,}559{,}809}{2{,}097{,}152}-\frac{1}{8} \\ &< \psi\left(\frac{30{,}719}{32{,}768}\right)-\varphi\left(M\left(\frac{1}{32},y\right)\right) \\ &\leq \psi\left(M_3\left(\frac{1}{32},y\right)\right)-\varphi\left(M_3\left(\frac{1}{32},y\right)\right). \end{aligned}$$

That is, (2.22) holds. Thus the conditions of Theorem 2.6 are satisfied. It follows from Theorem 2.6 that the mappings A, B, S, and T have a unique common fixed point $1 \in X$.

Example 2.3 Let $X = [0,1]$ be endowed with the Euclidean metric $d(x,y) = |x-y|$ for all $x,y \in X$. Let $A,B,S,T : X \to X$ and $\psi,\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$Ax = x^2, \qquad Bx = \frac{1}{2}x^2, \qquad Sx = 0, \quad \forall x \in X, \qquad Tx = \begin{cases} 0, & \forall x \in [0,1), \\ \frac{1}{4}, & x = 1 \end{cases}$$

and

$$\psi(t) = \begin{cases} 16t^2, & \forall t \in [0,\frac{1}{4}), \\ 8t-1, & \forall t \in [\frac{1}{4},+\infty), \end{cases} \qquad \varphi(t) = \begin{cases} 4t^2, & \forall t \in [0,\frac{1}{4}), \\ \frac{1}{6+2\sqrt{t}}, & \forall t \in [\frac{1}{4},+\infty). \end{cases}$$

It is easy to see that (2.1)-(2.3) hold, $(\psi,\varphi) \in \Phi_1 \times \Phi_2$, $\psi(t) \geq \varphi(t)$ for each $t \in \mathbb{R}^+$ and $\varphi(\mathbb{R}^+) \subset [0,\frac{1}{4})$. Let $x,y \in X$. In order to verify (2.23), we have to consider two possible cases as follows:

Case 1. $x \in X \setminus \{1\}$. It is clear that

$$\psi\big(d(Tx,Sy)\big) = \psi(0) = 0 \leq \psi\big(M_1(x,y)\big) - \varphi\big(M_1(x,y)\big);$$

Case 2. $x = 1$. It follows that

$$\begin{aligned} M_1(1,y) &= \max\left\{\left|1-\frac{y^2}{2}\right|,\frac{3}{4},\frac{y^2}{2},\frac{1}{2}\left(1+\left|\frac{1}{4}-\frac{y^2}{2}\right|\right),\frac{|\frac{1}{4}-\frac{y^2}{2}|}{1+|1-\frac{y^2}{2}|},\right. \\ &\qquad \left.\frac{\frac{3}{4}\cdot\frac{y^2}{2}}{1+|1-\frac{y^2}{2}|},\frac{1+1+|\frac{1}{4}-\frac{y^2}{2}|}{1+\frac{3}{4}+\frac{y^2}{2}}\cdot\frac{3}{4}\right\} \\ &\geq \frac{3}{4} \end{aligned}$$

and

$$\begin{aligned}\psi\big(d(T1,Sy)\big) &= \psi\left(\frac{1}{4}\right) = 1 < 5 - \frac{1}{4} \le \psi\left(\frac{3}{4}\right) - \varphi\big(M_1(1,y)\big) \\ &\le \psi\big(M_1(1,y)\big) - \varphi\big(M_1(1,y)\big).\end{aligned}$$

That is, (2.23) holds. It follows from Theorem 2.4 that the mappings A, B, S, and T have a unique common fixed point $0 \in X$. However, we neither use Theorem 1 in [1] nor employ Theorem 2.1 in [5] to show the existence of fixed points of the mapping T in X.

