Ahmad et al. *Fixed Point Theory and Applications* (2015) 2015:80
DOI 10.1186/s13663-015-0333-2

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH

Open Access

New fixed point theorems for generalized F-contractions in complete metric spaces

Jamshaid Ahmad[1], Ahmed Al-Rawashdeh[2*] and Akbar Azam[1]

*Correspondence:
aalrawashdeh@uaeu.ac.ae
[2]Department of Mathematical Sciences, UAE University, Al Ain, 15551, UAE
Full list of author information is available at the end of the article

Abstract

In this paper, owing to the concept of F-contraction, we define two new classes of functions $M(S,T)$ and $N(S,T)$, and we prove some new fixed point results for single-valued and multivalued mappings in complete metric spaces. Our results extend, generalize and unify several known results in the literature. We include an example to show that the generalization is proper.

MSC: Primary 30C45; 30C10; secondary 47B38

Keywords: fixed point; F-contractions; metric space; multivalued mappings

1 Introduction and preliminaries

In fixed point theory, the contractive conditions on underlying functions play an important role in finding solutions of fixed point problems. Banach contraction principle is a remarkable result in metric fixed point theory. Over the years, it has been generalized in different directions by several mathematicians (see [1–13] and [14–16]). In 2012, Wardowski [17] introduced a new concept of contraction, and he proved a fixed point theorem which generalizes the Banach contraction principle. Later on, Wardowski and Van Dung [18] gave the idea of F-weak contraction and proved a theorem concerning F-weak contraction. Afterwards, Abbas *et al.* [2] further generalized the concept of F-contraction and proved certain fixed point results. Hussain and Salimi [10] introduced an α-GF-contraction with respect to a general family of functions G and established Wardowski-type fixed point results in ordered metric spaces. Batra *et al.* [4, 5] extended the concept of F-contraction on graphs and altered distances. They proved some fixed point and coincidence point results by illustrating them with some examples. Recently, Cosentino and Vetro [7] followed the approach of F-contraction and obtained some fixed point theorems of Hardy-Rogers-type for self-mappings in complete metric spaces and complete ordered metric spaces. Then Sgroi and Vetro [19] extended this Hardy-Rogers-type fixed point result for multivalued mappings. The reader can see [1, 3, 8, 9, 11, 12, 18, 20] for recent results in this direction.

The aim of this article is to establish some new fixed point theorems and generalize the results of Beg and Azam [6], Cosentino and Vetro [7], Sgroi and Vetro [19] and Wardowski [17] by introducing a new type of contractions.

We recall some basic known definitions and results which will be used in the sequel. Throughout this article, $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$ denote the set of natural numbers, the set of positive real numbers and the set of real numbers, respectively.

To be consistent with Wardowski [17], we denote by $\mathcal{F}$ the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing;

(F2) for all sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) there exists $0 < k < 1$ such that $\lim_{a\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 1.1 [17] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exist $\tau \in \mathbb{R}^+$ and a function $F \in \mathcal{F}$ such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \quad \Rightarrow \quad \tau + F\big(d(Tx, Ty)\big) \le F\big(d(x, y)\big). \tag{1.1}$$

Example 1.2 [17] Let $F : \mathbb{R}_+ \to \mathbb{R}$ be defined by $F(\alpha) = \ln \alpha$. It is clear that F satisfies (F1)-(F3) for any $k \in (0, 1)$. Each mapping $T : X \to X$ satisfying (1.1) is an F-contraction such that

$$d(Tx, Ty) \le e^{-\tau} d(x, y) \quad \text{for all } x, y \in X, Tx \ne Ty.$$

It is clear that for $x, y \in X$ such that $Tx = Ty$, the inequality $d(Tx, Ty) \le e^{-\tau} d(x, y)$ also holds, *i.e.*, T is a Banach contraction.

Example 1.3 [17] If $F(\alpha) = \ln \alpha + \alpha$, $\alpha > 0$, then F satisfies (F1)-(F3). Then condition (1.1) satisfied by the mapping $T : X \to X$ is of the form

$$\frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty)-d(x,y)} \le e^{-\tau} \quad \text{for all } x, y \in X, Tx \ne Ty.$$

Remark 1.4 From (F1) and (1.1), it is easy to conclude that every F-contraction is necessarily continuous.

Wardowski [17] stated a modified version of the Banach contraction principle as follows.

Theorem 1.5 [17] *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point $z \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to z.*

Cosentino and Vetro in [7] proved the following Hardy-Rogers-type fixed point theorem for F-contractive condition in the setting of complete metric spaces.

Theorem 1.6 [7] *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping. If there exist $\tau > 0$ and reals $\alpha, \beta, \gamma, \delta, L \ge 0$ such that for all $x, y \in X$,*

$d(Tx, Ty) > 0$ *implies*

$$\begin{aligned} \tau + F\big(d(Tx, Ty)\big) \le F\big(&\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) \\ &+ \delta\big(d(x, Ty) + L d(y, Tx)\big)\big), \end{aligned} \tag{1.2}$$

where $F \in \mathcal{F}$ and $\alpha + \beta + \gamma + 2L = 1$ and $\gamma \ne 1$, then T has a unique fixed point.

2 Main results

In this section, we prove a common fixed point theorem for self-mappings regarding F-contraction, and we give an illustrative example. For a metric space (X,d) and two self-mappings $S,T:X\to X$, we denote by $M(S,T)$ the collection of all functions $\lambda:X\times X\to[0,1)$ such that

$$\lambda(TSx,y)\le\lambda(x,y)\quad\text{and}\quad\lambda(x,STy)\le\lambda(x,y).$$

Similarly $N(S,T)$ denotes the collection of all functions $\Lambda:X\to[0,1)$ for all $x,y\in X$ with

$$\Lambda(TSx)\le\Lambda(x).$$

In the following proposition, we discuss some properties of the above control functions belonging to the classes $M(S,T)$ and $N(S,T)$. This proposition plays an important role in the proofs of our main theorems.

Proposition 2.1 *Let (X,d) be a metric space and $S,T:X\to X$ be self-mappings. Let $x_0\in X$, we define the sequence $\{x_n\}$ by $x_{2n+1}=Sx_{2n}$, $x_{2n+2}=Tx_{2n+1}$ for all integers $n\ge 0$.*

If $\lambda\in M(S,T)$, then $\lambda(x_{2n},y)\le\lambda(x_0,y)$ and $\lambda(x,x_{2n+1})\le\lambda(x,x_1)$ for all $x,y\in X$ and integers $n\ge 0$.

Proof Let $x,y\in X$ and integers $n\ge 0$. Then we have

$$\lambda(x_{2n},y)=\lambda(TSx_{2n-2},y)\le\lambda(x_{2n-2},y)=\lambda(TSx_{2n-4},y)\le\cdots\le\lambda(x_0,y).$$

Similarly, we have

$$\lambda(x,x_{2n+1})=\lambda(x,STx_{2n-1})\le\lambda(x,x_{2n-1})=\lambda(x,STx_{2n-3})\le\cdots\le\lambda(x,x_1).\qquad\square$$

Now we establish a theorem regarding common fixed points of self-mappings $S,T:X\to X$ under some new contractive conditions and generalized Theorem 1.6 in the sense that instead of taking constants, we take control functions.

