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Approximating common fixed points of semigroups in metric spaces

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Abstract

In this paper, we investigate the common fixed points set of nonexpansive semigroups of nonlinear mappings $\{T_t\}_{t\geq 0}$, *i.e.*, a family such that $T_0(x) = x$, $T_{s+t} = T_s(T_t(x))$, where the domain is a metric space (M,d). In particular we prove that under suitable conditions, the common fixed points set is the same as the common fixed points set of two mappings from the family. Then we use the modified Mann iteration process to approximate such common fixed points.

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1 Introduction

The purpose of this paper is to prove the existence of common fixed points for semigroups of nonlinear mappings acting in metric spaces. Recently, Khamsi and Kozlowski presented a series of fixed point results for pointwise contractions, asymptotic pointwise contractions, pointwise nonexpansive and asymptotic pointwise nonexpansive mappings acting in modular function spaces [1, 2].

Let us recall that a family $\{T_t\}_{t\geq 0}$ of mappings forms a semigroup if $T_0(x) = x$ and $T_{s+t} = T_s \circ T_t$. Such a situation is quite typical in mathematics and applications. For instance, in the theory of dynamical systems, the vector function space would define the state space and the mapping $(t,x) \to T_t(x)$ would represent the evolution function of a dynamical system. The question about the existence of common fixed points, and about the structure of the set of common fixed points, can be interpreted as a question whether there exist points that are fixed during the state space transformation T_t at any given point of time t, and if yes - what the structure of a set of such points may look like. In the setting of this paper, the state space is a nonlinear metric space.

The existence of common fixed points for families of contractions and nonexpansive mappings in Banach spaces has been the subject of intense research since the early 1960s, as investigated by Belluce and Kirk [3, 4], Browder [5], Bruck [6], DeMarr [7], and Lim [8]. The asymptotic approach for finding common fixed points of semigroups of Lipschitzian (but not pointwise Lipschitzian) mappings has also been investigated, see, *e.g.*, Tan and Xu [9]. It is worthwhile mentioning the recent studies on the special case, when the parameter set for the semigroup is equal to $\{0,1,2,3,\dots\}$, and $T_n = T^n$, the nth iterate of an asymptotic

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pointwise nonexpansive mapping. Kirk and Xu [10] proved the existence of fixed points for asymptotic pointwise contractions and asymptotic pointwise nonexpansive mappings in Banach spaces, while Hussain and Khamsi [11] extended this result to metric spaces, and Khamsi and Kozlowski to modular function spaces [1, 2]. In the context of modular function spaces, Khamsi discussed in [12] the existence of nonlinear semigroups in Musielak-Orlicz spaces and considered some applications to differential equations.

2 Uniform convexity in metric spaces

Throughout this paper, (M,d) will stand for a metric space. Suppose that there exists a family $\mathcal{F}$ of metric segments such that any two points x, y in M are endpoints of a unique metric segment $[x,y] \in \mathcal{F}$ ($[x,y]$ is an isometric image of the real line interval $[0,d(x,y)]$). We shall denote by $(1-\beta)x \oplus \beta y$ the unique point z of $[x,y]$ which satisfies

$$d(x,z) = \beta d(x,y) \quad \text{and} \quad d(z,y) = (1-\beta)d(x,y),$$

where $\beta \in [0,1]$. Such metric spaces are usually called *convex metric spaces* [13]. Moreover, if we have

$$d\left(\frac{1}{2}p \oplus \frac{1}{2}x, \frac{1}{2}p \oplus \frac{1}{2}y\right) \leq \frac{1}{2}d(x,y)$$

for all p, x, y in M, then M is said to be a *hyperbolic metric space* (see [14]).

Obviously, normed linear spaces are hyperbolic spaces. As nonlinear examples, one can consider the Hadamard manifolds [15], the Hilbert open unit ball equipped with the hyperbolic metric [16], and the CAT(0) spaces [17–19] (see Example 2.1). We will say that a subset C of a hyperbolic metric space M is convex if $[x,y] \subset C$ whenever x, y are in C.

