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Attractive points and convergence theorems for normally generalized hybrid mappings in CAT(0) spaces

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Abstract

We prove Δ-convergence theorems of Mann type to the set of attractive points of normally generalized hybrid mappings in CAT(0) spaces. Consequently, our main result can be applied to the result of Takahashi, Wong and Yao (Journal of Nonlinear and Convex Analysis 15:1087-1103, 2014, Theorem 5.1).

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1 Introduction

In 2011, Takahashi and Takeuchi [1] introduced the notion of attractive points for nonlinear mappings in a Hilbert space: Let H be a Hilbert space and C be a nonempty subset of H. Let T be a mapping from C into H. Let $A(T)$ denote the set of all attractive points of T, *i.e.*

$$A(T) = \{z \in H : \|z - Ty\| \leq \|z - y\|, \forall y \in H\}.$$

In 2012, Takahashi *et al.* [2] introduced the class of normally generalized hybrid mappings in a Hilbert space which contains the class of generalized hybrid mappings and the class of contractive mappings.

Definition 1.1 A mapping $T : C \longrightarrow H$ is called normally generalized hybrid if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

(1) $\alpha + \beta + \gamma + \delta \geq 0$;
(2) $\alpha + \beta > 0$ or $\alpha + \gamma > 0$; and
(3) $\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0$, $\forall x, y \in C$.

Such a mapping T can be called an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping.

The authors also proved the weak convergence theorems of Mann type for normally generalized hybrid mappings in Hilbert spaces without convexity assumption on the domain of mappings.

Theorem 1.2 ([2], Theorem 5.1) *Let H be a Hilbert space and C be a convex subset of H. Let $T : C \to C$ be a normally generalized hybrid mapping with $A(T) \neq \emptyset$. Let $P_{A(T)}$ be the metric projection of H onto $A(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n$, $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to v in $A(T)$, where $v = \lim_{n\to\infty} P_{A(T)}x_n$.*

In 2013, Kakavandi [3] introduced the so-called half-space topology such that convergence in this topology is equivalent to Δ-convergence for any sequence in a complete CAT(0) space. The author's results answered positively open questions in [4]. Moreover, the concepts of $(\mathbb{S})$ property and $(\overline{Q_4})$ condition are also introduced in a complete CAT(0) space.

In this paper, motivated by [2], we consider the concept of attractive points in CAT(0) spaces and prove $\triangle$-convergence theorems of Mann type for normally generalized hybrid mappings in such spaces satisfying the $(\mathbb{S})$ property and the $(\overline{Q_4})$ condition.

2 Preliminaries

Let (X, d) be a metric space and C be a nonempty subset of X. A mapping $T : C \longrightarrow X$ is called nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in X.$$

In 2008, Kohsaka and Takahashi [5] introduced the class of nonspreading mappings in a Hilbert space H, *i.e.* a mapping $T : C \longrightarrow H$ is called nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In 2010, Takahashi [6] and Kocourek *et al.* [7] introduced the wider class of nonspreading mappings in Hilbert spaces as follows. A mapping $T : C \longrightarrow H$ is called hybrid [6] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

A mapping $T : C \longrightarrow H$ is called an (α, β)-generalized hybrid [7] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2, \quad \forall x, y \in C.$$

We can see that $(1, 0)$, $(2, 1)$ and $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mappings are nonexpansive mappings, nonspreading mappings and hybrid mappings, respectively. Moreover, if $\alpha + \beta = -\gamma - \delta = 1$, then an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping is a generalized hybrid mapping.

In 2003, Takahashi and Toyoda [8] proved the following result.

Lemma 2.1 ([8]) *Let C be a nonempty closed convex subset of a Hilbert space H. Let P be the metric projection from H onto C. Let $\{u_n\}$ be a sequence in H. If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for all $u \in C$ and $n \in \mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0 \in C$.*

The following result is obtained by Takahashi *et al.* [2].

Lemma 2.2 ([2]) *Let C be a nonempty subset of a real Hilbert space H. Let $T : C \to C$ be an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping. If $\{x_n\}$ converges weakly to z and $x_n - Tx_n \to 0$, then $z \in A(T)$.*

It is worth mentioning that the related recent results are obtained in [9, 10] and [11].