Theorem 2.2 *Let (X,d) be a complete metric space and $S,T:X\to X$ be self-mappings. If there exist $\tau>0$ and mappings $\lambda,\mu,\gamma,\delta,L\in M(S,T)$ such that for all $x,y\in X$,*

(a)

$$\lambda(x,y)+\mu(x,y)+\gamma(x,y)+2L(x,y)=1,\quad \gamma(x,y)\ne 1 \text{ and } \delta(x,y)\ge 0;$$
$$\lambda(x,y)+\mu(x,y)+\gamma(x,y)+2\delta(x,y)=1,\quad \gamma(x,y)\ne 1 \text{ and } L(x,y)\ge 0;$$

(b) *$d(Sx,Ty)>0$ implies*

$$\tau+F\bigl(d(Sx,Ty)\bigr)\le F\bigl(\lambda(x,y)d(x,y)+\mu(x,y)\bigl(d(x,Sx)\bigr)+\gamma(x,y)d(y,Ty)\\ +\delta(x,y)\bigl(d(x,Ty)+L(x,y)d(y,Sx)\bigr)\bigr),$$

where $F\in\mathcal{F}$, then S and T have a common fixed point.

Moreover, if

(c)

$$\lambda(x,y) + \delta(x,y) + L(x,y) \leq 1,$$

then the common fixed point of S and T is unique.

Proof Let $x_0 \in X$, we define the sequence $\{x_n\}$ by

$$x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n+2} = Tx_{2n+1}$$

for all integers $n \geq 0$. From Proposition 2.1, for all integers $n \geq 0$, we have

$$\begin{aligned}
\tau + F\big(d(x_{2n}, x_{2n+1})\big) &= \tau + F\big(d(Tx_{2n-1}, Sx_{2n})\big) = \tau + F\big(d(Sx_{2n}, Tx_{2n-1})\big) \\
&\leq F\big(\lambda(x_{2n}, x_{2n-1})d(x_{2n}, x_{2n-1}) \\
&\quad + \mu(x_{2n}, x_{2n-1})d(x_{2n}, Sx_{2n}) \\
&\quad + \gamma(x_{2n}, x_{2n-1})d(x_{2n-1}, Tx_{2n-1}) \\
&\quad + \delta(x_{2n}, x_{2n-1})d(x_{2n}, Tx_{2n-1}) \\
&\quad + L(x_{2n}, x_{2n-1})d(x_{2n-1}, Sx_{2n})\big) \\
&= F\big(\lambda(x_{2n}, x_{2n-1})d(x_{2n}, x_{2n-1}) + \mu(x_{2n}, x_{2n-1})d(x_{2n}, x_{2n+1}) \\
&\quad + \gamma(x_{2n}, x_{2n-1})d(x_{2n-1}, x_{2n}) + L(x_{2n}, x_{2n-1})d(x_{2n-1}, x_{2n+1})\big) \\
&\leq F\big(\lambda(x_0, x_{2n-1})d(x_{2n}, x_{2n-1}) + \mu(x_0, x_{2n-1})d(x_{2n}, x_{2n+1}) \\
&\quad + \gamma(x_0, x_{2n-1})d(x_{2n-1}, x_{2n})\big) \\
&\quad + L(x_0, x_{2n-1})\big(d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})\big) \\
&\leq F\big(\lambda(x_0, x_1)d(x_{2n}, x_{2n-1}) + \mu(x_0, x_1)d(x_{2n}, x_{2n+1}) \\
&\quad + \gamma(x_0, x_1)d(x_{2n-1}, x_{2n}) \\
&\quad + L(x_0, x_1)\big(d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})\big)\big) \\
&= F\big(\big(\lambda(x_0, x_1) + \gamma(x_0, x_1) + L(x_0, x_1)\big)d(x_{2n-1}, x_{2n}) \\
&\quad + \big(\mu(x_0, x_1) + L(x_0, x_1)\big)d(x_{2n}, x_{2n+1})\big).
\end{aligned} \tag{2.1}$$

Since F is strictly increasing, we deduce the following:

$$\begin{aligned}
d(x_{2n}, x_{2n+1}) &< \big(\lambda(x_0, x_1) + \gamma(x_0, x_1) + L(x_0, x_1)\big)d(x_{2n-1}, x_{2n}) \\
&\quad + \big(\mu(x_0, x_1) + L(x_0, x_1)\big)d(x_{2n}, x_{2n+1}).
\end{aligned}$$

Hence,

$$d(x_{2n}, x_{2n+1}) < \frac{\lambda(x_0, x_1) + \gamma(x_0, x_1) + L(x_0, x_1)}{1 - \mu(x_0, x_1) - L(x_0, x_1)} d(x_{2n-1}, x_{2n}) = d(x_{2n-1}, x_{2n}).$$

Consequently, from (2.1) we have

$$\tau + F\big(d(x_{2n}, x_{2n+1})\big) < F\big(d(x_{2n-1}, x_{2n})\big). \tag{2.2}$$

Similarly, we have

$$\begin{aligned}
\tau + F\big(d(x_{2n+1}, x_{2n+2})\big) &= \tau + F\big(d(Sx_{2n}, Tx_{2n+1})\big) \\
&\leq F\big(\lambda(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+1}) + \mu(x_{2n}, x_{2n+1})d(x_{2n}, Sx_{2n}) \\
&\quad + \gamma(x_{2n}, x_{2n+1})d(x_{2n+1}, Tx_{2n+1}) + \delta(x_{2n}, x_{2n+1})d(x_{2n}, Tx_{2n+1}) \\
&\quad + L(x_{2n}, x_{2n+1})\big(d(x_{2n+1}, Sx_{2n})\big)\big) \\
&= F\big(\lambda(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+1}) + \mu(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+1}) \\
&\quad + \gamma(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}) + \delta(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+2})\big) \\
&\leq F\big(\lambda(x_0, x_{2n+1})d(x_{2n}, x_{2n+1}) + \mu(x_0, x_{2n+1})d(x_{2n}, x_{2n+1}) \\
&\quad + \gamma(x_0, x_{2n+1})d(x_{2n+1}, x_{2n+2}) \\
&\quad + \delta(x_0, x_{2n+1})\big(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\big)\big) \\
&\leq F\big(\lambda(x_0, x_1)d(x_{2n}, x_{2n+1}) + \mu(x_0, x_1)d(x_{2n}, x_{2n+1}) \\
&\quad + \gamma(x_0, x_1)d(x_{2n+1}, x_{2n+2}) + \delta(x_0, x_1)\big(d(x_{2n}, x_{2n+1}) \\
&\quad + d(x_{2n+1}, x_{2n+2})\big)\big) \\
&= F\big(\big(\lambda(x_0, x_1) + \mu(x_0, x_1) + \delta(x_0, x_1)\big)d(x_{2n}, x_{2n+1}) \\
&\quad + \big(\gamma(x_0, x_1) + \delta(x_0, x_1)\big)d(x_{2n+1}, x_{2n+2})\big).
\end{aligned} \tag{2.3}$$

Since F is strictly increasing, we deduce

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) < {} & \big(\lambda(x_0, x_1) + \mu(x_0, x_1) + \delta(x_0, x_1)\big)d(x_{2n}, x_{2n+1}) \\
& + \big(\gamma(x_0, x_1) + \delta(x_0, x_1)\big)d(x_{2n+1}, x_{2n+2}).
\end{aligned}$$

Hence,

$$d(x_{2n+1}, x_{2n+2}) < \frac{\lambda(x_0, x_1) + \mu(x_0, x_1) + \delta(x_0, x_1)}{1 - \gamma(x_0, x_1) - \delta(x_0, x_1)} d(x_{2n}, x_{2n+1}) = d(x_{2n}, x_{2n+1}).$$