Definition 2.1 Let (M,d) be a hyperbolic metric space. We say that M is uniformly convex (in short, UC) if for any $a \in M$, for every $r > 0$, and for each $\epsilon > 0$,

$$\delta(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right); d(x,a) \leq r, d(y,a) \leq r, d(x,y) \geq r\varepsilon\right\} > 0.$$

The definition of uniform convexity finds its origin in Banach spaces [20]. To the best of our knowledge, the first attempt to generalize this concept to metric spaces was made in [21]. The reader may also consult [14, 16, 22].

From now onwards we assume that M is a hyperbolic metric space and if (M,d) is uniformly convex, then for every $s \geq 0$, $\epsilon > 0$, there exists $\eta(s,\epsilon) > 0$ depending on s and ϵ such that

$$\delta(r,\varepsilon) > \eta(s,\epsilon) > 0 \quad \text{for any } r > s.$$

Most of the results in this section may be found in [22].

Recall that the hyperbolic metric space (M,d) is said to be strictly convex if whenever

$$d(a,x) = d(a,y) = d\big(a, \lambda x \oplus (1-\lambda)y\big)$$

for any $a,x,y \in M$ and $\lambda \in (0,1)$, then we must have $x = y$.

Remark 2.1 [2, 22]

(i) Let us observe that $\delta(r, 0) = 0$, and $\delta(r, \varepsilon)$ is an increasing function of ε for every fixed r.

(ii) For $r_1 \leq r_2$ there holds

$$1 - \frac{r_2}{r_1}\left(1 - \delta\left(r_2, \varepsilon\frac{r_1}{r_2}\right)\right) \leq \delta(r_1, \varepsilon).$$

(iii) If (M, d) is uniformly convex, then (M, d) is strictly convex.

Lemma 2.1 [2, 22] *Assume that (M, d) is uniformly convex. Let $\{C_n\} \subset M$ be a sequence of nonempty, nonincreasing, convex, bounded and closed sets. Let $x \in M$ be such that*

$$0 < d = \lim_{n\to\infty} d(x, C_n) < \infty.$$

Let $x_n \in C_n$ be such that $d(x, x_n) \to d$. Then $\{x_n\}$ is a Cauchy sequence.

Recall that a hyperbolic metric space (M, d) is said to have the property (R) if any nonincreasing sequence of nonempty, convex, bounded and closed sets has a nonempty intersection [23].

Our next result deals with the existence and the uniqueness of the best approximants of convex, closed and bounded sets in a uniformly convex metric space. This result is of interest by itself as uniform convexity implies the property (R), which reduces to reflexivity in the linear case.

Theorem 2.1 [2, 22] *Assume that (M, d) is complete and uniformly convex. Let $C \subset M$ be nonempty, convex and closed. Let $x \in M$ be such that $d(x, C) < \infty$. Then there exists a unique best approximant of x in C, i.e., there exists a unique $x_0 \in C$ such that*

$$d(x, x_0) = d(x, C).$$

The following result gives the analogue of the well-known theorem that states any uniformly convex Banach space is reflexive (see Theorem 2.1 in [16]).

Theorem 2.2 [2, 22] *If (M, d) is complete and uniformly convex, then (M, d) has the property* (R).

Note that any hyperbolic metric space M which satisfies the property (R) is complete.

Example 2.1 Let (X, d) be a metric space. A *geodesic* from x to y in X is a mapping c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which will be denoted by $[x, y]$ and called the segment joining x to y.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [24]).

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following CAT(0) comparison axiom:

Let Δ be a geodesic triangle in X and let $\overline{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \overline{\Delta}$,

$$d(x, y) \leq d(\bar{x}, \bar{y}).$$

Complete CAT(0) spaces are often called *Hadamard spaces* (see [18]). If x, y_1, y_2 are points of a CAT(0) space and y_0 is the midpoint of the segment $[y_1, y_2]$, which will be denoted by $\frac{y_1 \oplus y_2}{2}$, then the CAT(0) inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2} d^2(x, y_1) + \frac{1}{2} d^2(x, y_2) - \frac{1}{4} d^2(y_1, y_2).$$

This inequality is the (CN) inequality of Bruhat and Tits [25]. As for the Hilbert space, the (CN) inequality implies that CAT(0) spaces are uniformly convex with

$$\delta(r, \varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

One may also find the modulus of uniform convexity via similar triangles.

