Alfuraidan *Fixed Point Theory and Applications* (2015) 2015:46
DOI 10.1186/s13663-015-0296-3

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

The contraction principle for mappings on a modular metric space with a graph

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Abstract
We give a generalization of the Banach contraction principle on a modular metric space endowed with a graph. The notion of a modular metric on an arbitrary set and the corresponding modular spaces, generalizing classical modulars over linear spaces like Orlicz spaces, were recently introduced. This paper can be seen as the modular metric version of Jachymski's fixed point result for mappings on a metric space with a graph.

MSC: Primary 47H09; secondary 46B20; 47H10; 47E10

Keywords: Δ_2-condition; fixed point; modular metric spaces; contraction mapping; connected graph

1 Introduction

Fixed point theorems for monotone single-valued mappings in a metric space endowed with a partial ordering have been widely investigated. These theorems are hybrids of the two most fundamental and useful theorems in fixed point theory: Banach's contraction principle [1], Theorem 2.1, and Tarski's fixed point theorem [2, 3]. The existence of fixed points for single-valued mappings in partially ordered metric spaces was initially considered by Ran and Reurings in [4] who proved the following result.

Theorem 1.1 [4] *Let $(X,\preceq)$ be a partially ordered set such that every pair $x,y \in X$ has an upper and lower bound. Let d be a metric on X such that (X,d) is a complete metric space. Let $f : X \to X$ be a continuous monotone (either order preserving or order reversing) mapping. Suppose that the following conditions hold:*

(1) *There exists a $k \in (0,1)$ with*

$$d\bigl(f(x),f(y)\bigr) \preceq kd(x,y), \quad \text{for all } x \succeq y.$$

(2) *There exists an $x_0 \in X$ with $x_0 \preceq f(x_0)$ or $x_0 \succeq f(x_0)$.*

Then f is a Picard operator (PO), that is, f has a unique fixed point $x^* \in X$ and for each $x \in X$, $\lim_{n\to\infty} f^n(x) = x^*$.

After this, different authors considered the problem of existence of a fixed point for contraction mappings in partially ordered sets; see [5–8] and references cited therein. Nieto *et al.* in [8], proved the following.

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Theorem 1.2 [8] *Let (X,d) be a complete metric space endowed with a partial ordering $\preceq$. Let $f: X \to X$ be an order preserving mapping such that there exists a $k \in [0,1)$ with*

$$d\big(f(x),f(y)\big) \preceq kd(x,y), \quad \text{for all } x \succeq y.$$

Assume that one of the following conditions holds:

(1) *f is continuous and there exists an $x_0 \in X$ with $x_0 \preceq f(x_0)$ or $x_0 \succeq f(x_0)$;*

(2) *$(X,d,\preceq)$ is such that for any nondecreasing $(x_n)_{n\in N}$, if $x_n \to x$, then $x_n \preceq x$ for $n \in N$, and there exists an $x_0 \in X$ with $x_0 \preceq f(x_0)$;*

(3) *$(X,d,\preceq)$ is such that for any nonincreasing $(x_n)_{n\in N}$, if $x_n \to x$, then $x_n \succeq x$ for $n \in N$, and there exists an $x_0 \in X$ with $x_0 \succeq f(x_0)$.*

Then f has a fixed point. Moreover, if $(X,\preceq)$ is such that every pair of elements of X has an upper or a lower bound, then f is a PO.

Generalizing the partial order concept of the fixed point theorems by using graphs was first established by Jachymski and Lukawska [9, 10]. Their works generalized and subsumed the works of [6, 8, 11, 12] to single-valued mapping in metric spaces with a graph. Jachymski [9] obtained the following result.