Consequently, from (2.3) we have

$$\tau + F\big(d(x_{2n+1}, x_{2n+2})\big) < F\big(d(x_{2n}, x_{2n+1})\big). \tag{2.4}$$

Thus

$$F\big(d(x_n, x_{n+1})\big) < F\big(d(x_{n-1}, x_n)\big) - \tau < F\big(d(x_{n-2}, x_{n-1})\big) - 2\tau < \cdots < F\big(d(x_0, x_1)\big) - n\tau \tag{2.5}$$

for all $n \in \mathbb{N}$. Since $F \in \digamma$, so by taking limit as $n \to \infty$ in (2.5) we have

$$\lim_{n\to\infty} F\big(d(x_n, x_{n+1})\big) = -\infty \quad \iff \quad \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.6}$$

Now, from (F3), there exists $0 < k < 1$ such that

$$\lim_{n\to\infty}\big[d(x_n,x_{n+1})\big]^k F\big(d(x_n,x_{n+1})\big) = 0. \tag{2.7}$$

By (2.5), we have

$$\begin{aligned} &d(x_n,x_{n+1})^k F\big(d(x_n,x_{n+1})\big) - d(x_n,x_{n+1})^k F\big(d(x_1,x_0)\big) \\ &\quad< d(x_n,x_{n+1})^k\big[F\big(d(x_0,x_1) - n\tau\big) - F\big(d(x_0,x_1)\big)\big] \\ &\quad= -n\tau\big[d(x_n,x_{n+1})\big]^k \leq 0. \end{aligned} \tag{2.8}$$

By taking limit as $n \to \infty$ in (2.8) and applying (2.6) and (2.7), we have

$$\lim_{n\to\infty} n\big[d(x_n,x_{n+1})\big]^k = 0. \tag{2.9}$$

It follows from (2.9) that there exists $n_1 \in \mathbb{N}$ such that

$$n\big[d(x_n,x_{n+1})\big]^k \leq 1 \tag{2.10}$$

for all $n > n_1$. This implies

$$d(x_n,x_{n+1}) \leq \frac{1}{n^{1/k}} \tag{2.11}$$

for all $n > n_1$. Now we prove that $\{x_n\}$ is a Cauchy sequence. For $m > n > n_1$, we have

$$d(x_n,x_m) \leq \sum_{i=n}^{m-1} d(x_i,x_{i+1}) \leq \sum_{i=n}^{m-1}\frac{1}{i^{1/k}}. \tag{2.12}$$

Since $0 < k < 1$, then $\sum_{i=1}^{\infty}\frac{1}{i^{1/k}}$ converges. Therefore, $d(x_n,x_m) \to 0$ as $m,n \to \infty$. Thus we proved that $\{x_n\}$ is a Cauchy sequence in X. The completeness of X ensures that there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. First we show that z is a fixed point of S. By Proposition 2.1, we have

$$\begin{aligned} \tau + F\big(d(Sz,x_{2n+2})\big) &= \tau + F\big(d(Sz,Tx_{2n+1})\big) \\ &\leq F\big(\lambda(z,x_{2n+1})d(z,x_{2n+1}) + \mu(z,x_{2n+1})d(z,Sz) \\ &\quad + \gamma(z,x_{2n+1})d(x_{2n+1},Tx_{2n+1}) + \delta(z,x_{2n+1})d(z,Tx_{2n+1}) \\ &\quad + L(z,x_{2n+1})d(x_{2n+1},Sz)\big) \\ &= F\big(\lambda(z,x_{2n+1})d(z,x_{2n+1}) + \mu(z,x_{2n+1})d(z,Sz) \\ &\quad + \gamma(z,x_{2n+1})d(x_{2n+1},x_{2n+2}) + \delta(z,x_{2n+1})d(z,x_{2n+2}) \\ &\quad + L(z,x_{2n+1})d(x_{2n+1},Sz)\big) \\ &\leq F\big(\lambda(z,x_1)d(z,x_{2n+1}) + \mu(z,x_1)d(z,Sz) + \gamma(z,x_1)d(x_{2n+1},x_{2n+2}) \\ &\quad + \delta(z,x_1)d(z,x_{2n+2}) + L(z,x_1)d(x_{2n+1},Sz)\big). \end{aligned}$$

Since F is strictly increasing, we deduce

$$\begin{aligned} d(Sz,x_{2n+2}) &< \lambda(z,x_1)d(z,x_{2n+1}) + \mu(z,x_1)d(z,Sz) + \gamma(z,x_1)d(x_{2n+1},x_{2n+2}) \\ &\quad + \delta(z,x_1)d(z,x_{2n+2}) + L(z,x_1)d(x_{2n+1},Sz). \end{aligned}$$

Letting $n \to +\infty$ in the previous inequality, we get

$$d(Sz,z) \leq \bigl(\mu(z,x_1) + L(z,x_1)\bigr)d(Sz,z)$$

as $\mu(z,x_1) + \gamma(z,x_1) < 1$. This implies $d(Sz,z) = 0$. Thus we have $z = Sz$. We also show that z is a fixed point of T. By Proposition 2.1, we have

$$\begin{aligned} \tau + F\bigl(d(x_{2n+1},Tz)\bigr) &= \tau + F\bigl(d(Sx_{2n},Tz)\bigr) \\ &\leq F\bigl(\lambda(x_{2n},z)d(x_{2n},z) + \mu(x_{2n},z)d(x_{2n},Sx_{2n}) \\ &\quad + \gamma(x_{2n},z)d(z,Tz) + \delta(x_{2n},z)d(x_{2n},Tz) \\ &\quad + L(x_{2n},z)d(z,Sx_{2n})\bigr) \\ &= F\bigl(\lambda(x_{2n},z)d(x_{2n},z) + \mu(x_{2n},z)d(x_{2n},x_{2n+1}) + \gamma(x_{2n},z)d(z,Tz) \\ &\quad + \delta(x_{2n},z)d(x_{2n},Tz) + L(x_{2n},z)d(z,x_{2n+1})\bigr) \\ &\leq F\bigl(\lambda(x_0,z)d(x_{2n},z) + \mu(x_0,z)d(x_{2n},x_{2n+1}) + \gamma(x_0,z)d(z,Tz) \\ &\quad + \delta(x_0,z)d(x_{2n},Tz) + L(x_0,z)d(z,x_{2n+1})\bigr). \end{aligned}$$

Since F is strictly increasing, we deduce

$$\begin{aligned} d(x_{2n+1},Tz) &< \lambda(x_0,z)d(x_{2n},z) + \mu(x_0,z)d(x_{2n},x_{2n+1}) \\ &\quad + \gamma(x_0,z)d(z,Tz) + \delta(x_0,z)d(x_{2n},Tz) + L(x_0,z)d(z,x_{2n+1}). \end{aligned}$$

Letting $n \to +\infty$ in the previous inequality, we get

$$d(z,Tz) \leq \bigl(\gamma(x_0,z) + \gamma(x_0,z)\bigr)\bigl(d(z,Tz)\bigr).$$

This implies $d(z,Tz) = 0$ and hence $z = Tz$. Therefore, z is a common fixed point of S and T.