3 Common fixed points of nonexpansive semigroups

Recall the definition of a nonexpansive mapping defined in metric spaces.

Definition 3.1 Let (M, d) be a metric space and $C \subset M$ be a nonempty subset. A mapping $T : C \to C$ is said to be nonexpansive if

$$d\big(T(x), T(y)\big) \leq d(x, y)$$

for any $x, y \in C$. A point $x \in C$ is called a fixed point of T if $T(x) = x$. The set of fixed points of T will be denoted by $\mathrm{Fix}(T)$.

This definition is now extended to a one-parameter family of mappings.

Definition 3.2 Let (M, d) be a metric space and $C \subset M$ be a nonempty subset. A one-parameter family $\mathcal{F} = \{T_t; t \geq 0\}$ of mappings from C into itself is said to be a nonexpansive semigroup on C if $\mathcal{F}$ satisfies the following conditions:

(i) $T_0(x) = x$ for $x \in C$;
(ii) $T_{t+s}(x) = T_t(T_s(x))$ for $x \in C$ and $t, s \in [0, \infty)$;
(iii) for each $t \geq 0$, T_t is a nonexpansive mapping.

Define the set of all common fixed points of $\mathcal{F}$ as

$$\mathrm{Fix}(\mathcal{F}) = \bigcap_{t \geq 0} \mathrm{Fix}(T_t).$$

First let us extend Bruck's result [6] to metric spaces.

Lemma 3.1 [6] *Let* (M,d) *be a hyperbolic metric space. Assume that* (M,d) *is strictly convex. Let* C *be a subset of* M. *Let* S *and* T *be nonexpansive mappings from* C *into* M *with a common fixed point. Then, for each* $\lambda \in (0,1)$, *the mapping* $U : C \to M$ *defined by* $U(x) = \lambda S(x) \oplus (1-\lambda)T(x)$ *for* $x \in C$ *is nonexpansive and* $\mathrm{Fix}(U) = \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$ *holds.*

Proof Clearly we have $\mathrm{Fix}(S) \cap \mathrm{Fix}(T) \subset \mathrm{Fix}(U)$. Let us now prove that $\mathrm{Fix}(U) \subset \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$. Let $x \in \mathrm{Fix}(U)$, *i.e.*, $U(x) = x$. Since S and T have a common fixed point, let $a \in \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$ be fixed. We have

$$d(a,x) = d\big(a, \lambda S(x) \oplus (1-\lambda)T(x)\big) \leq \lambda d\big(a, S(x)\big) + (1-\lambda) d\big(a, T(x)\big)$$

because of the hyperbolicity of M. Since a is a common fixed point of S and T which are nonexpansive, we get

$$d(a,x) = d\big(a, \lambda S(x) \oplus (1-\lambda)T(x)\big) \leq \lambda d(a,x) + (1-\lambda) d(a,x) = d(a,x).$$

Therefore we must have

$$d\big(a, S(x)\big) = d\big(a, T(x)\big) = d(a,x)$$

since $0 < \lambda < 1$. The strict convexity of M will then imply $S(x) = T(x)$. Clearly from $x = \lambda S(x) \oplus (1-\lambda)T(x)$ we conclude that $S(x) = T(x) = x$, *i.e.*, $x \in \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$. The proof of Lemma 3.1 is complete. □

The next result concerns continuous semigroups.

Definition 3.3 Let (M,d) be a metric space and $C \subset M$ be nonempty and closed. A one-parameter family $\mathcal{F} = \{T_t; t \geq 0\}$ of mappings from C into M is said to be continuous on C if for any $x \in C$, the mapping $t \to T_t(x)$ is continuous, *i.e.*, for any $t_0 \geq 0$, we have

$$\lim_{t \to t_0} d\big(T_t(x), T_{t_0}(x)\big) = 0$$

for any $x \in C$.

The following result is easy to prove.