Theorem 1.3 [9] *Let (X,d) be a complete metric space and let the triplet (X,d,G) have the following property*:

(P) *for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, if $x_n \to x$ as $n \to \infty$ and $(x_n,x_{n+1}) \in E(G)$, then $(x_n,x) \in E(G)$, for all n.*

Let $f: X \to X$ be a G-contraction. Then the following statements hold:

(1) *$F_f \neq \emptyset$ if and only if $X_f \neq \emptyset$;*

(2) *if $X_f \neq \emptyset$ and G is weakly connected, then f is a Picard operator, i.e., $F_f = \{x^*\}$ and sequence $\{f^n(x)\} \to x^*$ as $n \to \infty$, for all $x \in X$;*

(3) *for any $x \in X_f$, $f\,|_{[x]_{\widetilde{G}}}$ is a Picard operator;*

(4) *if $X_f \subseteq E(G)$, then f is a weakly Picard operator, i.e., $F_f \neq \emptyset$ and, for each $x \in X$, we have a sequence $\{f^n(x)\} \to x^*(x) \in F_f$ as $n \to \infty$.*

The aim of this paper is to discuss the existence of fixed points for single Lipschitzian mappings defined on some subsets of modular metric spaces X endowed with a graph G. These modular metric spaces were introduced in [13, 14]. However, the way we approached the concept of modular metric spaces is different. Indeed we look at these spaces as the nonlinear version of the classical modular spaces as introduced by Nakano [15] on vector spaces and modular function spaces introduced by Musielack [16] and Orlicz [17]. In [18] the authors have defined and investigated the fixed point property in the framework of modular metric space and introduced the analog of the Banach contraction principle theorem in modular metric space.

2 Preliminaries

Let X be a nonempty set. Throughout this paper for a function $\omega: (0,\infty) \times X \times X \to (0,\infty)$ we will write

$$\omega_\lambda(x,y) = \omega(\lambda,x,y),$$

for all $\lambda > 0$ and $x, y \in X$.

Definition 2.1 [13, 14] A function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a modular on X if it satisfies the following axioms:

(i) $x = y$ if and only if $\omega_\lambda(x,y) = 0$, for all $\lambda > 0$;

(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$, for all $\lambda > 0$, and $x,y \in X$;

(iii) $\omega_{\lambda+\mu}(x,y) \le \omega_\lambda(x,z) + \omega_\mu(z,y)$, for all $\lambda,\mu > 0$ and $x,y,z \in X$.

If instead of (i), we have only the condition (i′)

$$\omega_\lambda(x,x) = 0, \quad \text{for all } \lambda > 0, x \in X,$$

then ω is said to be a pseudomodular on X. A modular ω on X is said to be regular if the following weaker version of (i) is satisfied:

$$x = y \quad \text{if and only if} \quad \omega_\lambda(x,y) = 0, \quad \text{for some } \lambda > 0.$$

Finally, ω is said to be convex if for $\lambda,\mu > 0$ and $x,y,z \in X$, it satisfies the inequality

$$\omega_{\lambda+\mu}(x,y) \le \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z,y).$$

Note that for a pseudomodular ω on a set X, and any $x,y \in X$, the function $\lambda \to \omega_\lambda(x,y)$ is nonincreasing on $(0,\infty)$. Indeed, if $0 < \mu < \lambda$, then

$$\omega_\lambda(x,y) \le \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y).$$

Definition 2.2 [13, 14] Let ω be a pseudomodular on X. Fix $x_0 \in X$. The two sets

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x,x_0) \to 0 \text{ as } \lambda \to \infty\}$$

and

$$X_\omega^* = X_\omega^*(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x,x_0) < \infty\}$$

are said to be modular spaces (around x_0).

We obviously have $X_\omega \subset X_\omega^*$. In general this inclusion may be proper. It follows from [13, 14] that if ω is a modular on X, then the modular space X_ω can be equipped with a (nontrivial) distance, generated by ω and given by

$$d_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \le \lambda\},$$

for any $x,y \in X_\omega$. If ω is a convex modular on X, according to [13, 14] the two modular spaces coincide, *i.e.* $X_\omega^* = X_\omega$, and this common set can be endowed with the distance d_ω^* given by

$$d_\omega^*(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \le 1\},$$

for any $x,y \in X_\omega$. These distances will be called Luxemburg distances.