Now we show the uniqueness. Suppose that there exists another common fixed point u of S and T, that is, $u = Su = Tu$. Assume that $Su \neq Tz$, then from (b) we have

$$\begin{aligned} \tau + F\bigl(d(Su,Tz)\bigr) &\leq F\bigl(\lambda(u,z)d(u,z) + \mu(u,z)d(u,Su) + \gamma(u,z)d(z,Tz) \\ &\quad + \delta(u,z)d(u,Tz) + L(u,z)d(z,Su)\bigr) \\ &= F\bigl(\bigl(\lambda(u,z) + \delta(u,z) + L(u,z)\bigr)d(u,z)\bigr). \end{aligned}$$

Since F is strictly increasing, we deduce

$$d(Su,Tz) \leq \bigl(\lambda(u,z) + \delta(u,z) + L(u,z)\bigr)d(u,z) = \bigl(\lambda(u,z) + \delta(u,z) + L(u,z)\bigr)d(Su,Tz).$$

It implies that $d(Su, Tz) = 0$, that is, $Su = Tz$. It is a contradiction. Thus S and T have a unique common fixed point, which ends the proof. □

Consequently, we have the following results.

Corollary 2.3 *Let (X,d) be a complete metric space and $S,T : X \to X$ be self-mappings. If there exist $\tau > 0$ and mappings $\lambda,\mu,\gamma \in M(S,T)$ such that for all $x,y \in X$,*

(a)

$$\lambda(x,y) + 2\mu(x,y) + \gamma(x,y) = 1;$$

(b) $d(Sx,Ty) > 0$ *implies*

$$\begin{aligned}\tau + F\bigl(d(Sx,Ty)\bigr) \le F\bigl(&\lambda(x,y)d(x,y) + \mu(x,y)\bigl(d(x,Sx) + d(y,Ty)\bigr)\\ &+ \gamma(x,y)\bigl(d(x,Ty) + d(y,Sx)\bigr)\bigr),\end{aligned}$$

where $F \in \mathcal{F}$, then S and T have a unique common fixed point.

Corollary 2.4 *Let (X,d) be a complete metric space and $S,T : X \to X$ be self-mappings. If there exist $\tau > 0$ and mappings $\Lambda,\Theta,\Xi,\Delta,L \in N(S,T)$ such that for all $x,y \in X$,*

(a) $\Lambda(x) + \Theta(x) + \Xi(x) + 2L(x) = 1$, $\Xi(x) \ne 1$ *and* $\Delta(x) \ge 0$;
$\Lambda(x) + \Theta(x) + \Xi(x) + 2\Delta(x) = 1$, $\Xi(x) \ne 1$ *and* $L(x) \ge 0$;

(b) $d(Sx,Ty) > 0$ *implies*

$$\begin{aligned}\tau + F\bigl(d(Sx,Ty)\bigr) \le F\bigl(&\Lambda(x)d(x,y) + \Theta(x)d(x,Sx) + \Xi(x)d(y,Ty)\\ &+ \Delta(x)d(x,Ty) + L(x)d(y,Sx)\bigr),\end{aligned}$$

where $F \in \mathcal{F}$, then S and T have a common fixed point.

Moreover, if

$$\Lambda(x) + \Delta(x) + L(x) \le 1,$$

then the common fixed point of S and T is unique.

Proof Define $\lambda,\mu,\gamma,\delta,L : X \times X \to [0,1)$ by $\lambda(x,y) = \Lambda(x)$, $\mu(x,y) = \Theta(x)$, $\gamma(x,y) = \Xi(x)$, $\delta(x,y) = \Delta(x)$ and $L(x,y) = L(x)$ for all $x,y \in X$. Then, for all $x,y \in X$,

(a)

$$\begin{aligned}
&\lambda(TSx,y) = \Lambda(TSx) \le \Lambda(x) = \lambda(x,y) \quad \text{and} \quad \lambda(x,STy) = \Lambda(x) = \lambda(x,y);\\
&\mu(TSx,y) = \Theta(TSx) \le \Theta(x) = \mu(x,y) \quad \text{and} \quad \mu(x,STy) = \Theta(x) = \mu(x,y);\\
&\gamma(TSx,y) = \Xi(TSx) \le \Xi(x) = \gamma(x,y) \quad \text{and} \quad \gamma(x,STy) = \Xi(x) = \gamma(x,y);\\
&\delta(TSx,y) = \Delta(TSx) \le \Delta(x) = \gamma(x,y) \quad \text{and} \quad \delta(x,STy) = \Delta(x) = \delta(x,y);\\
&L(TSx,y) = L(TSx) \le L(x) = L(x,y) \quad \text{and} \quad L(x,STy) = L(x) = L(x,y);
\end{aligned}$$

(b)

$$\lambda(x,y) + \mu(x,y) + \gamma(x,y) + 2L(x,y) = \Lambda(x) + \Theta(x) + \Xi(x) + 2L(x) = 1;$$
$$\lambda(x,y) + \mu(x,y) + \gamma(x,y) + 2\delta(x,y) = \Lambda(x) + \Theta(x) + \Xi(x) + 2\Delta(x) = 1$$

and

$$\gamma(x,y) = \Xi(x) \neq 1;$$

(c) $d(Sx, Ty) > 0$ implies

$$\begin{aligned}\tau + F\big(d(Sx,Ty)\big) &\leq F\big(\Lambda(x)d(x,y) + \Theta(x)d(x,Sx) + \Xi(x)d(y,Ty)\\ &\quad + \Delta(x)d(x,Ty) + L(x)d(y,Sx)\big)\\ &= F\big(\lambda(x,y)d(x,y) + \mu(x,y)d(x,Sx) + \gamma(x,y)d(y,Ty)\\ &\quad + \delta(x,y)d(x,Ty) + L(x,y)d(y,Sx)\big).\end{aligned}$$

By Theorem 2.2, S and T have a unique common fixed point. □

By letting $\Lambda(\cdot) = \Lambda$, $\Theta(\cdot) = \Theta$, $\Xi(\cdot) = \Xi$, $\Delta(\cdot) = \Delta$ and $L(\cdot) = L$ in Corollary 2.4, we get the following result.

Corollary 2.5 *Let (X,d) be a complete metric space and $S,T : X \to X$ be self-mappings. If there exist $\tau > 0$ and a mapping $F : \mathbb{R}^+ \to \mathbb{R}$ such that for all $x,y \in X$,*
$d(Sx,Ty) > 0$ implies

$$\tau + F\big(d(Sx,Ty)\big) \leq F\big(\Lambda d(x,y) + \Theta d(x,Sx) + \Xi d(y,Ty) + \Delta d(x,Ty) + Ld(y,Sx)\big)$$

for all nonnegative reals $\Lambda, \Theta, \Xi, \Delta, L \in [0,1)$ with $\Lambda + \Theta + \Xi + 2\Delta = 1$, $\Xi \neq 1$ and $L \geq 0$, then S and T have a common fixed point. Moreover, if

$$\Lambda + \Delta + L \leq 1,$$

then the common fixed point of S and T is unique.

By setting $S = T$ in the above corollary, we get Theorem 3.1 of [7].

Corollary 2.6 [7] *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping. If there exist $\tau > 0$ and the mapping $F : \mathbb{R}^+ \to \mathbb{R}$ such that for all $x,y \in X$,*
$d(Tx,Ty) > 0$ implies

$$\tau + F\big(d(Tx,Ty)\big) \leq F\big(\Lambda d(x,y) + \Theta d(x,Tx) + \Xi d(y,Ty) + \Delta d(x,Ty) + Ld(y,Tx)\big)$$

for all nonnegative reals $\Lambda, \Theta, \Xi, \Delta, L \in [0,1)$ with $\Lambda + \Theta + \Xi + 2\Delta = 1$, $\Xi \neq 1$ and $L \geq 0$. Then T has a fixed point. Moreover, if $\Lambda + \Delta + L \leq 1$, then the fixed point of T is unique.