Proposition 3.1 *Let* (M,d) *be a metric space and* $C \subset M$ *be nonempty and closed. Let* $\mathcal{F} = \{T_t; t \geq 0\}$ *be a one-parameter semigroup of mappings from* C *into* M *which is continuous on* C. *Let* A *be a dense subset of* $[0,+\infty)$. *Then we have*

$$\mathrm{Fix}(\mathcal{F}) = \bigcap_{a \in A} \mathrm{Fix}(T_a).$$

Recall the following lemma which can be found in any introductory course on real analysis.

Lemma 3.2 [26] *Let G be a nonempty additive subgroup of $\mathbb{R}$. Then G is either dense in $\mathbb{R}$ or there exists $a > 0$ such that $G = a \cdot \mathbb{Z} = \{an, n \in \mathbb{Z}\}$. Therefore if α and β are two real numbers such that $\frac{\alpha}{\beta}$ is irrational, then the set*

$$G(\alpha,\beta) = \{\alpha n + \beta m; n, m \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$. In particular, the set $G(\alpha,\beta) \cap [0,+\infty)$ is dense in $[0,+\infty)$.

The following result will be useful to prove our main result of this section.

Proposition 3.2 *Let (M,d) be a metric space and $C \subset M$ be nonempty and closed. Let $\mathcal{F} = \{T_t; t \geq 0\}$ be a one-parameter semigroup of mappings from C into M. Let α and β be any two positive real numbers. Then we have*

$$\bigcap_{a\in G_+(\alpha,\beta)} \operatorname{Fix}(T_a) = \operatorname{Fix}(T_\alpha) \cap \operatorname{Fix}(T_\beta),$$

where $G_+(\alpha,\beta) = \{\alpha n + \beta m; n, m \in \mathbb{Z}\} \cap [0,+\infty)$.

Proof Clearly we have $\bigcap_{a\in G_+(\alpha,\beta)} \operatorname{Fix}(T_a) \subset \operatorname{Fix}(T_\alpha) \cap \operatorname{Fix}(T_\beta)$ since $\alpha, \beta \in G_+(\alpha,\beta)$. Conversely, let $x \in \operatorname{Fix}(T_\alpha) \cap \operatorname{Fix}(T_\beta)$. Let $a \in G_+(\alpha,\beta)$. Then there exist $n, m \in \mathbb{Z}$ such that $a = n\alpha + m\beta$. Assume first that both n and m are positive. Then

$$T_a(x) = T_{n\alpha+m\beta}(x) = T_{n\alpha}\big(T_{m\beta}(x)\big) = T_\alpha^n\big(T_\beta^m(x)\big) = x,$$

where we used the property T_0 is the identity map. Otherwise assume $a = n\alpha - \bar{m}\beta$, where both n and $\bar{m}$ are positive. Hence $a + \bar{m}\beta = n\alpha$. So

$$T_a(x) = T_a\big(T_\beta^{\bar{m}}(x)\big) = T_{a+\bar{m}\beta}(x) = T_{n\alpha}(x) = T_\alpha^n(x) = x.$$

Hence $x \in \operatorname{Fix}(T_a)$ for any $a \in G_+(\alpha,\beta)$. This completes the proof of Proposition 3.2. □

If we combine Propositions 3.1 and 3.2, we get the following result.

Theorem 3.1 [27] *Let (M,d) be a metric space and $C \subset M$ be nonempty and closed. Let $\mathcal{F} = \{T_t; t \geq 0\}$ be a one-parameter semigroup of mappings from C into M which is continuous on C. Let α and β be two positive real numbers such that $\frac{\alpha}{\beta}$ is irrational, then we have*

$$\operatorname{Fix}(\mathcal{F}) = \operatorname{Fix}(T_\alpha) \cap \operatorname{Fix}(T_\beta).$$

In particular, we have

$$\operatorname{Fix}(\mathcal{F}) = \operatorname{Fix}(T_1) \cap \operatorname{Fix}(T_{\sqrt{2}}) = \operatorname{Fix}(T_1) \cap \operatorname{Fix}(T_\pi).$$

If the metric space (M,d) is hyperbolic strictly convex, Lemma 3.1 allows us to get the following result.

