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# Common fixed point of multifunctions on partial metric spaces

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**Abstract**

In this paper, some multifunctions on partial metric space are defined and common fixed points of such multifunctions are discussed. The results presented in the paper generalize some of the existing results in the literature. Several conclusions of the main results are given.

**MSC:** 47H10; 54H10; 46T99

**Keywords:** common fixed point; partial metric space; multifunction

## 1 Introduction

The notion of a partial metric space (PMS) was introduced by Matthews [1] in 1992 (see also [2]). The PMS is a generalization of the usual metric space in which $d(x,x)$ is no longer necessarily zero. Recently, many authors have focused on the PMS and its topological properties(see for example [3–15]). Partial metric spaces have extensive application potential in the research area of computer domains and semantics (see [15–18]).

A partial metric is a function $p : X \times X \to [0,\infty)$ satisfying the following conditions:

(a) $p(x,y) = p(y,x)$ (symmetry),

(b) $x = y \iff p(x,x) = p(x,y) = p(y,y)$ (equality),

(c) $p(x,x) \leq p(x,y)$ (small self-distances),

(d) $p(x,y) \leq p(x,z) + p(y,z) - p(z,z)$ (triangularity),

for all $x,y,z \in X$. Then $(X,p)$ is called a partial metric space.

Each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$ with a base of the family of open $p$-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where

$$B_p(x,\varepsilon) = \left\{ y \in X : p(x,y) < p(x,x) + \varepsilon \right\},$$

for all $x \in X$ and $\varepsilon > 0$. For a partial metric $p$ on $X$, the function $d_p : X \times X \to [0,\infty)$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a (usual) metric on $X$. Another metric on $X$ induced by $p$ is defined in [19] as $d(x,y) = p(x,y)$ whenever $x \neq y$ and $d(x,y) = 0$ whenever $x = y$.

Some topological concepts and basic results on a PMS are defined as follows.

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A sequence $\{x_n\}_{n \geq 1}$ in a PMS $(X, p)$ converges to $x \in X$ if and only if $p(x, x) = \lim_{n \to \infty} p(x, x_n)$.

A sequence $\{x_n\}_{n \geq 1}$ is called a Cauchy sequence if and only if $\lim_{n,m \to \infty} p(x_n, x_m)$ exists and is finite.

A PMS $(X, p)$ is said to be complete whenever every Cauchy sequence $\{x_n\}_{n \geq 1}$ in $X$ converges to a point $x$ with respect to $\tau_p$, that is, $p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m)$.

**Lemma 1.1** ([3]) *Let $(X, p)$ be a partial metric space. Then*:

(i) *A sequence $\{x_n\}_{n \geq 1}$ is Cauchy in a PMS $(X, p)$ if and only if $\{x_n\}_{n \geq 1}$ is Cauchy in the metric space $(X, d_p)$.*

(ii) *A PMS $(X, p)$ is complete if and only if the metric space $(X, d_p)$ is complete. Moreover,*

$$\lim_{n \to \infty} d_p(x, x_n) = 0 \quad \Longleftrightarrow \quad p(x, x) = \lim_{n \to \infty} p(x, x_n) = \lim_{n,m \to \infty} p(x_n, x_m).$$

An interesting property of partial metric spaces is the nonuniqueness of limits of sequences. To emphasize this property we consider the following example.

**Example 1.1** Let $X = [0, \infty)$ and define a partial metric $p$ on $X$ as

$$p(x, y) = \max\{x, y\}.$$

Consider the sequence $\{x_n\} = \{1 + \frac{1}{n}\}$. Notice that

$$\lim_{n \to \infty} p(x_n, 1) = \lim_{n \to \infty} \max\left\{1 + \frac{1}{n}, 1\right\} = \lim_{n \to \infty} 1 + \frac{1}{n} = 1.$$

Also

$$\lim_{n \to \infty} p(x_n, 2) = \lim_{n \to \infty} \max\left\{1 + \frac{1}{n}, 2\right\} = \lim_{n \to \infty} 2 = 2.$$

Moreover, for any $a \geq 1$ we have

$$\lim_{n \to \infty} p(x_n, a) = a.$$

In what follows, we introduce the notions, notations, and assumptions used in the discussion. Throughout this paper, we suppose that $(X, p)$ is a partial metric space. We denote the family of all nonempty subsets of $X$ by $2^X$, the family of all closed subsets of $X$ by $C(X)$ and the family of all closed and bounded subsets of $X$ by $CB(X)$. The partial Hausdorff distance $H_p$ on $CB(X)$ was introduced by Aydi *et al.* [20] as follows:

$$H_p(A, B) = \max\left\{\sup_{a \in A} p(a, B), \sup_{b \in B} p(b, A)\right\}, \tag{1}$$

for all $A, B \in CB(X)$, where

$$p(x, A) = \inf_{a \in A} p(x, a). \tag{2}$$

Let $T : X \to 2^X$ be a multi-valued function (multifunction). We denote the set of fixed points of $T$ by $F(T)$, *i.e.*,

$$F(T) = \{x \in X : x \in Tx\}. \tag{3}$$

**Lemma 1.2** ([4]) *Let $(X, p)$ be a partial metric space, $A \subseteq X$, and $x \in X$. Then $x \in \overline{A}$ if and only if $p(x, A) = p(x, x)$.*

In 2012, Aydi *et al.* [20] proved the following fixed point theorem on partial metric space.