Putting $\Lambda = \Delta = L = 0$ and $\Theta + \Xi = 1$ with $\Theta \neq 0$ and $\Xi \neq 1$ in Corollary 2.6, we get Corollary 3.2 of [7] as follows.

Corollary 2.7 [7] *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping. If there exist $\tau > 0$ and a mapping $F : \mathbb{R}^+ \to \mathbb{R}$ such that for all $x, y \in X$,*
$d(Tx, Ty) > 0$ implies

$$\tau + F\big(d(Tx, Ty)\big) \leq F\big(\Theta d(x, Tx) + \Xi d(y, Ty)\big)$$

for all nonnegative reals $\Theta, \Xi, \in [0,1)$ with $\Theta + \Xi = 1$ and $\Xi \neq 1$, then T has a unique fixed point.

Putting $\Lambda = \Theta = \Xi = 0$ and $\Delta = \frac{1}{2}$ in Corollary 2.6, we get Corollary 3.3 of [7] as follows.

Corollary 2.8 [7] *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping. If there exist $\tau > 0$ and the mapping $F : \mathbb{R}^+ \to \mathbb{R}$ such that for all $x, y \in X$,*
$d(Tx, Ty) > 0$ implies

$$\tau + F\big(d(Tx, Ty)\big) \leq F\left(\frac{1}{2} d(x, Ty) + L d(y, Tx)\right)$$

for nonnegative real $L \in [0,1)$. Then T has a fixed point. Moreover, if $L \leq \frac{1}{2}$, then the fixed point of T is unique.

Remark 2.9 If $\lambda(x,y) = 1$, $\mu(x,y) = \gamma(x,y) = \delta(x,y) = L(x,y) = 0$ and $S = T$ in Theorem 2.2, we can get Theorem 2.1 of Wardowski [17].

3 Fixed point results for multivalued mappings

The fixed point theory of multivalued contraction mappings using the Hausdorff metric was initiated by Nadler [13], who extended the Banach contraction principle to multivalued mappings. Since then many authors have studied fixed points for multivalued mappings. The theory of multivalued mappings has many applications in control theory, convex optimization, differential equations and economics. Recently, Sgroi and Vetro have extended the concept of F-contraction for multivalued mapping and they proved the following theorem in [19].

Theorem 3.1 [19] *Let (X,d) be a complete metric space and $T : X \to \mathrm{CB}(X)$. If there exist a mapping $F \in \mathcal{F}$, $\tau > 0$ and real numbers $\alpha, \beta, \gamma, \delta, L \geq 0$ such that*

$$2\tau + F\big(H(Tx, Ty)\big) \leq F\big(\alpha d(x,y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta\big(d(x, Ty) + L d(y, Tx)\big)\big)$$

for all $x, y \in X$, with $Tx \neq Ty$, where $\alpha + \beta + \gamma + 2L = 1$ and $\gamma \neq 1$, then T has a unique fixed point.

In the present section, we recall the concept of F-contractions for multivalued mappings and prove a Suzuki-Hardy-Rogers-type fixed point theorem for such contractions. Our new result generalizes and improves Sgroi and Vetro's fixed point theorem, Nadler's fixed point theorem and the Banach contraction principle.

Theorem 3.2 *Let (X,d) be a metric space and let $T : X \to \mathrm{CB}(X)$ be a multivalued mapping. Assume that there exists a function $F \in \mathcal{F}$ which is continuous from right and $\tau \in \mathbb{R}^+$*

such that

$$\lambda d(x, Tx) \leq d(x, y) \tag{3.1}$$

implies

$$\begin{aligned} 2\tau + F\big(H(Tx, Ty)\big) \leq F\big(&a_1 d(x, y) + a_2\big(d(x, Tx)\big) + a_3 d(y, Ty) \\ &+ a_4\big(d(x, Ty) + a_5 d(y, Tx)\big)\big) \end{aligned} \tag{3.2}$$

for all $x, y \in X$, $Tx \neq Ty$, *where* a_i, $i = 1, 2, 3, 4, 5$ *are nonnegative numbers and* $a_1 + a_2 + a_3 + 2a_4 = 1$ *and* $a_4 \neq 1$. *Here* $\frac{1-a_3-a_4}{1+a_1-a_4+a_5} = \lambda < 1$. *Then* T *has a fixed point.*

Proof Let $x_0 \in X$ be an arbitrary point of X and choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T and the proof is completed. Assume that $x_1 \notin Tx_1$, then $Tx_0 \neq Tx_1$. Now

$$\lambda d(x_0, Tx_0) \leq \lambda d(x_0, x_1) < d(x_0, x_1).$$

From the assumption, we have

$$\begin{aligned} 2\tau + F\big(H(Tx_0, Tx_1)\big) &\leq F\big(a_1 d(x_0, x_1) + a_2 d(x_0, Tx_0) \\ &\qquad + a_3 d(x_1, Tx_1) + a_4 d(x_0, Tx_1) + a_5 d(x_1, Tx_0)\big) \\ &\leq F\big(a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, Tx_1) \\ &\qquad + a_4 d(x_0, Tx_1) + a_5 d(x_1, x_1)\big) \\ &= F\big((a_1 + a_2 + a_4) d(x_0, x_1) + (a_3 + a_4) d(x_1, Tx_1)\big). \end{aligned}$$

As F is continuous from the right, there exists a real number $h > 1$ such that

$$F\big(hH(Tx_0, Tx_1)\big) \leq F\big(H(Tx_0, Tx_1)\big) + \tau.$$

Now, from

$$d(x_1, Tx_1) \leq H(Tx_0, Tx_1) < hH(Tx_0, Tx_1)$$

we deduce that there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq hH(Tx_0, Tx_1).$$

Consequently, we get

$$F\big(d(x_1, x_2)\big) \leq F\big(hH(Tx_0, Tx_1)\big) < F\big(H(Tx_0, Tx_1)\big) + \tau,$$

which implies that

$$\begin{aligned} 2\tau + F\big(d(x_1, x_2)\big) &\leq 2\tau + F\big(H(Tx_0, Tx_1)\big) + \tau \\ &\leq F\big(a_1 d(x_0, x_1) + a_2 d(x_0, Tx_0) \end{aligned}$$

$$+ a_3 d(x_1, Tx_1) + a_4 d(x_0, Tx_1) + a_5 d(x_1, Tx_0)\big) + \tau.$$

Thus

$$\tau + F\big(d(x_1, x_2)\big) \le F\big((a_1 + a_2 + a_4)d(x_0, x_1) + (a_3 + a_4)d(x_1, x_2)\big). \tag{3.3}$$

Since F is strictly increasing, we deduce

$$d(x_1, x_2) < (a_1 + a_2 + a_4)d(x_0, x_1) + (a_3 + a_4)d(x_1, x_2),$$

and hence

$$d(x_1, x_2) < \left(\frac{a_1 + a_2 + a_4}{1 - a_3 - a_4}\right)d(x_0, x_1) = d(x_0, x_1).$$