First attempts to generalize the classical function spaces of the Lebesgue type L^p were made in the early 1930s by Orlicz and Birnbaum in connection with orthogonal expansions. Their approach consisted in considering spaces of functions with some growth properties different from the power type growth control provided by the L^p-norms. Namely, they considered the function spaces defined as follows:

$$L^\varphi = \left\{f : \mathbb{R} \to \mathbb{R}; \exists\lambda > 0 : \rho(\lambda f) = \int_{\mathbb{R}} \varphi\big(\lambda\big|f(x)\big|\big)\,dx < \infty\right\},$$

where $\varphi : [0,\infty] \to [0,\infty]$ was assumed to be a convex function increasing to infinity, *i.e.* the function which to some extent behaves similarly to power functions $\varphi(t) = t^p$. A modular function spaces L^φ furnishes a wonderful example of a modular metric space. Indeed define the function ω by

$$\omega_\lambda(f,g) = \rho\left(\frac{f-g}{\lambda}\right) = \int_{\mathbb{R}} \varphi\left(\frac{|f(x)-g(x)|}{\lambda}\right) dx,$$

for all $\lambda > 0$, and $f,g \in L^\varphi$. Then ω is a modular metric on L^φ. Moreover, the distance d^*_ω is exactly the distance generated by the Luxemburg norm on L^φ.

For more examples on modular function spaces, the reader may consult the book of Kozlowski [19], and for modular metric spaces [13, 14].

Definition 2.3 Let X_ω be a modular metric space.

(1) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω is said to be ω-convergent to $x \in X_\omega$ if and only if $\omega_1(x_n,x) \to 0$, as $n \to \infty$. x will be called the ω-limit of $\{x_n\}$.

(2) The sequence $\{x_n\}_{n\in N}$ in X_ω is said to be ω-Cauchy if $\omega_1(x_m,x_n) \to 0$, as $m,n \to \infty$.

(3) A subset M of X_ω is said to be ω-closed if the ω-limit of a ω-convergent sequence of M always belong to M.

(4) A subset M of X_ω is said to be ω-complete if any ω-Cauchy sequence in M is a ω-convergent sequence and its ω-limit is in M.

(5) A subset M of X_ω is said to be ω-bounded if we have

$$\delta_\omega(M) = \sup\big\{\omega_1(x,y); x,y \in M\big\} < \infty.$$

In general if $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$, for some $\lambda > 0$, then we may not have $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$, for all $\lambda > 0$. Therefore, as in modular function spaces, we will say that ω satisfies the Δ_2-condition

- If $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$, for some $\lambda > 0$ implies $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$, for all $\lambda > 0$.

In [13] and [14], one will find a discussion of the connection between ω-convergence and metric convergence with respect to the Luxemburg distances. In particular, we have

$$\lim_{n\to\infty} d_\omega(x_n,x) = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \omega_\lambda(x_n,x) = 0, \quad \text{for all } \lambda > 0,$$

for any $\{x_n\} \in X_\omega$ and $x \in X_\omega$. In particular we have ω-convergence and d_ω convergence are equivalent if and only if the modular ω satisfies the Δ_2-condition. Moreover, if the modular ω is convex, then we know that d^*_ω and d_ω are equivalent which implies

$$\lim_{n\to\infty} d^*_\omega(x_n,x) = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \omega_\lambda(x_n,x) = 0, \quad \text{for all } \lambda > 0,$$

for any $\{x_n\} \in X_\omega$ and $x \in X_\omega$ [13, 14].