We recall the concept of Banach limit which plays a major role in our results. Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of the dual metric space $(l^\infty)^*$ of the space l^∞. We denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. We sometimes denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. Moreover, a mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$ and we know that there exists a Banach limit on l^∞ (see [12] for more details). If μ is a Banach limit on l^∞, then for any $(x_1, x_2, x_3, \ldots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

Let (X, d) be a metric space. A geodesic path (or geodesic) joining x to y in X is a map c from a closed interval $[0, l] \subseteq \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(s), c(t)) = |s - t|$ for all $s, t \in [0, l]$. In particular, the mapping c is an isometry and $d(x, y) = l$. The image of c is called a geodesic segment joining x and y when it is unique and denoted by $[x, y]$. We denote the unique point $z \in [x, y]$ such that $d(x, z) = \alpha d(x, y)$ and $d(y, z) = (1 - \alpha)d(x, y)$ by $(1 - \alpha)x \oplus \alpha y$, where $0 \leq \alpha \leq 1$.

The metric space (X, d) is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of points (the edges of $\triangle$). A comparison triangle for $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$.

A geodesic space X is called a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom:

Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle}$ be a comparison triangle in $\mathbb{R}^2$. Then the triangle $\triangle$ is said to satisfy the CAT(0) inequality if

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}),$$

for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$.

It is well known that any complete simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Hilbert spaces, pre-Hilbert spaces, $\mathbb{R}$-trees (see [13]), Euclidean buildings (see [14]), the complex Hilbert ball with a hyperbolic metric (see [15]), and many others.

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment y_1, y_2, then the CAT(0) inequality implies the so-called (CN) inequality of Bruhat and Tits [16], *i.e.*

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

Moreover, a uniquely geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality (see [13] for more details).

The following results are important to our theorems.

Lemma 2.3 ([13]) *Let X be a* CAT(0) *space, $x_1, x_2, y_1, y_2 \in X$ and $\alpha \in [0,1]$. Then*

$$d\bigl(\alpha x_1 \oplus (1-\alpha)y_1, \alpha x_2 \oplus (1-\alpha)y_2\bigr) \le \alpha d(x_1, x_2) + (1-\alpha)d(y_1, y_2).$$

From the above lemma, it is easy to see that for $x, y, z \in X$ and $\alpha \in [0,1]$,

$$d\bigl(\alpha x \oplus (1-\alpha)y, z\bigr) \le \alpha d(x,z) + (1-\alpha)d(y,z).$$

Lemma 2.4 ([17]) *Let X be a* CAT(0) *space, $x, y, z \in X$ and $\alpha \in [0,1]$. Then*

$$d^2\bigl(\alpha x \oplus (1-\alpha)y, z\bigr) \le \alpha d^2(x,z) + (1-\alpha)d^2(y,z) - \alpha(1-\alpha)d^2(x,y).$$

In 2008, Kirk and Panyanak [4] specialized Lim's concept [18] of $\triangle$-convergence in a general metric spaces to CAT(0) spaces and showed that many results in Banach spaces involving weak convergence have precise analogs in this setting.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space (X, d). For $x \in X$, we set

$$r\bigl(x, \{x_n\}\bigr) := \limsup_n d(x, x_n).$$

The asymptotic radius of $\{x_n\}$ is given by

$$r\bigl(\{x_n\}\bigr) := \inf_{x\in X} r\bigl(x, \{x_n\}\bigr).$$

The asymptotic center of $\{x_n\}$ is given by

$$A\bigl(\{x_n\}\bigr) := \bigl\{x \in X : r\bigl(x, \{x_n\}\bigr) = r\bigl(\{x_n\}\bigr)\bigr\}.$$

It is known from [19] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\} \subset X$ is said to $\triangle$-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim_{n\to\infty} x_n = x$. The uniqueness of asymptotic center implies that the CAT(0) space X satisfies Opial's property, *i.e.* for given $\{x_n\} \subset X$ such that $\{x_n\}$ $\triangle$-converges to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

We also know the following results.

Lemma 2.5 ([4]) *Every bounded sequence in a complete* CAT(0) *space always has a $\triangle$-convergent subsequence.*

Lemma 2.6 ([20]) *If C is a closed convex subset of a complete* CAT(0) *space (X, d) and $\{x_n\}$ is a bounded sequence in C, then $A(\{x_n\}) \in C$.*

It is well known that a norm linear space satisfies the CAT(0) inequality if and only if it satisfies the parallelogram identity, *i.e.* it is a pre-Hilbert space; hence it is usual to have an inner product-like notion in complete CAT(0) spaces. In 2008, Berg and Nikolaev [21] introduced the concept of quasilinearization along these lines.

Let $\overrightarrow{ab}$ denote a pair $(a,b) \in X \times X$, and it is called a vector. Then the quasilinearization mapping $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}, \overrightarrow{uv} \rangle = \frac{1}{2}\bigl(d^2(a,v) + d^2(b,u) - d^2(a,u) - d^2(b,v)\bigr)$$

for any $a,b,u,v \in X$.

We say that (X,d) satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{uv} \rangle \le d(a,b)d(u,v) \quad \text{for all } a,b,u,v \in X.$$

The following lemma was also proved in [21].