Theorem 3.2 *Let (M,d) be a hyperbolic metric space. Assume that (M,d) is strictly convex and $C \subset M$ is nonempty and closed. Let $\mathcal{F} = \{T_t; t \geq 0\}$ be a one-parameter semigroup of nonexpansive mappings from C into M which is continuous on C. Let α and β be two positive real numbers such that $\frac{\alpha}{\beta}$ is irrational, then we have*

$$\mathrm{Fix}(\mathcal{F}) = \mathrm{Fix}\big(\lambda T_\alpha + (1-\lambda)T_\beta\big)$$

for any $\lambda \in (0,1)$.

4 Approximation of common fixed points of semigroups

In this section we use the previous results to investigate the behavior of Mann iterates generated by two mappings. These iterations will allow us to approximate common fixed points of a continuous semigroup.

Definition 4.1 Let $T: C \to C$ be a nonexpansive mapping and $\sigma \in (0,1)$. The Mann iteration process generated by the mapping T and the constant σ, denoted by $M(T,\sigma)$, is defined by the following iteration formula:

$$x_{n+1} = \sigma Tx_n \oplus (1-\sigma)x_n, \tag{4.1}$$

where x_1 is chosen arbitrarily in C.

The following technical lemmas will be useful throughout.

Lemma 4.1 [2, 22] *Let (M,d) be a uniformly convex hyperbolic metric space. Assume that there exists $R \in [0,+\infty)$ such that*

$$\limsup_{n\to\infty} d(x_n,a) \leq R, \qquad \limsup_{n\to\infty} d(y_n,a) \leq R, \quad \text{and} \quad \lim_{n\to\infty} d\big(a, \sigma x_n \oplus (1-\sigma)y_n\big) = R$$

for some $\sigma \in (0,1)$. Then $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Lemma 4.2 *Let (M,d) be a uniformly convex hyperbolic metric space. Let C be a nonempty, bounded, closed and convex subset of M. Let $T: C \to C$ be nonexpansive, and let $\sigma \in (0,1)$ and x_n be given by (4.1). Assume that ω is a fixed point of T. Then the limit of $\{d(x_n,\omega)\}$ exists.*

Proof We have

$$\begin{aligned} d(x_{n+1},\omega) &= d\big(\sigma Tx_n \oplus (1-\sigma)x_n, \omega\big) \\ &\leq \sigma d(Tx_n,\omega) + (1-\sigma)d(x_n,\omega) \\ &= \sigma d(Tx_n,T\omega) + (1-\sigma)d(x_n,\omega) \\ &\leq \sigma d(x_n,\omega) + (1-\sigma)d(x_n,\omega) \\ &= d(x_n,\omega) \end{aligned}$$

for any $n \geq 1$. Hence the sequence $\{d(x_n,\omega)\}$ is decreasing, which implies that it is convergent. □

Let us recall the definition of the asymptotic radius of a sequence.

Definition 4.2 Let (M,d) be a metric space and C be a nonempty subset of M. Let $\{x_n\}$ be a bounded sequence in M. Define $r(\cdot,\{x_n\}) : C \to [0,\infty)$ by

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius ρ_C of $\{x_n\}$ with respect to C is given by

$$\rho_C = \inf\{r(x,\{x_n\}) : x \in C\}.$$

ρ will denote the asymptotic radius of $\{x_n\}$ with respect to M. A point $\xi \in C$ is said to be an asymptotic center of $\{x_n\}$ with respect to C if $r(\xi,\{x_n\}) = r(C,\{x_n\}) = \min\{r(x,\{x_n\}) : x \in C\}$.

The set of all asymptotic centers of $\{x_n\}$ with respect to C will be denoted by $A(C,\{x_n\})$. When $C = M$, we use the notation $A(\{x_n\})$ instead of $A(M,\{x_n\})$. In general, the set $A(C,\{x_n\})$ of asymptotic centers of a bounded sequence $\{x_n\}$ may be empty or contain more than one point. Note that the asymptotic radius is also known in the literature as a type function. For more on this we refer to [22].

