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Some fixed point theorems for (α, ψ)-rational type contractive mappings

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Abstract

In this paper, we introduce the concept of (α, ψ)-rational type contractive mappings and provide sufficient conditions for the existence and uniqueness of a fixed point for such class of generalized nonlinear contractive mappings in the setting of generalized metric spaces. We also deduce several interesting corollaries.

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Keywords: fixed point; α-admissible; contraction mappings; metric space

1 Introduction and preliminaries

Fixed point theory has gained very large impetus due to its wide range of applications in various fields such as engineering, economics, computer science, and many others. It is well known that the contractive conditions are very indispensable in the study of fixed point theory, and Banach's fixed point theorem [1] for contraction mappings is one of the pivotal result in analysis. This theorem has been extended and generalized by various authors (see, *e.g.*, [2–28]) in various abstract spaces, one of which is generalized metric space.

As pointed out in [3], the topology of a generalized metric space has some disadvantages:

(T1) A generalized metric does not need to be continuous.
(T2) A convergent sequence in generalized metric spaces does not need to be Cauchy.
(T3) A generalized metric space does not need to be Hausdorff, and hence the uniqueness of the limits cannot be guaranteed.

In this paper, we introduce the concept of (α, ψ)-rational type contractive mappings and provide sufficient conditions for the existence and uniqueness of fixed points for such class of generalized nonlinear contractive mappings in the framework of generalized metric spaces by caring the problems (T1)-(T3) mentioned above. We also deduce several interesting corollaries. The proved results generalize and extend various well-known results in the literature. The techniques used in this paper have been studied and improved by various authors (see [3–9, 16] and references cited therein).

To start with, we give some notations and introduce some definitions which will be used in the sequel.

Definition 1.1 [2] Let X be a nonempty set and $d : X \times X \to [0, \infty)$ satisfy the following conditions, for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y:

(GMS1) $d(x,y) = 0$ if and only if $x = y$,
(GMS2) $d(x,y) = d(y,x)$,
(GMS3) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$.

Then the map d is called a generalized metric and abbreviated as GM. Here, the pair (X,d) is called a generalized metric space and abbreviated as GMS.

In the above definition, if d satisfies only (GMS1) and (GMS2), then it is called a semimetric (see, *e.g.*, [18]).

A sequence $\{x_n\}$ in a GMS (X,d) is GMS convergent to a limit x if and only if $d(x_n,x) \to 0$ as $n \to \infty$.

A sequence $\{x_n\}$ in a GMS (X,d) is GMS Cauchy if and only if for every $\epsilon > 0$ there exists a positive integer $N(\epsilon)$ such that $d(x_n,x_m) < \epsilon$, for all $n > m > N(\epsilon)$.

A GMS (X,d) is called complete if every GMS Cauchy sequence in X is GMS convergent.

A mapping $T : (X,d) \to (X,d)$ is continuous if for any sequence $\{x_n\}$ in X such that $d(x_n,x) \to 0$ as $n \to \infty$, we have $d(Tx_n, Tx) \to 0$ as $n \to \infty$.

The following assumption was suggested by Wilson [18] to replace the triangle inequality with the weakened condition.

(W) For each pair of (distinct) points u, v, there is a number $r_{u,v} > 0$ such that for every $z \in X$, $r_{u,v} < d(u,z) + d(z,v)$.

Proposition 1.1 [20] *In a semimetric space, the assumption* (W) *is equivalent to the assertion that the limits are unique.*

Proposition 1.2 [20] *Suppose that* $\{x_n\}$ *is a Cauchy sequence in a GMS* (X,d) *with* $\lim_{n\to\infty} d(x_n,u) = 0$, *where* $u \in X$. *Then* $\lim_{n\to\infty} d(x_n,z) = d(u,z)$, *for all* $z \in X$. *In particular, the sequence* $\{x_n\}$ *does not converge to* z *if* $z \neq u$.

Definition 1.2 Let X be a nonempty set, $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be two mappings. We say that T is an α-admissible mapping if $\alpha(x,y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$, for all $x, y \in X$.