Lemma 2.7 ([21]) *A geodesically connected metric space is a* CAT(0) *space if and only if it satisfies the Cauchy-Schwarz inequality.*

Consider the map $\Theta : \mathbb{R} \times X \times X \longrightarrow C(X;\mathbb{R})$ defined by

$$\Theta(t,a,b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle \quad \text{for all } x \in X,$$

where $C(X;\mathbb{R})$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwarz inequality implies that $\Theta(t,a,b)$ is the Lipschitz function with the Lipschitz semi-norm $L(\Theta(t,a,b)) = td(a,b)$ $(t \in \mathbb{R}, a,b \in X)$, where $L(\varphi) = \sup\{\frac{\varphi(x)-\varphi(y)}{d(x,y)} : x,y \in X, x \neq y\}$ is the Lipschitz semi-norm for any function $\varphi : X \longrightarrow \mathbb{R}$.

In 2010, Kakavandi and Amini [22] defined a pseudometric D on $\mathbb{R} \times X \times X$ by

$$D\bigl((t,a,b),(s,u,v)\bigr) = L\bigl(\Theta(t,a,b) - \Theta(s,u,v)\bigr).$$

Lemma 2.8 ([22]) *Let X be a complete* CAT(0) *space. Let $(t,a,b),(s,u,v) \in \mathbb{R} \times X \times X$. Then $D((t,a,b),(s,u,v)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{uv}, \overrightarrow{xy} \rangle$ for all $x,y \in X$.*

Therefore, D defines an equivalent relation on $\mathbb{R} \times X \times X$, where the equivalent class of (t,a,b) is

$$[t\overrightarrow{ab}] = \bigl\{ s\overrightarrow{uv} : t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{uv}, \overrightarrow{xy} \rangle, \forall x,y \in X \bigr\}.$$

Let $X^* = \{[t\overrightarrow{ab}] : (t,a,b) \in \mathbb{R} \times X \times X\}$. Then (X^*,D) is a metric space and it is called the dual metric space of (X,d).

In 2013, Kakavandi [3] introduced the concept of $(\mathbb{S})$ property for a complete CAT(0) space as follows.

Definition 2.9 ([3]) A complete CAT(0) space (X,d) satisfies the $(\mathbb{S})$ property if for any $(x,y) \in X \times X$ there exists a point $y_x \in X$ such that $[\overrightarrow{xy}] = [\overrightarrow{y_x x}]$.

There are many CAT(0) spaces satisfying the ($\mathbb{S}$) property; for example, Hilbert spaces and symmetric Hadamard manifolds including hyperbolic spaces [3]. Moreover, Kakavandi also proved the characterization of $\triangle$-convergence for CAT(0) spaces satisfying the ($\mathbb{S}$) property as follows.

Lemma 2.10 ([3]) *Let (X,d) be a complete* CAT(0) *space, $\{x_n\}$ be a bounded sequence in X and $x \in X$. If X satisfies the ($\mathbb{S}$) property, then $\triangle\text{-}\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle = 0$ for all $y \in X$.*

In 2008, Kirk and Panyanak [4] introduced a geometric condition on CAT(0) spaces called the (Q_4) condition.

(Q_4) A CAT(0) space (X,d) is said to satisfy the (Q_4) condition if $\forall x,y,p,q \in X$

$$d(p,x) < d(x,q) \quad \text{and} \quad d(p,y) < d(y,q) \quad \text{imply} \quad d(p,m) < d(m,q), \quad \forall m \in [x,y].$$

The authors mentioned that this condition holds in many CAT(0) spaces including Hilbert spaces and $\mathbb{R}$-trees. Furthermore, in 2009, Espínola and Fernández-León [23] proved that any CAT(0) space of constant curvature satisfies the (Q_4) condition but any CAT(0) gluing space containing two spaces of constant but different curvatures fails to satisfy this condition.

In 2013, Kakavandi [3] modified the (Q_4) condition as follows.

$(\overline{Q_4})$ A CAT(0) space (X,d) is said to satisfy the $(\overline{Q_4})$ condition if for any $x,y,p,q \in X$,

$$d(p,x) \le d(x,q) \quad \text{and} \quad d(p,y) \le d(y,q) \quad \text{imply} \quad d(p,m) \le d(m,q), \quad \forall m \in [x,y].$$

We can see that Hilbert spaces, $\mathbb{R}$-trees and every CAT(0) space of constant curvature satisfy the $(\overline{Q_4})$ condition. Anyway, since $(\overline{Q_4})$ implies (Q_4), there are some complete CAT(0) spaces that do not satisfy such a condition.

