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# An extension of the contraction mapping principle to Lipschitzian mappings

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**Abstract**

Very recently, Udo-utun (Fixed Point Theory Appl. 2014:65, 2014) established a property possessed by contractive operators with nonempty fixed point sets and used it to prove the existence of fixed points of nonexpansive maps. We extend these results and prove the existence of fixed points for $L$-Lipschitzian maps that possess this property. Our results generalize and unify results concerning asymptotic fixed point theory of contractive mappings. An application in ordinary differential equations is illustrated via a formulation of the global existence theorem for initial value problems.

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**Keywords:** $(\delta, k)$-weak contraction; fixed points; Krasnoselskii iteration; $L$-Lipschitzian

## 1 Introduction

The famous *Banach fixed point theorem*, called the *contraction mapping principle*, has been studied by numerous authors and numerous generalizations have been obtained by eminent mathematicians based on various contractive conditions. Our approach in this paper consists in applying a more unifying contractive condition derived by Berinde [1] to obtain asymptotic fixed point theorem for the class of Lipschitzian mappings satisfying a certain (boundedness-like) condition derived very recently by the first author [2]. Let $(X, d)$ be a metric space and $T : X \to X$ a mapping satisfying the condition

$$d(Tx, Ty) \le Ld(x, y), \quad x, y \in X. \tag{1}$$

Then $T$ is called an $L$-Lipschitzian operator where $L \ge 0$ is the least constant with this property. $T$ is called a contraction if $L \in [0, 1)$ and, for $T = 1$, $T$ is called a nonexpansive operator. $T$ is said to be expansive if $L > 1$.

In the sequel we shall denote by $\mathrm{Fix}(T)$ the set of fixed points of $T$. The study of fixed points of contractive and expansive mappings still attracts the attention of numerous researchers investigating extensions and generalizations of Banach fixed point theorem. The contraction mapping principle in a Banach space $(E, \|\cdot\|)$ asserts that if $T : K \subset E \to K$ is a contraction (*i.e.* for all $x, y \in K$, $\|Tx - Ty\| \le L\|x - y\|$, $L \in (0, 1)$) then:

p1. $T$ has a unique fixed point given by the limit $p = \lim_{n\to\infty} x_n = T^n x_0$ of the Picard iteration $\{x_n\}_{n=1}^{\infty}$ where $x_0$ is any initial guess in $K$.

p2. The following estimates hold: $\|x_n - p\| \leq \frac{L^n}{1-L}\|x_0 - x_1\|$, $\|x_n - p\| \leq \frac{L}{1-L}\|x_0 - x_1\|$.
p3. The rate of convergence of Picard iteration $\{x_{n+1}\} = \{Tx_n\}$ is given by
$\|x_n - p\| \leq L\|x_{n-1} - p\|$.

We recall that the contraction condition in a metric space $(X, d)$ is $d(Tx, Ty) \leq Ld(x, y)$, $0 \leq L < 1$ for all $x, y \in X$, $T$ is called *contractive* if

$$d(Tx, Ty) \leq d(x, y), \tag{2}$$

and $T$ is called a *quasicontraction* (Ciric [3]) if there exists $q \in (0, 1)$ such that

$$d(Tx, Ty) \leq q \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \tag{3}$$

Following Ciric [3], Berinde [1, 4] investigated all contractive conditions which make use of displacements of the forms $d(x, y)$, $d(x, Tx)$, $d(y, Ty)$, $d(x, Ty)$, and $d(y, Tx)$ under the concept of a $(\delta, k)$-*weak contraction* introduced by him as follows.

**Definition 1** [4] Let $(X, d)$ be a metric space, $\delta \in (0, 1)$, and $k \geq 0$, then a mapping $T : X \to X$ is called a $(\delta, k)$-weak contraction (or a weak contraction) if and only if

$$d(Tx, Tx) \leq \delta d(x, y) + Kd(y, Tx) \quad \text{for all } x, y \in X. \tag{4}$$

**Remark 1** A current terminology for the class of $(\delta, k)$-weak contractions is *almost contractions*.

The results of this paper constitute applications of this concept of $(\delta, k)$-weak contractions to obtain existence results for all Lipschitzian maps satisfying a certain boundedness notion as stated below. It is also shown that this condition is a nontrivial extension and modification of the Banach contraction mapping principle in Banach spaces.