Definition 1.3 Let (X,d) be a GMS and $\alpha : X \times X \to [0,\infty)$. X is called α-regular GMS if, for a sequence $\{x_n\}$ in X such that $x_n \to x$ and $\alpha(x_n,x_{n+1}) \geq 1$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x) \geq 1$ for all $k \in \mathbb{N}$.

Throughout the paper, $F(T)$ denotes the set of fixed points of the mapping T.

2 Main results

The contraction mappings considered in this paper are constructed via auxiliary functions defined below. Let Ψ be a family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following properties:

(i) ψ is upper semi-continuous, strictly increasing;
(ii) $\{\psi^n(t)\}_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$, for all $t > 0$;
(iii) $\psi(t) < t$, for every $t > 0$.

Definition 2.1 Let (X,d) be a GMS and $\alpha : X \times X \to [0,\infty)$. A self mapping $T : X \to X$ is said to be (α,ψ)-rational type-I contractive mapping if there exists a function $\psi \in \Psi$, such that for all $x, y \in X$ the following condition holds:

$$\alpha(x,y)d(Tx,Ty) \leq \psi\big(M(x,y)\big), \tag{2.1}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\right\}.$$

Next, we state and prove an existence and uniqueness theorem for fixed point of (α,ψ)-rational type-I contractive mappings.

Theorem 2.1 *Let (X,d) be a complete GMS, $T : X \to X$ be a self mapping and $\alpha : X \times X \to [0,\infty)$ a given function. Suppose that the following conditions are satisfied*:

(i) *T is an α-admissible mapping*;
(ii) *T is an (α,ψ)-rational type-I contractive mapping*;
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, $\alpha(x_0, T^2x_0) \geq 1$*;
(iv) *either T is continuous, or X is α-regular.*

Then T has a fixed point $x^* \in X$ and $\{T^nx_0\}$ converges to x^*. Further, if for all $x, y \in F(T)$, we have $\alpha(x,y) \geq 1$ then T has a unique fixed point in X.

Proof Let $x_0 \in X$ satisfies $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$. We construct the sequence $\{x_n\}$ in X as $x_n = T^nx_0 = Tx_{n-1}$, for $n \in \mathbb{N}$. It is obvious that if $x_{n_0} = x_{n_0+1}$, for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of T. Consequently, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Since T is α-admissible, $\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 \Longrightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1 \Longrightarrow$ and thus, $\alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1 \ldots$, and hence by induction, we get $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$.

By similar arguments, since $\alpha(x_0, T^2x_0) \geq 1$, we have $\alpha(x_0, x_2) = \alpha(x_0, T^2x_0) \geq 1$, $\alpha(Tx_0, Tx_2) = \alpha(x_1, x_3) \geq 1$. By induction, we get $\alpha(x_n, x_{n+2}) \geq 1$ for all $n \geq 0$. Consider (2.1) with $x = x_n$ and $y = x_{n+1}$. Clearly, we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\ &\leq \alpha(x_n, x_{n+1})d(Tx_n, Tx_{n+1}) \\ &\leq \psi\big(M(x_n, x_{n+1})\big), \end{aligned}$$

where

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\Bigg\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \\ &\qquad \frac{d(x_n, Tx_n)d(x_{n+1}, Tx_{n+1})}{1+d(x_n, x_{n+1})}, \frac{d(x_n, Tx_n)d(x_{n+1}, Tx_{n+1})}{1+d(Tx_n, Tx_{n+1})}\Bigg\} \\ &= \max\Bigg\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\ &\qquad \frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{1+d(x_n, x_{n+1})}, \frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{1+d(x_{n+1}, x_{n+2})}\Bigg\} \\ &= \max\big\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\big\}, \end{aligned} \tag{2.2}$$

since $\frac{d(x_n,x_{n+1})d(x_{n+1},x_{n+2})}{1+d(x_n,x_{n+1})} \leq d(x_{n+1},x_{n+2})$ and $\frac{d(x_n,x_{n+1})d(x_{n+1},x_{n+2})}{1+d(x_{n+1},x_{n+2})} \leq d(x_n,x_{n+1})$. If for some n, we have $M(x_n,x_{n+1}) = d(x_{n+1},x_{n+2})$, then