Definition 2.4 Let (X,ω) be a modular metric space. We will say that ω satisfies the Δ_2-type condition if for any $\alpha > 0$, there exists a $C > 0$ such that

$$\omega_{\lambda/\alpha}(x,y) \le C\omega_\lambda(x,y),$$

for any $\lambda > 0$, $x,y \in X_\omega$, with $x \neq y$.

Note that if ω satisfies the Δ_2-type condition, then ω satisfies the Δ_2-condition. The above definition will allow us to introduce the growth function in the modular metric spaces as was done in the linear case.

Definition 2.5 Let (X,ω) be a modular metric space. Define the growth function Ω by

$$\Omega(\alpha) = \sup\left\{ \frac{\omega_{\lambda/\alpha}(x,y)}{\omega_\lambda(x,y)}; \lambda > 0, x,y \in X_\omega, x \neq y \right\},$$

for any $\alpha > 0$.

The following properties were proved in [20].

Lemma 2.1 (Lemma 2.1, [20]) *Let (X,ω) be a modular metric space. Assume that ω is a convex regular modular metric which satisfies the Δ_2-type condition. Then*

1. $\Omega(\alpha) < \infty$, *for any* $\alpha > 0$,
2. Ω *is a strictly increasing function, and* $\Omega(1) = 1$,
3. $\Omega(\alpha\beta) \le \Omega(\alpha)\Omega(\beta)$, *for any* $\alpha,\beta \in (0,\infty)$,
4. $\Omega^{-1}(\alpha)\Omega^{-1}(\beta) \le \Omega^{-1}(\alpha\beta)$, *where* Ω^{-1} *is the function inverse of* Ω,
5. *for any* $x,y \in X_\omega$, $x \neq y$, *we have*

$$d^*_\omega(x,y) \le \frac{1}{\Omega^{-1}(1/\omega_1(x,y))}.$$

The following technical lemma will be useful later on in this work.

Lemma 2.2 [20] *Let (X,ω) be a modular metric space. Assume that ω is a convex regular modular metric which satisfies the Δ_2-type condition. Let $\{x_n\}$ be a sequence in X_ω such that*

$$\omega_1(x_{n+1},x_n) \le K\alpha^n, \quad n = 1,\ldots, \tag{1}$$

where K is an arbitrary nonzero constant and $\alpha \in (0,1)$. Then $\{x_n\}$ is Cauchy for both ω and d^*_ω.

Note that this lemma is crucial since the main assumption (1) on $\{x_n\}$ will not be enough to imply that $\{x_n\}$ is ω-Cauchy since ω fails the triangle inequality.

Let us finish this section with the needed graph theory terminology which will be used throughout.

Let (X,ω) be a modular metric space and M be a nonempty subset of X_ω. Let Δ denote the diagonal of the cartesian product $M \times M$. Consider a directed graph G_ω such that the set $V(G_\omega)$ of its vertices coincides with M, and the set $E(G_\omega)$ of its edges contains all loops, *i.e.*, $E(G_\omega) \supseteq \Delta$. We assume G_ω has no parallel edges (arcs), so we can identify G_ω with the pair $(V(G_\omega), E(G_\omega))$. Our graph theory notations and terminology are standard and can be found in all graph theory books, like [21] and [22]. Moreover, we may treat G_ω as a weighted graph (see [22], p.309) by assigning to each edge the distance between its vertices.

By G^{-1} we denote the conversion of a graph G, *i.e.*, the graph obtained from G by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(y,x) \mid (x,y) \in E(G)\}.$$

A digraph G is called an oriented graph if whenever $(u,v) \in E(G)$, then $(v,u) \notin E(G)$. The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

We call (V', E') a subgraph of G if $V' \subseteq V(G)$, $E' \subseteq E(G)$, and for any edge $(x,y) \in E'$, $x, y \in V'$.