Suppose that there exists $\varphi \in \Phi_1$ satisfying

$$d(Tx,Ty) \le d(x,y) - \varphi\big(d(x,y)\big), \quad \forall x,y \in X,$$

which implies

$$\frac{1}{4} = d\left(\frac{1}{4},0\right) = d\left(T1,T\frac{7}{8}\right) \le d\left(1,\frac{7}{8}\right) - \varphi\left(d\left(1,\frac{7}{8}\right)\right) = \frac{1}{8} - \varphi\left(\frac{1}{8}\right),$$

which means

$$0 < \varphi\left(\frac{1}{8}\right) \le \frac{1}{8} - \frac{1}{4} = -\frac{1}{8},$$

which is a contradiction.

Suppose that there exist $\psi,\varphi \in \Phi_1$ satisfying

$$\psi\big(d(Tx,Ty)\big) \le \psi\big(d(x,y)\big) - \varphi\big(d(x,y)\big), \quad \forall x,y \in X,$$

which yields

$$\begin{aligned}\psi\left(\frac{1}{4}\right) &= \psi\left(d\left(0,\frac{1}{4}\right)\right) = \psi\big(d(Tx,T1)\big) \le \psi\big(d(x,1)\big) - \varphi\big(d(x,1)\big) \\ &= \psi(1-x) - \varphi(1-x), \quad \forall x \in X \setminus \{1\},\end{aligned}$$

which gives

$$\begin{aligned}0 &< \psi\left(\frac{1}{4}\right) \le \limsup_{x\to 1}\big[\psi(1-x) - \varphi(1-x)\big] \\ &\le \limsup_{x\to 1}\psi(1-x) - \liminf_{x\to 1}\varphi(1-x) \le \psi(0) - \varphi(0) = 0,\end{aligned}$$

which is impossible.

Example 2.4 Let $X = [-1,1]$ be endowed with the Euclidean metric $d(x,y) = |x-y|$ for all $x,y \in X$. Let $A,B,S,T : X \to X$ and $\psi,\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$Ax = \frac{x^2}{2}, \qquad Tx = 0, \quad \forall x \in X,$$

$$Bx = \begin{cases} 0, & \forall x \in [-1,1), \\ \frac{1}{2}, & x = 1, \end{cases} \qquad Sx = \begin{cases} 0, & \forall x \in [-1,1), \\ \frac{1}{8}, & x = 1, \end{cases}$$

and

$$\psi(t)=\begin{cases}64t^3, & \forall t\in[0,\frac{1}{4}),\\ 32t^2-1, & \forall t\in[\frac{1}{4},+\infty),\end{cases}\qquad \varphi(t)=\begin{cases}128t^4, & \forall t\in[0,\frac{1}{4}),\\ \frac{1}{2}\sin\frac{\pi}{2+8t}, & \forall t\in[\frac{1}{4},+\infty).\end{cases}$$

Clearly, (2.1)-(2.3) holds, $(\psi,\varphi)\in\Phi_1\times\Phi_2$, $\psi(t)\geq\varphi(t)$ for each $t\in\mathbb{R}^+$ and $\varphi(t)\leq\frac{\sqrt{2}}{4}<\frac{1}{2}$ for all $t\in[\frac{1}{4},+\infty)$. Let $x,y\in X$. In order to verify (2.24), we have to consider two possible cases as follows:

Case 1. $y\in X\setminus\{1\}$. Obviously

$$\psi\bigl(d(Tx,Sy)\bigr)=\psi(0)=0\leq\psi\bigl(M_2(x,y)\bigr)-\varphi\bigl(M_2(x,y)\bigr);$$

Case 2. $y=1$. It follows that

$$\begin{aligned}M_2(x,1)&=\max\left\{\frac{1-x^2}{2},\frac{x^2}{2},\frac{1}{2}-\frac{1}{8},\frac{1}{2}\left(\left|\frac{x^2}{2}-\frac{1}{8}\right|+\frac{1}{2}\right),\frac{1+\frac{x^2}{2}}{1+\frac{1-x^2}{2}}\cdot\frac{3}{8},\right.\\&\qquad\left.\frac{1+\frac{3}{8}}{1+\frac{1-x^2}{2}}\cdot\frac{x^2}{2},\frac{1+|\frac{x^2}{2}-\frac{1}{8}|+\frac{1}{2}}{1+\frac{x^2}{2}+\frac{3}{8}}\cdot\frac{3}{8}\right\}\\&\geq\frac{3}{8}\end{aligned}$$