$$\begin{aligned} d(x_{n+1},x_{n+2}) &\leq \psi\big(M(x_n,x_{n+1})\big) \\ &= \psi\big(d(x_{n+1},x_{n+2})\big) \\ &< d(x_{n+1},x_{n+2}), \end{aligned} \tag{2.3}$$

which is impossible. Hence, $M(x_n,x_{n+1}) = d(x_n,x_{n+1})$, for all $n \in \mathbb{N}$,

$$\begin{aligned} d(x_{n+1},x_{n+2}) &\leq \psi\big(M(x_n,x_{n+1})\big) \\ &= \psi\big(d(x_n,x_{n+1})\big). \end{aligned} \tag{2.4}$$

From the property (iii) of ψ, we conclude that

$$d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1}), \tag{2.5}$$

for every $n \in \mathbb{N}$. Combining (2.4) and (2.5), we deduce $d(x_{n+1},x_{n+2}) \leq \psi^n(d(x_0,x_1))$, for all $n \in \mathbb{N}$. Using the property (ii) of ψ, it is clear that

$$\lim_{n\to\infty} d(x_{n+1},x_{n+2}) = 0. \tag{2.6}$$

Consider now (2.1) with $x = x_{n-1}$ and $y = x_{n+1}$. We have

$$\begin{aligned} d(x_n,x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}) \\ &\leq \alpha(x_{n-1},x_{n+1})d(Tx_{n-1}, Tx_{n+1}) \\ &\leq \psi\big(M(x_{n-1},x_{n+1})\big), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} M(x_{n-1},x_{n+1}) &= \max\bigg\{d(x_{n-1},x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1}), \\ &\qquad \frac{d(x_{n-1}, Tx_{n-1})d(x_{n+1}, Tx_{n+1})}{1 + d(x_{n-1},x_{n+1})}, \frac{d(x_{n-1}, Tx_{n-1})d(x_{n+1}, Tx_{n+1})}{1 + d(Tx_{n-1}, Tx_{n+1})}\bigg\} \\ &= \max\bigg\{d(x_{n-1},x_{n+1}), d(x_{n-1},x_n), d(x_{n+1},x_{n+2}), \\ &\qquad \frac{d(x_{n-1},x_n)d(x_{n+1},x_{n+2})}{1 + d(x_{n-1},x_{n+1})}, \frac{d(x_{n-1},x_n)d(x_{n+1},x_{n+2})}{1 + d(x_n,x_{n+2})}\bigg\}. \end{aligned} \tag{2.8}$$

From (2.5) we have $d(x_{n+1},x_{n+2}) < d(x_{n-1},x_n)$. Define $a_n = d(x_n,x_{n+2})$ and $b_n = d(x_n,x_{n+1})$. Then

$$M(x_{n-1},x_{n+1}) = \max\bigg\{a_{n-1}, b_{n-1}, \frac{b_{n-1}b_{n+1}}{1 + a_{n-1}}, \frac{b_{n-1}b_{n+1}}{1 + a_n}\bigg\}.$$

If $M(x_{n-1},x_{n+1}) = b_{n-1}$, or $\frac{b_{n-1}b_{n+1}}{1+a_{n-1}}$ or $\frac{b_{n-1}b_{n+1}}{1+a_n}$ then taking lim sup as $n \to \infty$ in (2.7) and using (2.6) and upper semi-continuity of ψ we get

$$0 \leq \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} \psi\big(M(x_{n-1},x_{n+1})\big) = \psi\Big(\limsup_{n\to\infty} M(x_{n-1},x_{n+1})\Big) = \psi(0) = 0,$$

and hence,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} d(x_n,x_{n+2}) = 0.$$