Specifically, we prove, in an arbitrary Banach space, the existence of fixed points for a class of Lipschitzian operators subject to the following central hypothesis:

$$\|y - Tx\| \leq M\|x - y\| \quad \text{whenever } \|x - y\| \leq \|y - Tx\| \tag{5}$$

for some $M \geq 1$ and for $x$ and $y$ in a certain subset of a closed convex subset $K$ of a Banach space $E$ with $x \neq y$.

## 2 Preliminary

Our method involves proving that for arbitrary $L(0, \infty)$, any $L$-Lipschitzian map satisfying (5) possesses a fixed point by showing that a corresponding averaged operator $S_\lambda x = \lambda x + (1 - \lambda)Tx$ is an example of $(\delta, k)$-weak contractions. In other words, as follows from [1, 4], the Krasnoselskii iteration scheme $x_{n+1} = \lambda x_n + (1 - \lambda)Tx_n$ of $T$ converges to a fixed point of $T$. Further, we demonstrate that, in Banach spaces, the hypothesis of the contraction mapping principle is a special case of (5) above.

Investigations concerning fixed points of expansive transformations include the work of Goebel and Kirk [5] on uniformly $L$-Lipschitzian mappings with $L < \sqrt{\frac{5}{2}}$. This estimate for $L$ was improved by Lifschitz [6] with $L < \sqrt{2}$. It is worth noting that recent research trends in uniformly $L$-Lipschitzian mappings are concerned with the normal structure and structure of fixed point sets $\mathrm{Fix}(T)$ of the semigroup of uniformly $L$-Lipschitzian mappings. Works in this direction include [7, 8] and [9]. Our results include the existence aspects of these studies as special cases.

Other examples of $(\delta,k)$-weak contractions are given in [10]. It is shown in [1, 4] that a lot of well-known contractive conditions in the literature are special cases of the almost contraction condition (4) as it does not require that $\delta + k < 1$, which is assumed in almost all fixed point theorems based on the contractive conditions which involve displacements of the forms $d(x,y)$, $d(x,Tx)$, $d(y,Ty)$, $d(x,Ty)$, and $d(y,Tx)$; see Berinde [1], Kannan [11], Rhoades [12], Zamfirescu [13] and references therein.

Examples of $L$-Lipschitzian operators satisfying (5) are illustrated below.

**Example 1** Let $E = (-\infty,\infty)$ and $K = [0,\infty)$, the operator $T : K \to K$ defined for any fixed $\tau > 0$, defined by $Tx = \tau e^{\tau - x}$, $x \in K$ is $L$-Lipschitzian, since $\|Tx - Ty\| = |\tau e^{\tau-x} - \tau e^{\tau-y}| \le \tau e^{\tau}|x - y|$, $x, y \in K$ and has the fixed point $x = \tau$. It is remarkable that $T$ satisfies the condition (5) *viz.*: $\frac{\|y-Tx\|}{\|x-y\|} = \frac{|y-Tx|}{|x-y|} \le 1 + \frac{|xe^{\tau}-\tau e^{\tau}|}{|x-y|}$. By the mean value theorem we obtain $\frac{\|y-Tx\|}{\|x-y\|} \le 1 + |x-\tau|\frac{|M_1-M_2|}{|x-y|}$ for some $M_1$, $M_2$ lying between $x$ and $\tau$. This shows that there exists $M \ge 1$ such that $\|y - Tx\| \le M\|x - y\|$ for all $x$ and $y$ in an appropriate closed neighborhood of the fixed point $\tau$ with $x \ne y$ and $x, y \notin \mathrm{Fix}(T)$.