3 Main results

In this section, we prove $\triangle$-convergence theorems of Mann type [24] to the set of attractive points of normally generalized hybrid mappings in CAT(0) spaces satisfying the ($\mathbb{S}$) property and the $(\overline{Q_4})$ condition. First of all, we consider the notion of the set of attractive points for any mapping $T : C \to X$, where X is a metric space and C is a nonempty subset of X defined as

$$A(T) = \{z \in X : d(z, Ty) \le d(z,y), \forall y \in C\}.$$

Moreover, in metric spaces, a normally generalized hybrid mapping is defined analogously to Definition 1.1 as follows:

A mapping $T : C \longrightarrow X$ is called normally generalized hybrid if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

(1) $\alpha + \beta + \gamma + \delta \ge 0$;
(2) $\alpha + \beta > 0$ or $\alpha + \gamma > 0$; and
(3) $\alpha d(Tx, Ty)^2 + \beta d(x, Ty)^2 + \gamma d(Tx, y)^2 + \delta d(x, y)^2 \le 0$, $\forall x, y \in C$.

Before proving our main theorem, we need the following lemmas which we proved in [25], but for the sake of completeness, we give a short proof.

Lemma 3.1 ([25]) *Let (X,d) be a complete* CAT(0) *space satisfying the $(\overline{Q_4})$ condition. Let C be a nonempty subset of X. Then, for any mapping $T: C \to X$, $A(T)$ is closed and convex.*

Proof Let $\{x_n\}$ be a sequence in $A(T)$ such that $x_n \longrightarrow x \in X$ as $n \longrightarrow \infty$.

For any $n \in \mathbb{N}$ and $y \in C$, we have $d(x_n, Ty) \leq d(x_n, y)$. Then

$$\lim_{n\to\infty} d(x_n, Ty) \leq \lim_{n\to\infty} d(x_n, y),$$
$$d(x, Ty) \leq d(x, y).$$

This implies that $A(T)$ is a closed subset of X. Let $z_1, z_2 \in A(T)$, then for any $y \in C$, we have $d(z_1, Ty) \leq d(z_1, y)$ and $d(z_2, Ty) \leq d(z_2, y)$. Since X satisfies the $(\overline{Q_4})$ condition, we have $d(m, Ty) \leq d(m, y)$ for all $m \in [z_1, z_2]$. This implies that $A(T)$ is a convex subset of X. Therefore, $A(T)$ is closed and convex. □

Lemma 3.2 ([25]) *Let C be a nonempty subset of a* CAT(0) *space X. Let $\{x_n\}$ be a bounded sequence in C. Let T be a mapping from C into itself such that $d(x_n, Tx_n) \longrightarrow 0$. Then*

(1) *the sequences $\{d(x_n, y)\}$ and $\{d(Tx_n, y)\}$ are bounded for all $y \in C$;*
(2) *$\mu_n d(x_n, y) = \mu_n d(Tx_n, y)$ for any Banach limit μ on l^∞.*

Proof Let $n \in \mathbb{N}$ and $y \in C$. By the triangle inequality, we have

$$d(x_n, y) \leq d(x_n, x_1) + d(x_1, y).$$

Since $\{x_n\}$ is bounded, the sequence $\{d(x_n, y)\}$ is bounded. Since $d(x_n, Tx_n) \longrightarrow 0$ and $\{d(x_n, y)\}$ is bounded, the sequence $\{d(Tx_n, y)\}$ is also bounded.

Let $y \in C$. Then

$$d(x_n, y) \leq d(x_n, Tx_n) + d(Tx_n, y) \tag{3.1}$$

and

$$d(Tx_n, y) \leq d(Tx_n, x_n) + d(x_n, y). \tag{3.2}$$

Since $d(x_n, Tx_n) \longrightarrow 0$, by applying a Banach limit μ to both sides of (3.1) and (3.2), we can conclude that

$$\mu_n d(x_n, y) = \mu_n d(Tx_n, y).$$ □

Next, we prove the analogous result to Lemma 2.2 in complete CAT(0) spaces satisfying the $(\mathbb{S})$ property.

Lemma 3.3 *Let (X,d) be a complete* CAT(0) *space satisfying the $(\mathbb{S})$ property. Let C be a nonempty subset of X and let $T: C \to C$ be an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid* *mapping. Let* $\{x_n\}$ *be a bounded sequence in* C *such that* $x_n \stackrel{\triangle}{\longrightarrow} z$ *and* $d(x_n, Tx_n) \longrightarrow 0$. *Then* $z \in A(T)$.

Proof Since T is an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping,

$$\alpha d^2(Tx, Ty) + \beta d^2(x, Ty) + \gamma d^2(Tx, y) + \delta d^2(x, y) \le 0, \quad \forall x, y \in C.$$

We have from Lemma 3.2 that $\{d(x_n, y)\}$ and $\{d(Tx_n, y)\}$ are bounded for all $y \in C$.