If $M(x_{n-1},x_{n+1}) = a_{n-1}$, then (2.7) implies

$$a_n \leq \psi(a_{n-1}) < a_{n-1},$$

due to the property (iii) of ψ. In other words, the sequence $\{a_n\}$ is positive monotone decreasing, and hence, it converges to some $t \geq 0$. Assume that $t > 0$. Now, by (2.7), we get

$$t = \limsup_{n\to\infty} a_n = \limsup_{n\to\infty} \psi(a_{n-1}) = \psi\Big(\limsup_{n\to\infty} a_{n-1}\Big) = \psi(t) < t,$$

which is a contradiction. Therefore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} d(x_n,x_{n+2}) = 0. \tag{2.9}$$

Now, we shall prove that $x_n \neq x_m$, for all $n \neq m$. Assume on the contrary that $x_n = x_m$, for some $m,n \in \mathbb{N}$ with $n \neq m$. Since $d(x_p,x_{p+1}) > 0$, for each $p \in \mathbb{N}$, without loss of generality, we may assume that $m > n+1$. Substitute again $x = x_n = x_m$ and $y = x_{n+1} = x_{m+1}$ in (2.1), which yields

$$\begin{aligned} d(x_n,x_{n+1}) &= d(x_n,Tx_n) = d(x_m,Tx_m) = d(Tx_{m-1},Tx_m) \\ &\leq \alpha(x_{m-1},x_m)d(Tx_{m-1},Tx_m) \leq \psi\big(M(x_{m-1},x_m)\big), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} M(x_{m-1},x_m) &= \max\bigg\{d(x_{m-1},x_m), d(x_{m-1},Tx_{m-1}), d(x_m,Tx_m), \\ &\qquad \frac{d(x_{m-1},Tx_{m-1})d(x_m,Tx_m)}{1+d(x_{m-1},x_m)}, \frac{d(x_{m-1},Tx_{m-1})d(x_m,Tx_m)}{1+d(Tx_{m-1},Tx_m)}\bigg\} \\ &= \max\bigg\{d(x_{m-1},x_m), d(x_{m-1},x_m), d(x_m,x_{m+1}), \\ &\qquad \frac{d(x_{m-1},x_m)d(x_m,x_{m+1})}{1+d(x_{m-1},x_m)}, \frac{d(x_{m-1},x_m)d(x_m,x_{m+1})}{1+d(x_m,x_{m+1})}\bigg\} \\ &= \max\big\{d(x_{m-1},x_m), d(x_m,x_{m+1})\big\}. \end{aligned} \tag{2.11}$$

If $M(x_{m-1},x_m) = d(x_{m-1},x_m)$, then (2.10) implies

$$d(x_n,x_{n+1}) \leq \psi\big(d(x_{m-1},x_m)\big) \leq \psi^{m-n}\big(d(x_n,x_{n+1})\big). \tag{2.12}$$

If on the other hand $M(x_{m-1},x_m) = d(x_m,x_{m+1})$, then from (2.10) we have

$$d(x_n,x_{n+1}) \leq \psi\big(d(x_m,x_{m+1})\big) \leq \psi^{m-n+1}\big(d(x_n,x_{n+1})\big). \tag{2.13}$$

Using the property (iii) of ψ, the two inequalities (2.12) and (2.13) imply

$$d(x_n,x_{n+1}) < d(x_n,x_{n+1}),$$

which is impossible.

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence, that is, $\lim_{n\to\infty} d(x_n,x_{n+k}) = 0$, for all $k \in \mathbb{N}$. We have already proved the cases for $k = 1$ and $k = 2$ in (2.6) and (2.9), respectively. Take arbitrary $k \geq 3$. We discuss two cases.