If x and y are vertices in a graph G, then a (directed) path in G from x to y of length N is a sequence $(x_i)_{i=1}^{i=N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, \ldots, N$. A graph G is connected if there is a directed path between any two vertices. G is weakly connected if $\widetilde{G}$ is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation $\mathcal{R}$ defined on $V(G)$ by the rule:

$y\mathcal{R}z$ if there is a (directed) path in G from y to z.

Clearly G_x is connected.

Definition 2.6 Let (X,ω) be a modular metric space and M be a nonempty subset of X_ω. A mapping $T : M \to M$ is called

(i) G_ω-contraction if T preserves edges of G_ω, *i.e.*,

$$\forall x, y \in M \quad \big((x,y) \in E(G_\omega) \Rightarrow \big(T(x), T(y)\big) \in E(G_\omega)\big),$$

and if there exists a constant $\alpha \in [0,1)$ such that

$$\omega_1\big(T(x), T(y)\big) \le \alpha\,\omega_1(x,y) \quad \text{for any } (x,y) \in E(G_\omega).$$

(ii) (ε,α)-G_ω-uniformly locally contraction if T preserves edges of G_ω and there exists a constant $\alpha \in [0,1)$ such that for any $(x,y) \in E(G_\omega)$

$$\omega_1\big(T(x), T(y)\big) \leq \alpha\omega_1(x,y) \quad \text{whenever } \omega_1(x,y) < \varepsilon.$$

Definition 2.7 A point $x \in M$ is called a fixed point of T whenever $x = T(x)$. The set of fixed points of T will be denoted by $\mathrm{Fix}(T)$.

3 Fixed points of G_ω-contractions

Throughout this section we assume that (X,ω) is a modular metric space, M be a nonempty subset of X_ω and G_ω is a directed graph such that $V(G_\omega) = M$ and $E(G_\omega) \supseteq \Delta$.

Our first result can be seen as an extension of Jachymski's fixed point theorem [9] to modular metric spaces. As Jachymski [9] did, we introduce the following property.

We say that the triple $(M, d^*_\omega, G_\omega)$ has Property (P) if

(P) For any sequence $\{x_n\}_{n\in\mathbb{N}}$ in M, if $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G_\omega)$, then $(x_n, x) \in E(G_\omega)$, for all n.

Note that property (P) is precisely the Nieto *et al.* [8] hypothesis relaxing continuity assumption as in Theorem 1.2((2) and (3)) rephrased in terms of edges.

Theorem 3.1 *Let (X,ω) be a modular metric space with a graph G_ω. Suppose that ω is a convex regular modular metric which satisfies the Δ_2-type condition. Assume that $M = V(G_\omega)$ is a nonempty ω-bounded, ω-complete subset of X_ω and the triple $(M, d^*_\omega, G_\omega)$ has property* (P). *Let $T : M \to M$ be G_ω-contraction map and $M_T := \{x \in M; (x, Tx) \in E(G_\omega)\}$. If $(x_0, T(x_0)) \in E(G_\omega)$, then the following statements hold*:

(1) *For any $x \in M_T$, $T|_{[x]_{\widetilde{G}_\omega}}$ has a fixed point.*
(2) *If G_ω is weakly connected, then T has a fixed point in M.*
(3) *If $M' := \bigcup\{[x]_{\widetilde{G}_\omega} : x \in M_T\}$, then $T|_{M'}$ has a fixed point in M.*

Proof (1) As $(x_0, T(x_0)) \in E(G_\omega)$, then $x_0 \in M_T$. Since T is a G_ω-contraction, there exists a constant $\alpha \in [0,1)$ such that $(T(x_0), T(T(x_0))) \in E(G_\omega)$ and

$$\omega_1\big(T(x_0), T\big(T(x_0)\big)\big) \leq \alpha\omega_1\big(x_0, T(x_0)\big).$$