If $\alpha + \beta > 0$, then

$$\begin{aligned} &\alpha d^2(Tx_n, Ty) + \beta d^2(x_n, Ty) + \gamma d^2(Tx_n, y) + \delta d^2(x_n, y) \le 0, \\ &(\alpha + \beta)\mu_n d^2(x_n, Ty) \le -(\gamma + \delta)\mu_n d^2(x_n, y) \end{aligned}$$

for all $y \in C$. Since $\alpha + \beta + \gamma + \delta \ge 0$ and $\alpha + \beta > 0$,

$$\begin{aligned} \mu_n d^2(x_n, Ty) &\le \frac{-(\gamma + \delta)}{\alpha + \beta}\mu_n d^2(x_n, y) \\ &\le \mu_n d^2(x_n, y). \end{aligned}$$

If $\alpha + \gamma > 0$, then

$$\begin{aligned} &\alpha d^2(Ty, Tx_n) + \beta d^2(y, Tx_n) + \gamma d^2(Ty, x_n) + \delta d^2(y, x_n) \le 0, \\ &(\alpha + \gamma)\mu_n d^2(Ty, x_n) \le -(\beta + \delta)\mu_n d^2(y, x_n) \end{aligned}$$

for all $y \in C$. It follows from $\alpha + \beta + \gamma + \delta \ge 0$ and $\alpha + \gamma > 0$ that

$$\begin{aligned} \mu_n d^2(Ty, x_n) &\le \frac{-(\beta + \delta)}{\alpha + \gamma}\mu_n d^2(y, x_n) \\ &\le \mu_n d^2(y, x_n). \end{aligned}$$

Therefore,

$$\mu_n d^2(x_n, Ty) \le \mu_n d^2(x_n, y) \tag{3.3}$$

for all $y \in C$. Furthermore, Lemma 2.10 and $x_n \stackrel{\triangle}{\longrightarrow} z$ imply that

$$\lim_{n\to\infty} \bigl(d^2(x_n, z) - d^2(x_n, y) + d^2(z, y)\bigr) = 0$$

for all $y \in X$. Thus,

$$\mu_n \bigl(d^2(x_n, z) - d^2(x_n, y) + d^2(z, y)\bigr) = 0 \tag{3.4}$$

for all $y \in X$. From (3.3), we have

$$-\mu_n d^2(x_n, y) \le -\mu_n d^2(x_n, Ty).$$

By adding $\mu_n(d^2(x_n,z) + d^2(z,y) + d^2(z,Ty))$ to the both sides of the above inequality, we assert that

$$\begin{aligned}
&-\mu_n d^2(x_n,y) + \mu_n\big(d^2(x_n,z) + d^2(z,y) + d^2(z,Ty)\big) \\
&\quad \le -\mu_n d^2(x_n,Ty) + \mu_n\big(d^2(x_n,z) + d^2(z,y) + d^2(z,Ty)\big), \\
&d^2(z,Ty) + \mu_n\big(d^2(x_n,z) - d^2(x_n,y) + d^2(z,y)\big) \\
&\quad \le d^2(z,y) + \mu_n\big(d^2(x_n,z) - d^2(x_n,Ty) + d^2(z,Ty)\big).
\end{aligned}$$

From (3.4), we have $d(z,Ty) \le d(z,y)$ and hence $z \in A(T)$. □

We prove the analogous result to Lemma 2.1 in complete CAT(0) spaces as follows.

Lemma 3.4 *Let (X,d) be a complete* CAT(0) *space and C be a closed convex subset of X. Let $\{x_n\}$ be a bounded sequence in X. If $d(x_{n+1},z) \le d(x_n,z)$ for all $z \in C$, then $P_C x_n \longrightarrow z_0 \in C$, where P_C is the metric projection from X onto C.*

Proof Since $d(x_{n+1},z) \le d(x_n,z)$ for all $z \in C$,

$$\begin{aligned}
d(x_{n+1},P_C x_{n+1}) &\le d(x_{n+1},P_C x_n) \\
&\le d(x_n,P_C x_n).
\end{aligned}$$

Then $\{d(x_n,P_C x_n)\}$ is decreasing and bounded, which implies that $\lim_{n\to\infty} d(x_n,P_C x_n)$ exists.