Case 1. Suppose that $k = 2m+1$, where $m \geq 1$. Using the quadrilateral inequality (GMS3), we have

$$\begin{aligned} d(x_n,x_{n+k}) &= d(x_n,x_{n+2m+1}) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{n+2m},x_{n+2m+1}) \\ &\leq \sum_{p=n}^{n+2m} \psi^p\big(d(x_0,x_1)\big) \\ &\leq \sum_{p=n}^{+\infty} \psi^p\big(d(x_0,x_1)\big) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.14}$$

Case 2. Suppose that $k = 2m$, where $m \geq 2$. Again, by applying the quadrilateral inequality, we have

$$\begin{aligned} d(x_n,x_{n+k}) &= d(x_n,x_{n+2m}) \leq d(x_n,x_{n+2}) + d(x_{n+2},x_{n+3}) + \cdots + d(x_{n+2m-1},x_{n+2m}) \\ &\leq d(x_n,x_{n+2}) + \sum_{p=n+2}^{n+2m-1} \psi^p\big(d(x_0,x_1)\big) \\ &\leq d(x_n,x_{n+2}) + \sum_{p=n}^{+\infty} \psi^p\big(d(x_0,x_1)\big) \to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{2.15}$$

since $\lim_{n\to\infty} = 0$ because of (2.9). In both of the above cases, we have $\lim_{n\to\infty} d(x_n,x_{n+k}) = 0$, for all $k \geq 3$. Hence we conclude that $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d\big(x_n,x^*\big) = 0. \tag{2.16}$$

We will show next that the limit x^* of the sequence $\{x_n\}$ is a fixed point of T. First, we suppose that T is continuous. Then from (2.16) we have

$$\lim_{n\to\infty} d\big(Tx_n,Tx^*\big) = \lim_{n\to\infty} d\big(x_{n+1},Tx^*\big) = 0.$$

Due to Proposition 1.2, we conclude that

$$x^* = Tx^*,$$

that is, x^* is a fixed point of T.

Now, we suppose that X is α-regular. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k-1}, x^*) \geq 1$ for all $k \in \mathbb{N}$. Now, from inequality (2.1) with $x = x_{n_k}$ and $y = x^*$, we obtain

$$\begin{aligned} d(x_{n_k+1}, Tx^*) &= d(Tx_{n_k}, Tx^*) \\ &\leq \alpha(x_{n_k}, x^*) d(Tx_{n_k}, Tx^*) \\ &\leq \psi(M(x_{n_k}, x^*)), \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} M(x_{n_k}, x^*) &= \max\biggl\{ d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), \\ &\qquad \frac{d(x_{n_k}, Tx_{n_k}) d(x^*, Tx^*)}{1 + d(x_{n_k}, x^*)}, \frac{d(x_{n_k}, Tx_{n_k}) d(x^*, Tx^*)}{1 + d(Tx_{n_k}, Tx^*)} \biggr\} \\ &= \max\biggl\{ d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), d(x^*, Tx^*), \\ &\qquad \frac{d(x_{n_k}, x_{n_k+1}) d(x^*, Tx^*)}{1 + d(x_{n_k}, x^*)}, \frac{d(x_{n_k}, x_{n_k+1}) d(x^*, Tx^*)}{1 + d(x_{n_k+1}, Tx^*)} \biggr\}. \end{aligned} \tag{2.18}$$

Letting $k \to \infty$ in (2.18), we obtain $M(x_{n_k}, x^*) = d(x^*, Tx^*)$. Therefore, upon taking the limit as $k \to \infty$, in inequality (2.17), we have $d(x^*, Tx^*) \leq \psi(d(x^*, Tx^*)) < d(x^*, Tx^*)$, which implies $x^* = Tx^*$, that is, x^* is a fixed point of T.

Finally, suppose that x^* and y^* are two fixed points of T such that $x^* \neq y^*$. Then by the hypothesis, $\alpha(x^*, y^*) \geq 1$. Hence, from (2.1) with $x = x^*$ and $y = y^*$ we have

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\leq \alpha(x^*, y^*) d(Tx^*, Ty^*) \\ &\leq \psi(M(x^*, y^*)), \end{aligned}$$

where

$$\begin{aligned} M(x^*, y^*) &= \max\biggl\{ d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*), \\ &\qquad \frac{d(x^*, Tx^*) d(y^*, Ty^*)}{1 + d(x^*, y^*)}, \frac{d(x^*, Tx^*) d(y^*, Ty^*)}{1 + d(Tx^*, Ty^*)} \biggr\} \\ &= d(x^*, y^*). \end{aligned} \tag{2.19}$$