By induction, we construct a sequence $\{x_n\}$ such that $x_{n+1} := T(x_n)$, $(x_n, x_{n+1}) \in E(G_\omega)$, and

$$\omega_1(x_{n+1}, x_n) \leq \alpha\omega_1(x_n, x_{n-1}) \leq \alpha^n\omega_1(x_0, x_1),$$

for any $n \geq 1$. Since M is ω-bounded, we have

$$\omega_1(x_{n+1}, x_n) \leq \delta_\omega(M)\alpha^n$$

for any $n \geq 1$. The technical Lemma 2.2 implies that $\{x_n\}$ is ω-Cauchy. Since M is ω-complete, therefore $\{x_n\}$ ω-converges to some point $x \in M$. By property (P), $(x_n, x) \in E(G_\omega)$ for all n, and hence

$$\omega_1\big(x_{n+1}, T(x)\big) \leq \alpha\omega_1(x_n, x).$$

We conclude that $\lim_{n\to\infty}\omega_1(x_{n+1},T(x))=0$. Using the properties of ω, we have

$$\omega_2\big(x,T(x)\big)\leq\omega_1(x,x_{n+1})+\omega_1\big(x_{n+1},T(x)\big),$$

for all $n\geq 1$. This implies $\omega_2(x,T(x))=0$. Therefore, $x=T(x)$, *i.e.*, x is a fixed point of T. As $(x_0,x)\in E(G_\omega)$, we have $x\in[x_0]_{\widetilde{G}_\omega}$.

(2) Since $M_T\neq\emptyset$, there exists an $x_0\in M_T$, and since G_ω is weakly connected, then $[x_0]_{\widetilde{G}_\omega}=M$ and by (1), mapping T has a fixed point.

(3) It follows easily from (1) and (2). □

Edelstein [23] has extended the classical fixed point theorem for contractions to the case when X is a complete ε-chainable metric space and the mapping $T:X\to X$ is an (ε,k)-uniformly locally contraction. Here we investigate Edelstein's result in modular metric spaces endowed with a graph. First let us introduce the ε-chainable concept in modular metric spaces with a graph. Our definition is slightly different from the one used in the classical metric spaces since the modulars fail in general the triangle inequality (see also [24]).

Definition 3.1 Let (X,ω) be a modular metric space, $M=V(G_\omega)$ be a nonempty subset of X_ω. M is said to be finitely ε-chainable (where $\varepsilon>0$ is fixed) if and only if there exists an $N\geq 1$ such that for any $a,b\in M$ with $(a,b)\in E(G_\omega)$ there is an N,ε-chain from a to b (that is, a finite set of vertices $x_0,x_1,\ldots,x_N\in V(G_\omega)=M$ such that $x_0=a$, $x_N=b$, $(x_i,x_{i+1})\in E(G_\omega)$ and $\omega_1(x_i,x_{i+1})<\varepsilon$, for all $i=0,1,2,\ldots,N-1$).

We have the following result.

Theorem 3.2 *Let (X,ω) be a modular metric space. Suppose that ω is a convex regular modular metric which satisfies the Δ_2-type condition. Assume that $M=V(G_\omega)$ is a nonempty ω-complete and ω-bounded subset of X_ω which is finitely ε-chainable, for some fixed $\varepsilon>0$. Suppose that the triple (M,d^*_ω,G_ω) has property* (P). *Let $T:M\to M$ be (ε,α)-G_ω-uniformly locally contraction map. Then T has a fixed point in the vertex set of the graph M.*

Proof Since M is finitely ε-chainable, there exists an $N\geq 1$ such that for any $a,b\in M$ with $(a,b)\in E(G_\omega)$ there is a finite set of vertices $x_0,x_1,\ldots,x_N\in M$ such that $x_0=a$, $x_N=b$, $(x_i,x_{i+1})\in E(G_\omega)$, and $\omega_1(x_i,x_{i+1})<\varepsilon$, for all $i=0,1,2,\ldots,N-1$. For any $x,y\in M$ define