Over the years many people were successful in defining the analogue of linear properties in metric spaces. The weak-topology is still hard to define in the context of metric spaces. An approach to weak-convergence was offered by Kuczumow [28] and Lim [8] which they called Δ-convergence. Their approach was very successful in the case of CAT(0) spaces.

Definition 4.3 Let (M,d) be a metric space. A bounded sequence $\{x_n\}$ in M is said to Δ-converge to $x \in M$ if and only if x is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$. We write $x_n \stackrel{\Delta}{\to} x$ whenever $\{x_n\}$ Δ-converges to x.

In this section, we study the iteration scheme (4.1) for nonexpansive mappings. In particular, we investigate the Δ-convergence in uniformly convex hyperbolic spaces. Note that similar conclusions proved in Banach spaces require the Fréchet differentiability of the norm.

In the sequel, the following results will be needed.

Lemma 4.3 [22, 29] *Let (M,d) be a hyperbolic metric space. Assume that M is uniformly convex. Let C be a nonempty, closed and convex subset of M. Then every bounded sequence $\{x_n\} \in M$ has a unique asymptotic center with respect to C.*

Lemma 4.4 [22, 29] *Let (M,d) be a hyperbolic metric space. Assume that M is uniformly convex. Let C be a nonempty, closed and convex subset of M. Let $\{x_n\}$ be a bounded sequence in C such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is a sequence in C such that $\lim_{m\to\infty} r(y_m,\{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

The following result is similar to the demi-closed principle discovered by Göhde in uniformly convex Banach spaces [30].

Lemma 4.5 [29] *Let C be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space (M,d). Let $T : C \to C$ be a nonexpansive mapping. Let $\{x_n\} \in C$ be an approximate fixed point sequence of T, i.e., $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. If $x \in C$ is the asymptotic center of $\{x_n\}$ with respect to C, then x is a fixed point of T. In particular, if $\{x_n\} \in C$ is an approximate fixed point sequence of T such that $x_n \stackrel{\Delta}{\to} x$, then x is a fixed point of T.*

The following theorem is necessary to discuss the behavior of the iterates defined by (4.1).

Theorem 4.1 [29] *Let (M,d) be a hyperbolic metric space. Assume that M is uniformly convex. Let C be a nonempty, closed and convex subset of M. Let $T : C \to C$ be a nonexpansive mapping with a nonempty fixed points set. Let $\sigma \in (0,1)$, $x_1 \in C$ and generate $\{x_n\}$ by* (4.1). *Then we have*

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0,$$

i.e., $\{x_n\}$ is an approximate fixed point sequence for T.

Proof Let ω be a fixed point of T. From Lemma 4.2, there exists $r \in \mathbb{R}$ such that

$$\lim_{n\to\infty} d(x_n, \omega) = r. \tag{4.2}$$

Without loss of generality, we may assume $r > 0$. Note that

$$\limsup_{n\to\infty} d(Tx_n, \omega) = \limsup_{n\to\infty} d(Tx_n, T\omega) \le \limsup_{n\to\infty} d(x_n, \omega) = r \tag{4.3}$$

and

$$\begin{aligned} r &= \limsup_{n\to\infty} d(x_n, \omega) = \limsup_{n\to\infty} d\big(\sigma Tx_n \oplus (1-\sigma)x_n, \omega\big) \\ &\le \limsup_{n\to\infty} \sigma d(Tx_n, \omega) + (1-\sigma) d(x_n, \omega) \\ &\le \limsup_{n\to\infty} \sigma d(x_n, \omega) + (1-\sigma) d(x_n, \omega) \\ &= \limsup_{n\to\infty} d(x_n, \omega) = r. \end{aligned}$$

Hence

$$\limsup_{n\to\infty} d\big(\sigma Tx_n \oplus (1-\sigma)x_n, \omega\big) = r.$$

Using Lemma 4.1, we conclude

$$\limsup_{n\to\infty} d(Tx_n, x_n) = 0. \qquad \square$$

Note that the assumption that T has a fixed point may be relaxed if we assume C is bounded (for more see [22]). Next we discuss the Δ-convergence of the Mann iterates defined by (4.1).