Consequently, from (3.3) we have

$$\tau + F\big(d(x_1, x_2)\big) < F\big(d(x_0, x_1)\big).$$

Continuing in this manner, we can define a sequence $\{x_n\} \subset X$ such that $x_n \notin Tx_n$, $x_{n+1} \in Tx_n$ and

$$\tau + F\big(d(Tx_{n-1}, Tx_n)\big) < F\big(d(x_{n-1}, x_n)\big) \tag{3.4}$$

for all $n \in \mathbb{N} \cup \{0\}$.Therefore

$$F\big(d(x_n, x_{n+1})\big) < F\big(d(x_{n-1}, x_n)\big) - \tau < F\big(d(x_{n-2}, x_{n-1})\big) - 2\tau < \cdots \le F\big(d(x_0, x_1)\big) - n\tau \tag{3.5}$$

for all $n \in \mathbb{N}$. Since $F \in \mathcal{F}$, so by taking limit as $n \to \infty$ in (3.5), we have

$$\lim_{n\to\infty} F\big(d(x_n, x_{n+1})\big) = -\infty \quad \Longleftrightarrow \quad \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.6}$$

Now, from (F3), there exists $0 < k < 1$ such that

$$\lim_{n\to\infty} \big[d(x_n, x_{n+1})\big]^k F\big(d(x_n, x_{n+1})\big) = 0. \tag{3.7}$$

By (3.5), we have

$$\begin{aligned} &d(x_n, x_{n+1})^k F\big(d(x_n, x_{n+1})\big) - d(x_n, x_{n+1})^k F\big(d(x_0, x_1)\big) \\ &\quad < d(x_n, x_{n+1})^k \big[F\big(d(x_0, x_1) - n\tau\big) - F\big(d(x_0, x_1)\big)\big] \\ &\quad = -n\tau\big[d(x_n, x_{n+1})\big]^k \le 0. \end{aligned} \tag{3.8}$$

By taking limit as $n \to \infty$ in (3.8) and applying (3.6) and (3.7), we have

$$\lim_{n\to\infty} n\big[d(x_n, x_{n+1})\big]^k = 0. \tag{3.9}$$

It follows from (3.9) that there exists $n_1 \in \mathbb{N}$ such that

$$n\big[d(x_n, x_{n+1})\big]^k \leq 1 \tag{3.10}$$

for all $n > n_1$. This implies

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/k}} \tag{3.11}$$

for all $n > n_1$. Now we prove that $\{x_n\}$ is a Cauchy sequence. For $m > n > n_1$, we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}}. \tag{3.12}$$

Since, $0 < k < 1$, then $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ converges. Therefore, $d(x_n, x_m) \to 0$ as $m, n \to \infty$. Thus $\{x_n\}$ is a Cauchy sequence. Completeness of X ensures that there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. If there exists an increasing sequence $\{n_k\} \subset \mathbb{N}$ such that $x_{n_k} \in Tz$ for all $k \in \mathbb{N}$ since Tz is closed and $x_{n_k} \to z$, we get $z \in Tz$ and the proof is completed. So we can assume that there exists $n_0 \in \mathbb{N}$ such that $x_n \notin Tz$ for all $n_0 \in \mathbb{N}$ with $n \geq n_0$. Then we assume that $Tx_{n-1} \neq Tz$ for all $n \geq n_0$. Now we show that

$$\lambda d(z, Tx) \leq d(z, x)$$

for all $x \in X \backslash \{z\}$. Since $x_n \to z$, so there exists $n_0 \in \mathbb{N}$ such that

$$d(z, x_n) \leq \frac{1}{3} d(z, x)$$

for all $n \in \mathbb{N}$ with $n \geq n_0$. Then we have

$$\begin{aligned} \lambda d(x_n, Tx_n) &< d(x_n, Tx_n) \leq d(x_n, x_{n+1}) \\ &\leq d(x_n, z) + d(z, x_{n+1}) \\ &\leq \frac{2}{3} d(x, z) = d(x, z) - \frac{1}{3} d(x, z) \\ &\leq d(x, z) - d(z, x_n) \leq d(x, x_n). \end{aligned}$$

Thus, by assumption, we get

$$\begin{aligned} 2\tau + F\big(H(Tx_n, Tx)\big) &\leq F\big(a_1 d(x_n, x) + a_2 d(x_n, Tx_n) + a_3 d(x, Tx) \\ &\quad + a_4 d(x_n, Tx) + a_5 d(x, Tx_n)\big) \\ &\leq F\big(a_1 d(x_n, x) + a_2 d(x_n, x_{n+1}) + a_3 d(x, Tx) \\ &\quad + a_4 d(x_n, Tx) + a_5 d(x, x_{n+1})\big). \end{aligned}$$

Since F is continuous from the right, so there exists a real number $h > 1$ such that

$$F\big(hH(Tx_n, Tx)\big) < F\big(H(Tx_n, Tx)\big) + \tau.$$

Now, from

$$d(x_{n+1}, Tx) \le H(Tx_n, Tx) < hH(Tx_n, Tx)$$

we get

$$F\big(d(x_{n+1}, Tx)\big) \le F\big(hH(Tx_n, Tx)\big) < F\big(H(Tx_n, Tx)\big) + \tau.$$

Thus we have

$$\begin{aligned} 2\tau + F\big(d(x_{n+1}, Tx)\big) &\le 2\tau + F\big(H(Tx_n, Tx)\big) + \tau \\ &\le F\big(a_1 d(x_n, x) + a_2 d(x_n, x_{n+1}) + a_3 d(x, Tx) \\ &\quad + a_4 d(x_n, Tx) + a_5 d(x, x_{n+1})\big) + \tau. \end{aligned}$$

Since F is strictly increasing, we have

$$d(x_{n+1}, Tx) < a_1 d(x_n, x) + a_2 d(x_n, x_{n+1}) + a_3 d(x, Tx) + a_4 d(x_n, Tx) + a_5 d(x, x_{n+1}).$$

Letting n tend to ∞, we obtain

$$\begin{aligned} d(z, Tx) &\le a_1 d(z, x) + a_3 d(x, Tx) + a_4 d(z, Tx) + a_5 d(x, z) \\ &\le \left(\frac{a_1 + a_3 + a_5}{1 - a_3 - a_4}\right) d(z, x) \end{aligned}$$

for all $x \in X \setminus \{z\}$. We prove that

$$2\tau + F\big(H(Tz, Tx)\big) \le F\big(a_1 d(x, z) + a_2 d(x, Tx) + a_3 d(z, Tz) + a_4 d(x, Tz) + a_5 d(z, Tx)\big)$$

for all $x \in X$. Then, for every $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that

$$d(z, y_n) \le d(z, Tx) + \frac{1}{n} d(z, x).$$

So we have the following:

$$\begin{aligned} d(x, Tx) &\le d(x, y_n) \\ &\le d(x, z) + d(z, y_n) \\ &\le d(x, z) + d(z, Tx) + \frac{1}{n} d(z, x) \\ &\le d(x, z) + \frac{a_1 + a_3 + a_5}{1 - a_3 - a_4} d(z, x) + \frac{1}{n} d(z, x) \\ &= \left(1 + \frac{a_1 + a_3 + a_5}{1 - a_3 - a_4} + \frac{1}{n}\right) d(x, z) \end{aligned}$$

for all $n \in \mathbb{N}$, and hence $\lambda d(x, Tx) \le d(x, z)$. Thus, by assumption, we get

$$2\tau + F\big(H(Tx, Tz)\big) \le F\big(a_1 d(x, z) + a_2 d(x, Tx) + a_3 d(z, Tz) + a_4 d(x, Tz) + a_5 d(z, Tx)\big).$$