In order to show that $\{P_C x_n\}$ is a Cauchy sequence in C, let $m,n \in \mathbb{N}$ be such that $m < n$. Put $y = \frac{1}{2}P_C x_m \oplus \frac{1}{2}P_C x_n$. By the (CN) inequality, we have

$$d^2(x_n,y) \le \frac{1}{2}d^2(x_n,P_C x_m) + \frac{1}{2}d^2(x_n,P_C x_n) - \frac{1}{4}d^2(P_C x_m,P_C x_n).$$

Then

$$d^2(P_C x_m,P_C x_n) \le 2d^2(x_n,P_C x_m) + 2d^2(x_n,P_C x_n) - 4d^2(x_n,y).$$

Since $-d(x_n,y) \le -d(x_n,P_C x_n)$,

$$\begin{aligned}
d^2(P_C x_m,P_C x_n) &\le 2d^2(x_n,P_C x_m) + 2d^2(x_n,P_C x_n) - 4d^2(x_n,P_C x_n) \\
&= 2d^2(x_n,P_C x_m) - 2d^2(x_n,P_C x_n).
\end{aligned}$$

By the assumption, we can conclude that

$$d^2(P_C x_m,P_C x_n) \le 2d^2(x_m,P_C x_m) - 2d^2(x_n,P_C x_n).$$

Since $\lim_{n\to\infty} d(x_n,P_C x_n)$ exists, $\{P_C x_n\}$ is a Cauchy sequence.

Therefore, $P_C x_n \longrightarrow z_0$ for some $z_0 \in C$. □

In 2012, Dehghan and Rooin [26] proved the following lemma. Now, we give another proof.

Lemma 3.5 ([26]) *Let (X,d) be a complete* CAT(0) *space and C be a closed convex subset of X. Let $x \in X$ and $y \in C$. Then $y = P_C x$ if and only if $\langle \overrightarrow{xy}, \overrightarrow{yz} \rangle \geq 0$ for all $z \in C$.*

Proof Let $z \in C$. Suppose that $y = P_C x$. Let $0 < \lambda < 1$ and $m = \lambda z \oplus (1-\lambda)y$.

By Lemma 2.4, we have

$$d^2(x,m) \leq \lambda d^2(x,z) + (1-\lambda)d^2(x,y) - \lambda(1-\lambda)d^2(z,y).$$

Then

$$\frac{d^2(x,m)}{\lambda} - \frac{1-\lambda}{\lambda}d^2(x,y) + (1-\lambda)d^2(z,y) \leq d^2(x,z). \tag{3.5}$$

From (3.5) and $2\langle \overrightarrow{xy}, \overrightarrow{yz} \rangle = d^2(x,z) - d^2(x,y) - d^2(z,y)$, we have

$$\begin{aligned} 2\langle \overrightarrow{xy}, \overrightarrow{yz} \rangle &\geq \frac{d^2(x,m)}{\lambda} - \frac{1-\lambda}{\lambda}d^2(x,y) + (1-\lambda)d^2(z,y) - d^2(x,y) - d^2(z,y) \\ &= \frac{d^2(x,m)}{\lambda} - \frac{d^2(x,y)}{\lambda} - \lambda d^2(z,y). \end{aligned}$$

Since $y = P_C x$, $d(x,y) \leq d(x,m)$. Therefore,

$$\begin{aligned} 2\langle \overrightarrow{xy}, \overrightarrow{yz} \rangle &\geq \frac{d^2(x,y)}{\lambda} - \frac{d^2(x,y)}{\lambda} - \lambda d^2(z,y) \\ &= -\lambda d^2(z,y). \end{aligned}$$

By letting $\lambda \longrightarrow 0$, we assert that $\langle \overrightarrow{xy}, \overrightarrow{yz} \rangle \geq 0$ for all $z \in C$.

Conversely, suppose that $\langle \overrightarrow{xy}, \overrightarrow{yz} \rangle \geq 0$ for all $z \in C$.

For any $z \in C$, we have

$$\begin{aligned} 0 &\leq 2\langle \overrightarrow{xy}, \overrightarrow{yz} \rangle \\ &= d^2(x,z) - d^2(x,y) - d^2(y,z) \\ &\leq d^2(x,z) - d^2(x,y). \end{aligned}$$

Then $d(x,y) \leq d(x,z)$ and hence $y = P_C x$. □

We are now in a position to prove our main theorem.

Theorem 3.6 *Let (X,d) be a complete* CAT(0) *space satisfying the $(\mathbb{S})$ property and the $(\overline{Q_4})$ condition. Let C be a convex subset of X and $T : C \to C$ be a normally generalized hybrid mapping with $A(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)Tx_n, \quad n \in \mathbb{N}.$$

Then $x_n \overset{\triangle}{\longrightarrow} v \in A(T)$, where $v = \lim_{n\to\infty} P_{A(T)} x_n$ and $P_{A(T)}$ is the metric projection from X onto $A(T)$.

Proof *Claim* 1. We show that $\lim_{n\to\infty} d(u,x_n)$ exists for all $u \in A(T)$.