Hence, we get $d(x^*, y^*) \leq \psi(d(x^*, y^*)) < d(x^*, y^*)$, which is possible only if $d(x^*, y^*) = 0$, that is, $x^* = y^*$. Hence T has a unique fixed point. □

Definition 2.2 Let (X, d) be a generalized metric space and $\alpha : X \times X \to \mathbb{R}^+$. A mapping $T : X \to X$ is said to be (α, ψ)-rational type-II contractive mapping if there exists a $\psi \in \Psi$, such that, for all $x, y \in X$, the following condition holds:

$$\alpha(x, y) d(Tx, Ty) \leq \psi(M(x, y)), \tag{2.20}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)+d(x,Ty)+d(y,Tx)}, \frac{d(x,Ty)d(x,y)}{1+d(x,Tx)+d(y,Tx)+d(y,Ty)}\right\}.$$

For this class of mappings we state a similar existence and uniqueness theorem.

Theorem 2.2 *Let (X,d) be a complete generalized metric space, $T : X \to X$ be a self mapping, and $\alpha : X \times X \to \mathbb{R}$. Suppose that the following conditions are satisfied*:

(i) *T is an α-admissible mapping*;
(ii) *T is an (α,ψ)-rational type-II contractive mapping*;
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$*;
(iv) *either T is continuous, or X is α-regular.*

Then T has a fixed point $x^* \in X$ and $\{T^nx_0\}$ converges to x^*. Further, if for all $x, y \in F(T)$, we have $\alpha(x,y) \geq 1$, then T has a unique fixed point in X.

Proof The proof can be done by following the lines of the proof of Theorem 2.1. □

The following example illustrating Theorem 2.1 is inspired by [4].

Example 2.1 Let X be a finite set defined as $X = \{1,2,3,4\}$. Define $d : X \times X \to [0,\infty)$ as:

$$d(1,1) = d(2,2) = d(3,3) = d(4,4) = 0,$$

$$d(1,2) = d(2,1) = 3,$$

$$d(2,3) = d(3,2) = d(1,3) = d(3,1) = 1,$$

$$d(1,4) = d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = 4.$$

The function d is not a metric on X. Indeed, note that

$$3 = d(1,2) \geq d(1,3) + d(3,2) = 1 + 1 = 2,$$

that is, the triangle inequality is not satisfied. However, d is a generalized metric on X and, moreover, (X,d) is a complete generalized metric space. Define $T : X \to X$ as

$$T1 = T2 = T3 = 2, \qquad T4 = 3,$$

$\alpha(x,y)$ as $\alpha(x,y) = 1$ and $\psi(t) = \frac{t}{2}$. Then, for $x = 1,2,3$ and $y = 1,2,3$, we have

$$\alpha(x,y)d(Tx,Ty) = 0 \leq \psi\bigl(M(x,y)\bigr) = 0.$$

On the other hand, for $x = 1,2,3$ and $y = 4$ we obtain

$$\alpha(x,4)d(Tx,T4) = d(2,3) = 1$$

and

$$M(x,4) = \max\left\{d(x,4), d(x,T4), d(y,T4), \frac{d(x,Tx)d(y,Ty)}{1+d(x,4)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,T4)}\right\} = 4,$$

and hence

$$\alpha(x,4)d(Tx,T4) = 1 \leq \frac{4}{2} = 2.$$

For $x = 4$, $y = 4$, the contraction condition is obvious. Clearly, T satisfies the conditions of Theorem 2.1 and has a unique fixed point $x = 2$.

3 Some consequences

In this section we give some consequences of the main results presented above. Specifically, we apply our results to generalized metric spaces endowed with a partial order.

Definition 3.1 Let $(X,\preceq)$ be a partially ordered set. A mapping $T : X \to X$ is said to be nondecreasing with respect to $\preceq$ if for every $x, y \in X$ $x \preceq y$ implies $Tx \preceq Ty$.

Definition 3.2 Let $(X,d,\preceq)$ be a partially ordered GMS. X is called regular GMS if, whenever $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $x_n \preceq x_{n+1}$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$.