$$\omega^*(x,y)=\inf\left\{\sum_{i=0}^{i=N-1}\omega_1(x_i,x_{i+1})\right\},$$

where the infimum is taken over all N,ε-chains $x_0,x_1,\ldots,x_N$ from x to y. Since M is finitely ε-chainable it follows that $\omega^*(x,y)<\infty$, for any $x,y\in M$. Using the basic properties of ω, we get

$$\omega_N(x,y)\leq\omega^*(x,y),$$

for any $x,y\in M$ with $(x,y)\in E(G_\omega)$. Moreover, if $\omega_1(x,y)<\varepsilon$, then we have $\omega^*(x,y)\leq\omega_1(x,y)$, for any $x,y\in M$ with $(x,y)\in E(G_\omega)$. Fix $x\in M$. Set $z_0=x$ and $z_1=T(z_0)$ with

$(z_0, z_1) \in E(G_\omega)$. Let $x_0, \dots, x_N$ be an N, ε-chain from z_0 to z_1. Such an N, ε-chain exists since M is finitely ε-chainable. Since T is (ε, α)-G_ω-uniformly locally contraction map, there exists a constant $\alpha \in [0, 1)$ such that

$$\omega_1\big(T(x_i), T(x_{i+1})\big) \leq \alpha\omega_1(x_i, x_{i+1}) < \alpha\varepsilon < \varepsilon,$$

for every i. Clearly this implies that $T(x_0), T(x_1), \dots, T(x_N)$ is N, ε-chain from $T(z_0)$ to $T(z_1)$ and

$$\omega^*(z_1, z_2) \leq \alpha\omega^*(z_0, z_1),$$

where $z_2 = T(z_1)$. By induction, we construct the sequence $\{z_n\} \in M$ with $(z_n, z_{n+1}) \in E(G_\omega)$ such that

$$\omega^*(z_n, z_{n+1}) \leq \alpha\omega^*(z_{n-1}, z_n),$$

for any $n \geq 1$, where $z_{n+1} = T(z_n)$. Obviously we have $\omega^*(z_n, z_{n+1}) \leq \alpha^n\omega^*(z_0, z_1)$, for any $n \geq 1$. Since ω satisfies the Δ_2-type condition, there exists $C > 0$ such that

$$\omega_1(z_n, z_{n+1}) \leq C\omega_N(z_n, z_{n+1}) \leq C\omega^*(z_n, z_{n+1}) \leq C\alpha^n\omega^*(z_0, z_1),$$

for any $n \geq 1$. Lemma 2.2 implies that $\{z_n\}$ is ω-Cauchy. Since M is ω-complete, then $\{z_n\}$ ω-converges to some $z \in M$. We claim that z is a fixed point of T. By property (P), $(z_n, z) \in E(G_\omega)$ for any $n \geq 1$. Using the ideas developed above, we have

$$\omega^*\big(z_{n+1}, T(z)\big) \leq \alpha\omega^*(z_n, z)$$

for any $n \geq 1$. Since

$$\omega_{N+1}\big(T(z), z\big) \leq \omega_1(z_{n+1}, z) + \omega_N\big(z_{n+1}, T(z)\big)$$

and

$$\omega_N\big(z_{n+1}, T(z)\big) \leq \omega^*(z_{n+1}, z) \leq \alpha\omega^*(z_n, z),$$

for any $n \geq 1$, we conclude that $\omega_{N+1}(T(z), z) = 0$. Indeed there exists n_0 such that for any $n \geq n_0$ we have $\omega_1(z_n, z) < \varepsilon$. By definition of ω^* we have $\omega^*(z_n, z) < \omega_1(z_n, z)$, for $n \geq n_0$. Hence $\lim_{n\to\infty} \omega^*(z_n, z) = 0$. Therefore $\omega_{N+1}(T(z), z) = 0$ holds. Since ω is regular we get $z = T(z)$, *i.e.*, z is a fixed point of T as claimed. □

Competing interests

The author declares that he has no competing interests.