By Lemma 2.3, we have

$$\begin{aligned} d(u,x_{n+1}) &= d\bigl(u,\alpha_n x_n \oplus (1-\alpha_n)Tx_n\bigr) \\ &\le \alpha_n d(u,x_n) + (1-\alpha_n)d(u,Tx_n). \end{aligned}$$

Since $u \in A(T)$,

$$\begin{aligned} d(u,x_{n+1}) &\le \alpha_n d(u,x_n) + (1-\alpha_n)d(u,x_n) \\ &= d(u,x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Then $\{d(u,x_n)\}$ is a decreasing and bounded sequence.

Therefore, $\lim_{n\to\infty} d(u,x_n)$ exists and $\{x_n\}$ is bounded.

Claim 2. We show that $d(x_n,Tx_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Choose $z \in A(T)$.

By Lemma 2.4, we have

$$\begin{aligned} d^2(z,x_{n+1}) &= d^2\bigl(z,\alpha_n x_n \oplus (1-\alpha_n)Tx_n\bigr) \\ &\le \alpha_n d^2(z,x_n) + (1-\alpha_n)d^2(z,Tx_n) - \alpha_n(1-\alpha_n)d^2(x_n,Tx_n). \end{aligned}$$

Then

$$\alpha_n(1-\alpha_n)d^2(x_n,Tx_n) \le \alpha_n d^2(z,x_n) + (1-\alpha_n)d^2(z,Tx_n) - d^2(z,x_{n+1}).$$

Since $z \in A(T)$,

$$\begin{aligned} \alpha_n(1-\alpha_n)d^2(x_n,Tx_n) &\le \alpha_n d^2(z,x_n) + (1-\alpha_n)d^2(z,x_n) - d^2(z,x_{n+1}) \\ &= d^2(z,x_n) - d^2(z,x_{n+1}). \end{aligned}$$

By the existence of $\lim_{n\to\infty} d(z,x_n)$, it follows that

$$\liminf_{n\to\infty} \alpha_n(1-\alpha_n)d^2(x_n,Tx_n) \le \liminf_{n\to\infty}\bigl(d^2(z,x_n) - d^2(z,x_{n+1})\bigr) = 0.$$

Since $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we have $d(x_n,Tx_n) \longrightarrow 0$.

Claim 3. The sequence $\{x_n\}$ $\triangle$-converges to an element in $A(T)$. Indeed, since $\{x_n\}$ is bounded, we can assume that $A(\{x_n\}) = \{v\}$ for some $v \in X$. It is sufficient to show that $A(\{x_{n_k}\}) = \{v\}$ for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ with $A(\{x_{n_k}\}) = \{w\}$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k'}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k'}} \xrightarrow{\triangle} z$ for some $z \in X$.

By Lemma 3.3 and Claim 2, we have $z \in A(T)$ and hence $\lim_{n\to\infty} d(z,x_n)$ exists. If $z \neq w$, then it follows from the uniqueness of asymptotic center that

$$\begin{aligned} \lim_{n\to\infty} d(z,x_n) &= \limsup_{k'\to\infty} d(z,x_{n_{k'}}) \\ &< \limsup_{k'\to\infty} d(w,x_{n'_k}) \end{aligned}$$

$$
\begin{aligned}
&\leq \limsup_{k\to\infty} d(w, x_{n_k}) \\
&< \limsup_{k\to\infty} d(z, x_{n_k}) \\
&= \lim_{n\to\infty} d(z, x_n),
\end{aligned}
$$

which is a contradiction. Therefore, $w = z \in A(T)$.

Suppose that $v \neq w$. Then

$$
\begin{aligned}
\lim_{n\to\infty} d(w, x_n) &= \limsup_{k\to\infty} d(w, x_{n_k}) \\
&< \limsup_{k\to\infty} d(v, x_{n_k}) \\
&\leq \limsup_{n\to\infty} d(v, x_n) \\
&< \limsup_{n\to\infty} d(w, x_n) \\
&= \lim_{n\to\infty} d(w, x_n).
\end{aligned}
$$

This leads to a contradiction and hence $v = w \in A(T)$.

Therefore, $x_n \stackrel{\triangle}{\longrightarrow} v \in A(T)$.

Claim 4. We show that $v = \lim_{n\to\infty} P_{A(T)} x_n$.

We can conclude from Claim 1 that $d(x_{n+1}, z) \leq d(x_n, z)$ for all $z \in A(T)$ and $n \in \mathbb{N}$. Furthermore, we obtain from Lemma 3.4 that $P_{A(T)} x_n \longrightarrow p$ for some $p \in A(T)$. It is sufficient to show that $\langle \overrightarrow{vp}, \overrightarrow{pz} \rangle \geq 0$ for all $z \in A(T)$.