Similarly, on the other hand the operator $T : K \to K$ defined by $Tx = e^{\tau-x}$ is $L$-Lipschitzian with Lipschitz constant $L = e^{\tau}$ and has no fixed point. As a counterexample, it is interesting that $T$ does not satisfy property (5). This follows from the fact that $\frac{\|y-Tx\|}{\|x-y\|} = \frac{|y-Tx|}{|x-y|} \le 1 + \max\{e^{-x}, e^{-\tau}\}\frac{|1-\tau|}{|x-y|} > M$ for all $M > 0$, *i.e.* when $x$ and $y$ are very close.

In [4] Berinde proved the following theorem, the *first almost contraction mapping principle*.

**Theorem 1** [1] *Let $(X,d)$ be a complete metric space and $T : X \to X$ be a $(\delta,k)$-weak contraction (i.e. almost contraction). Then*:

(1) $\mathrm{Fix}(T) = \{x \in X : Tx = x\} \ne \emptyset$.

(2) *For any $x_0 \in X$ the Picard iteration $\{x_n\}$ given by $x_{n+1} = T^n x_0$, $n = 0,1,2,\ldots,$ converges to some $x^* \in \mathrm{Fix}(T)$.*

(3) *The estimates*

$$d(x_n,x^*) \le \frac{\delta^n}{1-\delta} d(x_0,x_1), \quad n = 0,1,2,\ldots,$$

$$d(x_n,x^*) \le \frac{\delta}{1-\delta} d(x_{n-1},x_n), \quad n = 0,1,2,\ldots,$$

*hold, where $\delta$ is the constant appearing in* (4).

(4) *Under the additional condition that there exist $\theta \in (0,1)$ and some $k_1 \ge 0$ such that*

$$d(Tx,Ty) \le \theta d(x,y) + k_1 d(x,Tx) \quad \text{for all } x,y \in X, \tag{6}$$

*the fixed point $x^*$ is unique and the Picard iteration converges at the rate $d(x_n,x^*) \le \theta d(x_{n-1},x^*)$, $n \in \mathbb{N}$.*

Observe that:

(a) to establish our claim it suffices to prove the averaged operator $S_\lambda$ given by $S_\lambda x = \lambda x + (1-\lambda)Tx$ is a $(\delta, k)$-weak contraction and then apply Theorem 1 to obtain fixed points of $S_\lambda$;

(b) for $\lambda \in [0,1)$ the fixed point set $\mathrm{Fix}(S_\lambda)$ of $S_\lambda = \lambda I + (1-\lambda)T$ coincides with $\mathrm{Fix}(T)$.

In the sequel we shall make use of the Archimedean property and a recent result in [2] below.

**Lemma 1** [2] *Let $V$ be a normed linear space and $T: V \to V$ a map. If $\|x-y\| \leq \|y-Tx\|$ then $Tx \neq Ty$ for any distinct $x, y \in V$ satisfying $x, y \notin \mathrm{Fix}(T)$.*

## 3 Main results

**Theorem 2** *Let $K$ be a closed convex subset of a real Banach space $E$ and $T: K \to K$ a nonexpansive mapping. If $\mathrm{Fix}(T) \neq \emptyset$, then there exists an open subset $K_1 \subset K$ such that $T$ satisfies the condition*

$$\|y - Tx\| \leq M\|x-y\| \quad \textit{whenever } \|x-y\| \leq \|y-Tx\|$$

*for some $M \geq 1$ and for all $x, y \in K_1$; $x \neq y$, $x, y \notin \mathrm{Fix}(T)$.*

*Proof* Given a closed convex subset $K$ of real Banach space $E$ and $T: K \to K$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $Y$ denote the collection of elements of $K$ satisfying $\|x-y\| \leq \|y-Tx\|$, $x \neq y$, $x, y \notin \mathrm{Fix}(T)$. We shall derive an open subset $K_1 \subset K$ in which (5) is satisfied. For $x, y \in Y$ nonexpansiveness of $T$ yields the following:

$$\|Tx - Ty\| \leq \|x-y\|, \tag{7}$$

$$\|x-y\| \leq \|y-Tx\|,$$

$$\|Tx - Ty\| \leq \|y-Tx\|. \tag{8}$$

Adding (7) and (8) yields

$$\begin{aligned} &2\|Tx-Ty\| \leq \|x-y\| + \|y-Tx\| \\ &\quad\Longrightarrow\quad \|y-Tx\| - \|y-Ty\| \leq \frac{1}{2}\|x-y\| + \frac{1}{2}\|y-Tx\| \\ &\quad\Longrightarrow\quad \|y-Tx\| \leq \|x-y\| + 2\|y-Ty\|. \end{aligned} \tag{9}$$