Theorem 4.2 *Let (M,d) be a complete hyperbolic metric space. Assume that M is uniformly convex. Let C be a nonempty, closed and convex subset of M. Let $T : C \to C$ be a nonexpansive mapping with a nonempty fixed points set. Let $\sigma \in (0,1)$, $x_1 \in C$ and generate $\{x_n\}$ by* (4.1). *Then $\{x_n\}$ Δ-converges to x which is a fixed point of T, i.e., $x \in \mathrm{Fix}(T)$.*

Proof In [22] it is shown that $\mathrm{Fix}(T)$ is convex and closed. Theorem 4.1 implies that $\{x_n\}$ is an approximate fixed point sequence of T, *i.e.*, $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Let x be the unique asymptotic center of $\{x_n\}$. Then Lemma 4.5 implies that $x \in \mathrm{Fix}(T)$. Next we show that $\{x_n\}$ Δ-converges to x. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. Let z be the unique asymptotic center of $\{x_{n_i}\}$. Again since $\{x_{n_i}\}$ is an approximate fixed point sequence of T, we get $z \in \mathrm{Fix}(T)$. Hence

$$\limsup_{n_i\to\infty} d(x_{n_i}, z) \le \limsup_{n_i\to\infty} d(x_{n_i}, x).$$

Since $x, z \in \mathrm{Fix}(T)$, we get

$$\limsup_{n_i\to\infty} d(x_{n_i}, z) = \lim_{n\to\infty} d(x_n, z), \quad \text{and} \quad \limsup_{n_i\to\infty} d(x_{n_i}, x) = \lim_{n\to\infty} d(x_n, x).$$

Since x is the unique asymptotic center of $\{x_n\}$, we get $x = z$. This proves that $\{x_n\}$ Δ-converges to x. □

If we combine Theorem 3.2 and Theorem 4.2, we obtain the following result.

Theorem 4.3 *Let (M,d) be a complete hyperbolic metric space. Assume that M is uniformly convex. Let C be a nonempty, closed and convex subset of M. Let $\mathcal{F} = \{T_t; t \ge 0\}$ be a one-parameter semigroup of nonexpansive mappings from C into C. Assume that $\mathcal{F}$ is continuous and has a nonempty common fixed points set. Let α and β be two positive real numbers such that $\frac{\alpha}{\beta}$ is irrational. Fix $\lambda, \mu \in (0,1)$ such that $\lambda + \mu < 1$. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = (\mu + \lambda)\bigl(\mu T_\alpha(x_n) \oplus \lambda T_\beta(x_n)\bigr) \oplus (1 - \mu - \lambda)x_n \tag{4.4}$$

for any $n \ge 1$. Then $\{x_n\}$ Δ-converges to a common fixed point of the semigroup $\mathcal{F}$.

Proof Set

$$S = \frac{\mu}{\mu+\lambda} T_\alpha \oplus \frac{\lambda}{\mu+\lambda} T_\beta.$$

Observe that $S : C \to C$ is nonexpansive. Clearly we have $\mathrm{Fix}(\mathcal{F}) \subset \mathrm{Fix}(S)$. Let $x_1 \in C$ and $\{x_n\}$ be the sequence generated by (4.4). Then

$$x_{n+1} = \sigma S(x_n) + (1 - \sigma)x_n,$$

where $\sigma = \mu + \lambda \in (0,1)$. Clearly $\{x_n\}$ is the same sequence generated by (4.1) for the map S. Theorem 4.2 implies that $\{x_n\}$ Δ-converges to some $x \in \mathrm{Fix}(S)$. From Theorem 3.2, we know that

$$\mathrm{Fix}(\mathcal{F}) = \mathrm{Fix}(S).$$

Hence $x \in \mathrm{Fix}(\mathcal{F})$, which completes the proof of Theorem 4.3. □

Using the techniques developed in [29], one may show that the conclusion of Theorem 4.3 is still valid if we consider the modified Mann iteration

$$x_{n+1} = \alpha_n T_\alpha \left(\beta_n T_\beta(x_n) \oplus (1-\beta_n)x_n\right) \oplus (1-\alpha_n)x_n,$$

where $\alpha_n, \beta_n \in (0,1)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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