Acknowledgements

The author acknowledges King Fahd University of Petroleum and Minerals for supporting this research.

Received: 3 December 2014 Accepted: 6 March 2015 Published online: 01 April 2015

References

1. Kirk, WA, Goebel, K: Topics in Metric Fixed Point Theory. Cambridge University Press, Cambridge (1990)
2. Granas, A, Dugundji, J: Fixed Point Theory. Springer, New York (2003)
3. Tarski, A: A lattice theoretical fixed point and its application. Pac. J. Math. **5**, 285-309 (1955)
4. Ran, ACM, Reurings, MCB: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**, 1435-1443 (2004)
5. Beg, I, Butt, AR: Fixed point for set valued mappings satisfying an implicit relation in partially ordered metric spaces. Nonlinear Anal. **71**, 3699-3704 (2009)
6. Drici, Z, McRae, FA, Devi, JV: Fixed point theorems in partially ordered metric space for operators with PPF dependence. Nonlinear Anal. **67**, 641-647 (2007)
7. Harjani, J, Sadarangani, K: Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations. Nonlinear Anal. **72**, 1188-1197 (2010). doi:10.1016/j.na.2009.08.003
8. Nieto, JJ, Pouso, RL, Rodriguez-Lopez, R: Fixed point theorems in ordered abstract spaces. Proc. Am. Math. Soc. **135**, 2505-2517 (2007)
9. Jachymski, J: The contraction principle for mappings on a metric space with a graph. Proc. Am. Math. Soc. **1**(136), 1359-1373 (2008)
10. Lukawska, GG, Jachymski, J: IFS on a metric space with a graph structure and extension of the Kelisky-Rivlin theorem. J. Math. Anal. Appl. **356**, 453-463 (2009)
11. O'Regan, D, Petrusel, A: Fixed point theorems for generalized contraction in ordered metric spaces. J. Math. Anal. Appl. **341**, 1241-1252 (2008)
12. Petrusel, A, Rus, IA: Fixed point theorems in ordered *L*-spaces. Proc. Am. Math. Soc. **134**, 411-418 (2005)
13. Chistyakov, VV: Modular metric spaces, I. Basic concepts. Nonlinear Anal. **72**(1), 1-14 (2010)
14. Chistyakov, VV: Modular metric spaces, II. Application to superposition operators. Nonlinear Anal. **72**(1), 15-30 (2010)
15. Nakano, H: Modulared Semi-Ordered Linear Spaces, i+288 pp. Maruzen, Tokyo (1950)
16. Musielak, J: Orlicz Spaces and Modular Spaces. Lecture Notes in Math., vol. 1034. Springer, Berlin (1983)
17. Orlicz, W: Collected Papers. Part I, II. PWN, Warsaw (1988)
18. Abdou, AN, Khamsi, MA: Fixed points results of pointwise contractions in modular metric spaces. Fixed Point Theory Appl. **2013**, 163 (2013)
19. Kozlowski, WM: Modular Function Spaces. Series of Monographs and Textbooks in Pure and Applied Mathematics, vol. 122. Dekker, New York (1988)
20. Abdou, AN, Khamsi, MA: Fixed points of multivalued contraction mappings in modular metric spaces. Fixed Point Theory Appl. **2014**, 249 (2014). doi:10.1186/1687-1812-2014-249
21. Chartrand, G, Lesniak, L, Zhang, P: Graphs & Digraphs. CRC Press, New York (2011)
22. Johnsonbaugh, R: Discrete Mathematics. Prentice Hall, New York (1997)
23. Edelstein, M: An extension of Banach's contraction principle. Proc. Am. Math. Soc. **12**, 7-10 (1961)
24. Ben-El-Mechaiekh, H: The Ran-Reurings fixed point theorem without partial order: a simple proof. J. Fixed Point Theory Appl. (2015). doi:10.1007/s11784-015-0218-3

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