Let $\{x_n\}_{n=0}^\infty$ be the sequence generated from the Krasnoselskii iteration scheme $x_{n+1} = \lambda x_n + (1-\lambda)Tx_n$, $\lambda \in [0,1)$; it is well known that, for some initial guess $x_0 \in K$, the sequence of Krasnoselskii iterations $\{x_n\}_{n=1}^\infty = \{S_\lambda^n x_0\}_{n=1}^\infty$ converges to a fixed point of the nonexpansive mapping $T$ with nonempty fixed point set $\mathrm{Fix}(T)$. Given that $\mathrm{Fix}(T) \neq \emptyset$ we set $K_0 = Y \cap \{S_\lambda^n x_0\}_{n=1}^\infty$ and observe that $K_0$ is a nonempty bounded set. This follows since we can always find $n, m \in \mathbb{N}$ such that $\|S_\lambda^n x_0 - S_\lambda^m x_0\| \leq \|S_\lambda^m x_0 - S_\lambda^{n+1} x_0\|$ for any $x_0 \in K$. By Lemma 1 $S_\lambda^m x_0 \neq S_\lambda^{n+1} x_0$ if $x_0, S_\lambda x_0 \notin \mathrm{Fix}(S_\lambda)$. Since $\mathrm{Fix}(T) = \mathrm{Fix}(S_\lambda)$ it follows that $T^m x_0 \neq T^{n+1} x_0$ if $x_0, Tx_0 \notin \mathrm{Fix}(T)$.

In this case $\|y - S_\lambda y\|$ takes the form $\|x_m - x_{m+1}\|$ where $x_k = S_\lambda^k x_0$, while $\|x-y\| \leq \|y-Tx\|$ takes the form $\|x_n - x_m\| \leq \|x_m - x_{n+1}\|$. Clearly, $\|x_m - x_{m+1}\| \leq \|x_n - x_m\|$ since the condition

$\|x_n - x_m\| \leq \|x_m - x_{n+1}\|$ implies that $n \geq m$ for $n$ and $m$ large enough. This means that $\|y - S_\lambda y\| \leq \|x - y\|$ whenever $\|x - y\| \leq \|y - S_\lambda x\|$ in $K_0$. This yields $\|y - Ty - \lambda(y - Ty)\| \leq \|x - y\|$ for all $\lambda \in [0, \infty)$. Therefore, since $\lambda$ can be made as small as we please, also $\|y - Ty\| \leq \|x - y\|$ in $K_0$, so (9) yields $\|y - Tx\| \leq 3\|x - y\|$ in $K_0$. Let $K_1$ be the smallest open set in $K$ containing $K_0$ and, considering the continuity of $T$, we conclude that condition (5) is satisfied by nonexpansive mappings $T$ for which $\mathrm{Fix}(T) \neq \emptyset$. □

**Theorem 3** *Let $K$ be a closed convex subset of a Banach space $E$ and $T : K \to K$ an $L$-Lipschitzian operator. Suppose there exists an open subset $K_1 \subset K$ such that $T$ satisfies the condition*

$$\|y - Tx\| \leq M\|x - y\| \quad \textit{whenever } \|x - y\| \leq \|y - Tx\| \tag{10}$$

*for some $M \geq 1$ for all $x, y \in K_1$; $x \neq y$, $x, y \notin \mathrm{Fix}(T)$. Then $T$ has a fixed point in $K$ and the Krasnoselskii iteration scheme $x_{n+1} = \lambda x_n + (1 - \lambda)T^n x_0$, $n \geq 0$; $x_0 \in K_1$ converges to a fixed point of $T$ in $K$.*