First, we show that $\lim_{n\to\infty} d(x_n, p) = \lim_{n\to\infty} d(x_n, P_{A(T)} x_n)$.

From Claim 1, we have that $\lim_{n\to\infty} d(p, x_n)$ exists.

Since

$$
\begin{aligned}
d(x_{n+1}, P_{A(T)} x_{n+1}) &\leq d(x_{n+1}, P_{A(T)} x_n) \\
&\leq d(x_n, P_{A(T)} x_n)
\end{aligned}
$$

for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} d(x_n, P_{A(T)} x_n)$ exists. By the triangle inequality, we have

$$
\begin{aligned}
d(x_n, p) &\leq d(x_n, P_{A(T)} x_n) + d(P_{A(T)} x_n, p) \\
&\leq d(x_n, p) + d(P_{A(T)} x_n, p).
\end{aligned}
$$

Since $\lim_{n\to\infty} d(P_{A(T)} x_n, p) = 0$, we have

$$
\lim_{n\to\infty} d(x_n, p) \leq \lim_{n\to\infty} d(x_n, P_{A(T)} x_n) \leq \lim_{n\to\infty} d(x_n, p).
$$

This implies that

$$
\lim_{n\to\infty} d(x_n, p) = \lim_{n\to\infty} d(x_n, P_{A(T)} x_n).
$$

Moreover, we can apply Lemma 2.10 to conclude that

$$
\lim_{n\to\infty} \bigl(d^2(x_n, v) - d^2(x_n, z) + d^2(v, z) \bigr) = 0
$$

for all $z \in X$. Consider

$$\begin{aligned} 0 &= \lim_{n\to\infty} \bigl(d^2(x_n, v) - d^2(x_n, p) + d^2(v, p)\bigr), \\ -d^2(v, p) &= \lim_{n\to\infty} \bigl(d^2(x_n, v) - d^2(x_n, p)\bigr), \\ -2d^2(v, p) &= \lim_{n\to\infty} \bigl(d^2(x_n, v) - d^2(x_n, p) - d^2(v, p)\bigr) \\ &= \lim_{n\to\infty} \bigl(d^2(x_n, v) - d^2(x_n, P_{A(T)}x_n) - d^2(v, P_{A(T)}x_n)\bigr) \\ &= \lim_{n\to\infty} \langle \overrightarrow{x_n P_{A(T)}x_n}, \overrightarrow{P_{A(T)}x_n v}\rangle. \end{aligned}$$

By Lemma 3.5, we have

$$\langle \overrightarrow{x_n P_{A(T)}x_n}, \overrightarrow{P_{A(T)}x_n v}\rangle \geq 0,$$

which implies that $d^2(v, p) \leq 0$ and hence $v = p$.

Therefore, $x_n \xrightarrow{\triangle} v$, where $v = \lim_{n\to\infty} P_{A(T)}x_n$. □

It is known that a Hilbert space satisfies both the ($\mathbb{S}$) property and the ($\overline{Q_4}$) condition. Furthermore, Δ-convergence and weak convergence are the same in a Hilbert space. Consequently, we can conclude that the result of Takahashi, Wong and Yao [2], Theorem 5.1, is an application of Theorem 3.6.

Moreover, the following example shows that there is a CAT(0) space satisfying both the ($\mathbb{S}$) property and the ($\overline{Q_4}$) condition but it is not a Hilbert space. Thus, our main theorem is a strict generalization of the result of Takahashi, Wong and Yao [2], Theorem 5.1.

Example 3.7 Consider the subset $\mathcal{H}$ of $\mathbb{R}^2$

$$\mathcal{H} = \bigl\{(x, y) \in \mathbb{R}^2 | y^2 - x^2 = 1 \text{ and } y > 0\bigr\}.$$

In other words, $\mathcal{H}$ is the upper curve of the hyperbolic $\{(x, y) \in \mathbb{R}^2 | y^2 - x^2 = 1\}$. Let d be a metric defined by the function $d : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ that assigns to each pair of vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ the unique nonnegative number $d(u, v) \geq 0$ such that

$$\cosh d(u, v) = u_2 v_2 - u_1 v_1.$$

It is known that, in general, the metric space $(\mathcal{H}, d)$ is a CAT(0) space and also a one-dimensional hyperbolic space viewed from a hyperboloid model (for more details, see [13]). Then $(\mathcal{H}, d)$ satisfies the ($\mathbb{S}$) property. Furthermore, it is easy to see that $(\mathcal{H}, d)$ is an $\mathbb{R}$-tree. Hence $(\mathcal{H}, d)$ satisfies the ($\overline{Q_4}$) condition.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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