Taking $x = x_{n+1}$, we have

$$\begin{aligned}2\tau + F\bigl(d(x_{n+1}, Tz)\bigr) &\leq 2\tau + F\bigl(H(Tx_n, Tz)\bigr)\\ &\leq F\bigl(a_1 d(x_n, z) + a_2 d(x_n, Tx_n) + a_3 d(z, Tz)\\ &\quad + a_4 d(x_n, Tz) + a_5 d(z, Tx_n)\bigr).\end{aligned}$$

Since F is strictly increasing, we have

$$d(x_{n+1}, Tz) < a_1 d(x_n, z) + a_2 d(x_n, Tx_n) + a_3 d(z, Tz) + a_4 d(x_n, Tz) + a_5 d(z, Tx_n).$$

Letting $n \to +\infty$, we get

$$d(z, Tz) \leq (a_3 + a_4) d(z, Tz)$$

as $a_3 + a_4 < 1$. Thus we get $d(z, Tz) = 0$. Since Tz is closed, we obtain $z \in Tz$. Thus z is a fixed point of T. □

Corollary 3.3 *Let (X, d) be a metric space and let $T : X \to \mathrm{CB}(X)$ be a multivalued mapping. Assume that there exists a function $F \in \digamma$ that is continuous from right and $\tau \in \mathbb{R}^+$ such that*

$$\beta d(x, Tx) \leq d(x, y)$$

implies

$$2\tau + F\bigl(H(Tx, Ty)\bigr) \leq F\bigl(r_1 d(x, y) + r_2\bigl(d(x, Ty) + r_3 d(y, Tx)\bigr)\bigr)$$

for all $x, y \in X$, $Tx \neq Ty$, where a_i, $i = 1, 2, 3$ are nonnegative numbers and $r_1 + 2r_2 = 1$ and $r_2 \neq 1$. Here $\frac{1-r_2}{1+r_1-r_2+r_3} = \beta < 1$. Then T has a fixed point.

Proof By taking $a_2 = a_3 = 0$ in previous result. □

Now we present the following example which illustrates our results.

Example 3.4 Let $X = [0, 1]$, $T : X \to \mathrm{CB}(X)$ be defined as $Tx = [0, \frac{x}{4}]$ and d be the usual metric on X. Taking $F(t) = \ln(t) + t$ for all $t \in \mathbb{R}^+$ and $\tau = \ln(\sqrt{2})$. Without loss of generality, we take $x < y$. Then, for all $x, y \in X$, $d(Tx, Ty) > 0$ and $d(x, y) > 0$. Now

$$\lambda d(x, Tx) = 0 < d(x, y)$$

implies that

$$\begin{aligned}2\tau + F\bigl(H(Tx, Ty)\bigr) &= \ln(2) + \ln\bigl(H(Tx, Ty)\bigr) = \ln(2) + \ln\left(\frac{1}{4}|y - x|\right) + \frac{1}{4}|y - x|\\ &\leq \ln(2) + \ln\left(\frac{3}{8}|y - x|\right) + \frac{3}{4}|y - x|\end{aligned}$$

$$
\begin{aligned}
&\le \ln(2) + \ln\left(\frac{1}{2}\right) + \ln\left(\frac{3}{4}|y-x|\right) + \frac{3}{4}|y-x| \\
&= \ln\left(\frac{1}{2}|y-x| + \frac{1}{8}|y-x| + \frac{1}{8}|y-x|\right) \\
&\quad + \left(\frac{1}{2}|y-x| + \frac{1}{8}|y-x| + \frac{1}{8}|y-x|\right) \\
&\le \ln\left(\frac{1}{2}|y-x| + \frac{1}{4}\left|y-\frac{x}{2}\right| + \frac{1}{4}\left|x-\frac{y}{2}\right|\right) \\
&\quad + \left(\frac{1}{2}|y-x| + \frac{1}{4}\left|y-\frac{x}{2}\right| + \frac{1}{4}\left|x-\frac{y}{2}\right|\right) \\
&= F\bigl(a_1 d(x,y) + a_2\bigl(d(x,Ty) + a_3 d(y,Tx)\bigr)\bigr),
\end{aligned}
$$

where $a_1 + 2a_2 = 1$ and $a_2 \neq 1$. Thus all conditions of the above corollary are satisfied and 0 is a fixed point of T.

Now we prove a new fixed point theorem for Kannan-type multivalued F-contractions, which is a generalization of the results of Beg and Azam [6].

Theorem 3.5 *Let (X,d) be a complete metric space and let $T : X \to \mathrm{CB}(X)$. Assume that there exist a function $F \in \digamma$ which is continuous from right, $\tau > 0$ and $\varphi_i : \mathbb{R} \to [0,1)$ ($i = 1,2$) such that*

$$
2\tau + F\bigl(H(Tx,Ty)\bigr) \le F\bigl(\varphi_1\bigl(d(x,Tx)\bigr)d(x,Tx) + \varphi_2\bigl(d(y,Ty)\bigr)d(y,Ty)\bigr) \tag{3.13}
$$

for all $x, y \in X$, with $Tx \neq Ty$, where $\varphi_1(d(x,Tx)) + \varphi_2(d(y,Ty)) = 1$. Then T has a fixed point.

Proof Let $x_0 \in X$ be an arbitrary point of X and choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T and the proof is completed. Assume that $x_1 \notin Tx_1$, then $Tx_0 \neq Tx_1$. From (3.13), we have

$$
\begin{aligned}
2\tau + F\bigl(H(Tx_0,Tx_1)\bigr) &\le F\bigl(\varphi_1\bigl(d(x_0,Tx_0)\bigr)d(x_0,Tx_0) + \varphi_2\bigl(d(x_1,Tx_1)\bigr)d(x_1,Tx_1)\bigr) \\
&\le F\bigl(\varphi_1\bigl(d(x_0,x_1)\bigr)d(x_0,x_1) + \varphi_2\bigl(d(x_1,x_2)\bigr)d(x_1,x_2)\bigr).
\end{aligned}
$$

As F is continuous from the right, there exists a real number $h > 1$ such that

$$
F\bigl(hH(Tx_0,Tx_1)\bigr) \le F\bigl(H(Tx_0,Tx_1)\bigr) + \tau.
$$

Now, from

$$
d(x_1,Tx_1) \le H(Tx_0,Tx_1) < hH(Tx_0,Tx_1),
$$

we deduce that there exists $x_2 \in Tx_1$ such that

$$
d(x_1,x_2) \le hH(Tx_0,Tx_1).
$$

Consequently, we get

$$
F\bigl(d(x_1,x_2)\bigr) \le F\bigl(hH(Tx_0,Tx_1)\bigr) < F\bigl(H(Tx_0,Tx_1)\bigr) + \tau,
$$

which implies that

$$\begin{aligned} 2\tau + F\big(d(x_1,x_2)\big) &\leq 2\tau + F\big(H(Tx_0, Tx_1)\big) + \tau \\ &\leq F\big(\varphi_1\big(d(x_0,x_1)\big)d(x_0,x_1) + \varphi_2\big(d(x_1,x_2)\big)d(x_1,x_2)\big) + \tau. \end{aligned}$$

Thus

$$\tau + F\big(d(x_1,x_2)\big) \leq F\big(\varphi_1\big(d(x_0,x_1)\big)d(x_0,x_1) + \varphi_2\big(d(x_1,x_2)\big)d(x_1,x_2)\big).$$