*Further, condition* (10) *generalizes contraction condition* (*i.e.* (1) *with* $L \in (0, 1)$) *in Banach spaces.*

*Proof* The proof entails applications of Theorem 1 and Theorem 2. Let $S_\lambda$ denote the averaged operator $S_\lambda = \lambda I + (1 - \lambda)T$, $\lambda \in [0, 1)$, we obtain

$$\begin{aligned} \|S_\lambda x - S_\lambda y\| &= \bigl\|\lambda x + (1 - \lambda)Tx - \bigl[\lambda y + (1 - \lambda)Ty\bigr]\bigr\| \\ &\leq \bigl\|y - \bigl[\lambda x + (1 - \lambda)Tx\bigr]\bigr\| + (1 - \lambda)\|y - Ty\| \\ &\leq \|y - S_\lambda x\| + (1 - \lambda)\|y - Tx\| + (1 - \lambda)\|Tx - Ty\|. \end{aligned} \tag{11}$$

If $\|y - Tx\| \leq \|x - y\|$ then (11) yields $\|S_\lambda x - S_\lambda y\| \leq (1 - \lambda)(1 + L)\|x - y\| + \|y - S_\lambda x\|$. Clearly, if $\lambda \in (\frac{L}{L+1}, 1]$ then (11) yields

$$\|S_\lambda x - S_\lambda y\| \leq \delta\|x - y\| + \|y - S_\lambda x\|, \quad \text{where } \lambda \text{ and } \delta \text{ satisfy } 0 < \frac{\delta - (1 - \lambda)}{1 - \lambda} < L.$$

So in this case $S_\lambda$ is a $(\delta, k)$-weak contraction with $k = 1$ and $\delta$ is constrained by $\lambda$ and $L$ as shown above.

On the other hand if (or when) $\|x - y\| \leq \|y - Tx\|$ then, by (10), (11) yields $\|S_\lambda x - S_\lambda y\| \leq (1 - \lambda)(L + M)\|x - y\| + \|y - S_\lambda x\|$. So choosing $\lambda \in (0, 1)$ such that $(1 - \lambda) < \min\{\frac{1}{1+L}, \frac{1}{M+L}\}$ and $k = 1$ and by Theorem 1 we conclude that $S_\lambda$ has a fixed point in $K$. Therefore $T$ has a fixed point in $K$.

To complete the proof we need to establish that condition (10) generalizes, in the Banach space context, the contraction condition, *viz.*: $\|Tx - Ty\| \leq L\|x - y\|$; $L \in (0, 1)$. We note that since the collection of all contractions is a proper subclass of the class of nonexpansive mappings with nonempty fixed point set then by Theorem 2 all contractions satisfy condition (10) since the fixed point set of contractions is nonempty, which ends the proof. □

Next, using Theorem 2 and putting $L = 1$ in Theorem 3 we obtain the following fixed point results for nonexpansive mappings in arbitrary Banach spaces.

**Corollary 1** *Let $K \subseteq E$ be a closed convex subset of a real Banach space $E$ and $T : K \to K$ a nonexpansive operator. Then $T$ has a fixed point in $K$ if and only if there exists an open subset $K_1 \subset K$ such that $T$ satisfies the condition below*:

$$\|y - Tx\| \leq M\|x - y\| \quad \textit{whenever } \|x - y\| \leq \|y - Tx\|$$

*for some $M \geq 1$ for all $x, y \in K_1$; $x \neq y$, $x, y \notin \mathrm{Fix}(T)$. Further, the Krasnoselskii iteration scheme $x_{n+1} = \lambda x_n + (1 - \lambda)T^n x_0$, $n \geq 0$; $x_0 \in K_1$, converges to a fixed point of $T$ in $K$.*

The proof follows from the proof of Theorem 3 by putting $L = 1$. An alternative proof of sufficiency was given in Udo-utun [2].

## 4 Application to global existence theory

In this section we illustrate applicability of our main results by formulating the global existence condition which constitutes a nontrivial improvement of the well-known Picard-Lindelof result, see for example [14, 15], the theorem for the initial value problem $x' = f(t, x)$; $x(t_0) = x_0$ where $f : G \to \mathbb{R}$, $G$ is the rectangular region $G$: $a \leq t \leq b$, $\alpha \leq x \leq \beta$ and $t_0 \in [a, b]$.