and

$$\begin{aligned}\psi\bigl(d(Tx,S1)\bigr)&=\psi\left(\frac{1}{8}\right)=64\times\frac{1}{8^3}=\frac{1}{8}<3=32\times\left(\frac{3}{8}\right)^2-1-\frac{1}{2}\\&<\psi\bigl(M_2(x,1)\bigr)-\varphi\bigl(M_2(x,1)\bigr).\end{aligned}$$

That is, (2.24) holds. Consequently, Theorem 2.5 guarantees that the mappings A,B,S, and T have a unique common fixed point $0\in X$. However, we do not invoke that Theorem 2.1 in [5] proves the existence of fixed points of the mapping S in X. Otherwise there exist $\psi,\varphi\in\Phi_1$ satisfying

$$\psi\bigl(d(Sx,Sy)\bigr)\leq\psi\bigl(d(x,y)\bigr)-\varphi\bigl(d(x,y)\bigr),\quad\forall x,y\in X,$$

which yields

$$\begin{aligned}\psi\left(\frac{1}{8}\right)&=\psi\left(d\left(S\frac{31}{32},S1\right)\right)\leq\psi\left(d\left(\frac{31}{32},1\right)\right)-\varphi\left(d\left(\frac{31}{32},1\right)\right)\\&=\psi\left(\frac{1}{32}\right)-\varphi\left(\frac{1}{32}\right)<\psi\left(\frac{1}{32}\right)\\&\leq\psi\left(\frac{1}{8}\right),\end{aligned}$$

which is a contradiction.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Liaoning Normal University, Dalian, Liaoning 116029, People's Republic of China. [2]Zhuanghe Senior High School, Zhuanghe, Liaoning 116400, People's Republic of China. [3]Department of Mathematics, Changwon National University, Changwon, 641-773, Korea. [4]Department of Mathematics and RINS, Gyeongsang National University, Jinju, 660-701, Korea.

Acknowledgements
The authors are very grateful to the referees for their valuable comments and suggestions. This research was supported by the Science Research Foundation of Educational Department of Liaoning Province (L2012380) and the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT & Future Planning (2013R1A1A2057665).

Received: 14 November 2014 Accepted: 23 January 2015 Published online: 10 February 2015

References
1. Rhoades, BE: Some theorems on weakly contractive maps. Nonlinear Anal. **47**, 2683-2693 (2001)
2. Abbas, M, Dorić, D: Common fixed point theorem for four mappings satisfying generalized weak contractive condition. Filomat **24**, 1-10 (2010)
3. Abbas, M, Khan, MA: Common fixed point theorem of two mappings satisfying a generalized weak contractive condition. Int. J. Math. Math. Sci. **2009**, Article ID 131068 (2009)
4. Dorić, D: Common fixed point for generalized (ψ,φ)-weak contractions. Appl. Math. Lett. **22**, 1896-1900 (2009)
5. Dutta, PN, Choudhury, BS: A generalization of contraction principle in metric spaces. Fixed Point Theory Appl. **2008**, Article ID 406368 (2008)
6. Liu, Z: Compatible mappings and fixed points. Acta Sci. Math. **65**, 371-383 (1999)
7. Moradi, S, Fathi, Z, Analouee, E: The common fixed point of single-valued generalized φ_f-weakly contractive mappings. Appl. Math. Lett. **24**, 771-776 (2011)
8. Popescu, O: Fixed points for (ψ,φ)-weak contractions. Appl. Math. Lett. **24**, 1-4 (2011)
9. Pasicki, L: Fixed point theorems for contracting mappings in partial metric spaces. Fixed Point Theory Appl. **2014**, 185 (2014)
10. Jungck, G, Rhoades, BE: Fixed point for set valued functions without continuity. Indian J. Pure Appl. Math. **29**, 227-238 (1998)