Theorem 3.1 *Let $(X,d,\preceq)$ be a partially ordered complete generalized metric space and $T : X \to X$ be a nondecreasing self mapping. Suppose that the following conditions are satisfied*:

(i) *There exists a function $\psi \in \Psi$ for which*

$$d(Tx,Ty) \leq \psi\big(M(x,y)\big), \tag{3.1}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\right\}$$

for all $x, y \in X$ with $x \preceq y$.

(ii) *There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and $x_0 \preceq T^2x_0$.*

(iii) *Either T is continuous, or X is regular.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*.

Proof Define a mapping $\alpha : X \times X \to [0,\infty)$ as follows.

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \preceq y \text{ or } y \preceq x, \\ 0, & \text{otherwise.} \end{cases}$$

Then the existence conditions of Theorem 2.1 hold and hence T has a fixed point which is the limit of the sequence $\{T^n x_0\}$. □

Theorem 3.2 *Let $(X, d, \preceq)$ be a partially ordered complete generalized metric space and $T : X \to X$ be a nondecreasing self mapping. Suppose that the following conditions are satisfied*:

(i) *There exist a function $\psi \in \Psi$ for which*

$$d(Tx, Ty) \leq \psi\bigl(M(x,y)\bigr), \tag{3.2}$$

where

$$M(x,y) = \max\biggl\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)+d(x,Ty)+d(y,Tx)}, \frac{d(x,Ty)d(x,y)}{1+d(x,Tx)+d(y,Tx)+d(y,Ty)}\biggr\}$$

for all $x, y \in X$ with $x \preceq y$;

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and $x_0 \preceq T^2x_0$*;

(iii) *either T is continuous, or X is regular.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*.

Proof Employing again a mapping $\alpha : X \times X \to [0, \infty)$ defined as

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \preceq y \text{ or } y \preceq x, \\ 0, & \text{otherwise,} \end{cases}$$

we observe that the existence conditions of Theorem 2.2 hold and hence, T has a fixed point which is the limit of the sequence $\{T^n x_0\}$. □

Several particular cases can also be deduced from the above results.

Corollary 3.1 *Let (X, d) be a complete generalized metric space, T be a self mapping, $T : X \to X$, and $\alpha : X \times X \to \mathbb{R}$. Suppose that the following conditions are satisfied*:

(i) *T is an α-admissible mapping*;

(ii) *T satisfies*

$$d(Tx, Ty) \leq kM(x,y), \tag{3.3}$$

where

$$M(x,y) = \max\biggl\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)+d(x,Ty)+d(y,Tx)}, \frac{d(x,Ty)d(x,y)}{1+d(x,Tx)+d(y,Tx)+d(y,Ty)}\biggr\}$$

for some $k \in [0,1)$;

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, $\alpha(x_0, T^2x_0) \geq 1$*;

(iv) *either T is continuous, or X is α-regular.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*. Further, if, for all $x, y \in F(T)$, we have $\alpha(x,y) \geq 1$, then T has a unique fixed point in X.

Proof Define $\psi(t) = kt$. Clearly, $\psi \in \Psi$. By Theorem 2.2, T has a unique fixed point. □

Corollary 3.2 *Let $(X, d, \preceq)$ be a partially ordered complete generalized metric space and $T : X \to X$ be a nondecreasing mapping. Suppose that the following conditions are satisfied:*

(i)

$$d(Tx, Ty) \leq kM(x, y), \tag{3.4}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)+d(x,Ty)+d(y,Tx)}, \frac{d(x,Ty)d(x,y)}{1+d(x,Tx)+d(y,Tx)+d(y,Ty)}\right\}$$

for all $x, y \in X$ with $x \preceq y$ and some $k \in [0, 1)$;

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and $x_0 \preceq T^2x_0$;*

(iii) *either T is continuous, or X is regular.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*.

Proof Define $\alpha : X \times X \to [0, \infty)$ as

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \preceq y \text{ or } y \preceq x, \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 3.1 implies that T has a fixed point. □

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final version of manuscript.

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