Since F is strictly increasing, we deduce

$$d(x_1,x_2) < \varphi_1\big(d(x_0,x_1)\big)d(x_0,x_1) + \varphi_2\big(d(x_1,x_2)\big)d(x_1,x_2),$$

and hence

$$d(x_1,x_2) < \frac{\varphi_1(d(x_0,x_1))}{1-\varphi_2(d(x_1,x_2))}d(x_0,x_1) = d(x_0,x_1).$$

Consequently,

$$\tau + F\big(d(x_1,x_2)\big) \leq F\big(d(x_0,x_1)\big).$$

Continuing in this manner, we can define a sequence $\{x_n\} \subset X$ such that $x_n \notin Tx_n$, $x_{n+1} \in Tx_n$ and

$$\tau + F\big(d(Tx_{n-1}, Tx_n)\big) \leq F\big(d(x_{n-1},x_n)\big)$$

for all $n \in \mathbb{N} \cup \{0\}$. Proceeding as in the proof of Theorem 3.2, we obtain that $\{x_n\}$ is a Cauchy sequence. Since X is a complete space, so there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. If there exists an increasing sequence $\{n_k\} \subset \mathbb{N}$ such that $x_{n_k} \in Tz$ for all $k \in \mathbb{N}$, since Tz is closed and $x_{n_k} \to z$, we get $z \in Tz$ and the proof is completed. So we can assume that there exists $n_0 \in \mathbb{N}$ such that $x_n \notin Tz$ for all $n_0 \in \mathbb{N}$ with $n \geq n_0$. Then we assume that $Tx_{n-1} \neq Tz$ for all $n \geq n_0$. Thus, by assumption, we have

$$\begin{aligned} 2\tau + F\big(d(x_{n+1}, Tz)\big) &\leq 2\tau + F\big(H(Tx_n, Tz)\big) \\ &\leq F\big(\varphi_1\big(d(x_n, Tx_n)\big)d(x_n, Tx_n) + \varphi_2\big(d(z, Tz)\big)\big(d(z, Tz)\big)\big). \end{aligned}$$

Since F is strictly increasing, we have

$$d(x_{n+1}, Tz) < \varphi_1\big(d(x_n, Tx_n)\big)d(x_n, Tx_n) + \varphi_2\big(d(z, Tz)\big)\big(d(z, Tz)\big).$$

Letting $n \to +\infty$, we get

$$d(z, Tz) \leq \varphi_2\big(d(z, Tz)\big)\big(d(z, Tz)\big)$$

as $\varphi_2(d(z, Tz)) < 1$. Thus we get $d(z, Tz) = 0$. Since Tz is closed, we obtain $z \in Tz$. Thus z is a fixed point of T, and hence the proof is completed. □

4 Conlusion

Wardowski [17] very recently exploited the idea of F-contraction and proved a significant result concerning the existence of fixed points for such contractions in complete metric spaces. We continue his investigations and define two new classes of functions $M(S,T)$ and $N(S,T)$. In the present project, some unique common fixed point theorems for single-valued mappings and fixed point theorems of multivalued mappings under generalized contractive conditions in a complete metric space (X,d) have been discussed. All the main results in this article are of some value for solving problems in complete metric spaces. Our results may be the motivation to other authors to extend and improve these results to be suitable tools for their applications.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan.
[2]Department of Mathematical Sciences, UAE University, Al Ain, 15551, UAE.

Acknowledgements
The authors thank the editors and the referees for their valuable comments and suggestions which improved greatly the quality of this paper. The second author (the corresponding one) acknowledges with thanks the Research Affairs at the UAEU for their financial support. This project is partially supported by the Research Affairs at UAEU, No. COS/IRG-14/13: 215070.

Received: 17 December 2014 Accepted: 17 May 2015 Published online: 06 June 2015

References

1. Acar, Ö, Altun, I: A fixed point theorem for multivalued mappings with δ-distance. Abstr. Appl. Anal. **2014**, Article ID 497092 (2014)
2. Abbas, M, Ali, B, Romaguera, S: Fixed and periodic points of generalized contractions in metric spaces. Fixed Point Theory Appl. **2013**, 243 (2013)
3. Ahmad, A, Al-Rawashdeh, A, Azam, A: Fixed point results for $\{\alpha,\xi\}$-expansive locally contractive mappings. J. Inequal. Appl. **2014**, 364 (2014)
4. Batra, R, Vashistha, S: Fixed points of an F-contraction on metric spaces with a graph. Int. J. Comput. Math. **91**, 1-8 (2014)
5. Batra, R, Vashistha, S, Kumar, R: A coincidence point theorem for F-contractions on metric spaces equipped with an altered distance. J. Math. Comput. Sci. **4**(5), 826-833 (2014)
6. Beg, I, Azam, A: Fixed points of asymptotically regular multivalued mappings. Bull. Aust. Math. Soc. **53**, 313-326 (1992)
7. Cosentino, M, Vetro, P: Fixed point results for F-contractive mappings of Hardy-Rogers-type. Filomat **28**(4), 715-722 (2014)
8. An, TV, Dung, NV, Kadelburg, Z, Radenović, S: Various generalizations of metric spaces and fixed point theorems. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **109**, 175-198 (2015)
9. Dung, NV, Hang, VTL: A fixed point theorem for generalized F-contractions on complete metric spaces. Vietnam J. Math. (2015). doi:10.1007/s10013-015-0123-5
10. Hussain, N, Salimi, P: Suzuki-Wardowski type fixed point theorems for α-GF-contractions. Taiwan. J. Math. (2014). doi:10.11650/tjm.18.2014.4462
11. Hussain, N, Ahmad, J, Azam, A: On Suzuki-Wardowski type fixed point theorems. J. Nonlinear Sci. Appl. (in press)
12. Klim, D, Wardowski, D: Fixed points of dynamic processes of set-valued F-contractions and application to functional equations. Fixed Point Theory Appl. **2015**, 22 (2015)
13. Nadler, SB Jr.: Multi-valued contraction mappings. Pac. J. Math. **30**, 475-488 (1969)
14. Shatanawi, W, Rajić, V, Radenović, S, Al-Rawashdeh, A: Mizoguchi-Takahashi-type theorems in tvs-cone metric space. Fixed Point Theory Appl. **2012**, 106 (2012)
15. Shatanawi, W, Al-Rawashdeh, A: Common fixed points of almost generalized (ψ,ϕ)-contractive mappings in ordered metric spaces. Fixed Point Theory Appl. **2012**, 80 (2012)
16. Shatanawi, W, Al-Rawashdeh, A, Aydi, H, Nashine, HK: On a fixed point for generalized contractions in generalized metric spaces. Abstr. Appl. Anal. **2012**, Article ID 246085 (2012)
17. Wardowski, D: Fixed points of a new type of contractive mappings in complete metric spaces. Fixed Point Theory Appl. **2012**, 94 (2012)
18. Wardowski, D, Van Dung, N: Fixed points of F-weak contractions on complete metric space. Demonstr. Math. **XLVII**(1), 146-155 (2014)
19. Sgroi, M, Vetro, C: Multi-valued F-contractions and the solution of certain functional and integral equations. Filomat **27**(7), 1259-1268 (2013)
20. Secelean, NA: Iterated function systems consisting of F-contractions. Fixed Point Theory Appl. **2013**, 277 (2013)