**Theorem 4** *Let $f$ be a Lipschitz function on the rectangular region $G$: $a \leq t \leq b$, $\alpha \leq x \leq \beta$. If the inequality*

$$\frac{\sup_{t\in[a,b]} |y(t) - f(t, x(t))|}{\sup_{t\in[a,b]} |x(t) - y(t)|} \leq M_1 \tag{12}$$

*is satisfied for some $M_1 \geq 1$ and for all pairs of distinct continuous functions $x(t)$ and $y(t)$ defined on the closed interval $[a, b]$, then the initial value problem $x' = f(t, x)$; $x(t_0) = x_0$, has a unique solution defined on $[a, b]$ for each $t_0 \in [a, b]$.*

*Proof* Using (12) we obtain

$$\begin{aligned}
&\frac{\sup_{t\in[a,b]} |(b - a)y(t) - (b - a)f(t, x(t))|}{(b - a)\sup_{t\in[a,b]} |x(t) - y(t)|} \leq M_1 \\
&\quad\Longrightarrow \quad \frac{\sup_{t\in[a,b]} |(b - a)y(t) - \int_{t_0}^{t} f(s, x(s))\,ds|}{(b - a)\sup_{t\in[a,b]} |x(t) - y(t)|} \leq M_1 \\
&\quad\Longrightarrow \quad \frac{\sup_{t\in[a,b]} |y(t) - \frac{1}{b-a}\int_{t_0}^{t} f(s, x(s))\,ds|}{\sup_{t\in[a,b]} |x(t) - y(t)|} \leq M_1.
\end{aligned} \tag{13}$$

On defining an operator $T : C[a, b] \to C[a, b]$ by $(Tz)(t) = \int_{t_0}^{t} f(s, z(s))\,ds$ (13) yields, in the Banach space $C[a, b]$, $\|(b - a)y - Tx\| \leq (b - a)M_1\|x - y\|$ for all $x, y \in C[a, b]$, $x \neq y$ where $\|z\| = \sup_{t\in[a,b]} |z(t)|$. Equivalently, we have

$$\left\| y - \frac{1}{b - a}Tx \right\| \leq M_1\|x - y\|, \quad x, y \in C[a, b], x \neq y. \tag{14}$$

But $\frac{1}{b-a}y \in C[a,b]$ whenever $y \in C[a,b]$, so applying (14) we obtain, for $y \in C[a,b]$, $x \neq y$, $\|\frac{1}{b-a}y - \frac{1}{b-a}Tx\| \leq M_1\|x-y\|$, which yields

$$\begin{aligned} &\|y - Tx\| \leq (b-a)M_1\|x-y\| \\ &\quad\implies \quad \|y - Tx\| \leq (b+1-a)M_1\|x-y\|, \quad x, y \in [a,b], x \neq y. \end{aligned} \tag{15}$$

This means that $T$ satisfies (10) with $M = (b+1-a)M_1$ so by Theorem 3 the Lipschitzian operator $T$ has a fixed point. Therefore the initial value problem has a unique solution since $f$ satisfies the Lipschitz condition. □

## 5 Conclusion

In conclusion we make the following remarks:

1. Our fixed point results do not guarantee the uniqueness of fixed points, however, an appropriate application of condition (6) of Theorem 1 yields the uniqueness of fixed points.
2. We observe that Theorem 2 can be investigated for a characterization of Lipschitzian mappings $T$ with nonempty fixed point sets $\mathrm{Fix}(T)$.
3. It is obvious that condition (5) can be invaluable in ordinary differential and integral equations for investigations of continuation of solutions, asymptotic properties of solutions, absolute stability, and instability of solutions.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
MYB and ZUS suggested the problem, XAU formulated the results and proofs, while MYB and XAU contributed the counterexample. All authors contributed in the corrections and proofreading. All authors read and approved the final manuscript.

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