**Theorem 1.3** *Let $(X, p)$ be a complete partial metric space and $T : X \to CB(X)$ a multifunction. Suppose that there exist $k \in (0, 1)$ such that*

$$H_p(Tx, Ty) \le kp(x, y), \tag{4}$$

*for all $x, y \in X$. Then $T$ has a fixed point.*

Some fixed point theorems for multifunctions on metric space are given next (see [6, 21, 22]).

**Theorem 1.4** *Let $(X, d)$ be a complete metric space and $T : X \to CB(X)$ a multifunction. Assume that there exists $r \in [0, 1)$ such that*

$$\frac{1}{1 + r} d(x, Tx) \le d(x, y) \quad implies \quad H(Tx, Ty) \le rd(x, y), \tag{5}$$

*for all $x, y \in X$. Then $T$ has a fixed point.*

**Theorem 1.5** *Let $(X, d)$ be a complete metric space and $T : X \to C(X)$ a multifunction. Assume that there exist $a, b, c \in [0, 1)$ such that $a + b + c < 1$ and*

$$\frac{(1 - b - c)}{1 + a} d(x, Tx) \le d(x, y) \tag{6}$$

*implies*

$$H(Tx, Ty) \le ad(x, y) + bd(x, Tx) + cd(y, Ty), \tag{7}$$

*for all $x, y \in X$. Then $T$ has a fixed point.*

The aim of this paper is to provide a new, more general condition for the multifunction $T$ which guarantees the existence of its fixed point. Our results generalize some of the existing ones.

In what follows, we consider two classes of functions, namely, $R_1$ and $R_2$ as defined below.

**Definition 1.2** Let $R_1$ be the set of all continuous functions $g : [0, \infty)^5 \to [0, \infty)$, satisfying the conditions:

(i) $g(t, t, t, 2t, t) < t$, for all $t \in [0, \infty)$,

(ii) $g$ is subhomogeneous, *i.e.*, $g(\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \alpha x_5) \leq \alpha g(x_1, x_2, x_3, x_4, x_5)$, for all $\alpha \geq 0$,

(iii) if $x_i, y_i \in [0, \infty)$, $x_i \leq y_i$ for $i = 1, \ldots, 5$ we have $g(x_1, x_2, x_3, x_4, x_5) \leq g(y_1, y_2, y_3, y_4, y_5)$.

Let $R_2$ be the set of all continuous function $g : [0, \infty)^5 \to [0, \infty)$ satisfying the following conditions:

(a) $g(t, t, t, t, t) < t$, for all $t \in [0, \infty)$,

(b) $g$ is subhomogeneous,

(c) if $x_i, y_i \in [0, \infty)$, $x_i \leq y_i$ for $i = 1, \ldots, 5$ we have $g(x_1, x_2, x_3, x_4, x_5) \leq g(y_1, y_2, y_3, y_4, x_5)$,

(d) for all $0 \leq a \leq x_4$, $g(x_1, x_2, x_3, x_4 - a, a) = g(x_1, x_2, x_3, x_4, 0)$.

**Remark 1.1** It is easy to see that if $g \in R_1$, then $g(1, 1, 1, 2, 1) = h \in (0, 1)$. Indeed, if $g \in R_1$, the conditions (i) and (ii) give

$$g(t, t, t, 2t, t) \leq tg(1, 1, 1, 2, 1) < t, \tag{8}$$

which implies $g(1, 1, 1, 2, 1) = h \in (0, 1)$. In addition, if $g \in R_2$, then by a similar argument we observe that $g(1, 1, 1, 1, 1) = h \in (0, 1)$

Examples of functions from both classes are given below.

**Example 1.3** The function $g(x_1, x_2, x_3, x_4, x_5) = k \max\{x_i\}_{i=1}^5$ for $k \in (0, \frac{1}{2})$ is in class $R_1$.

**Example 1.4** The function $g(x_1, x_2, x_3, x_4, x_5) = k \max\{x_1, x_2, x_3, \frac{x_4 + x_5}{2}\}$ for $k \in (0, 1)$ belongs to $R_2$.

The following results are quite trivial.

**Proposition 1.6** *If $g \in R_1$ and $u, v \in [0, \infty)$ are such that*

$$u \leq \max\{g(v, u, v, u, v), g(v, u, v, v + u, v)\}, \tag{9}$$

*then $u \leq hv$, where $h = g(1, 1, 1, 2, 1)$.*

*Proof* From (iii) it is clear that $g(v, u, v, u, v) \leq g(v, u, v, v + u, v)$ and hence $u \leq \max\{g(v, u, v, u, v), g(v, u, v, v + u, v)\} = g(v, u, v, v + u, v)$. If $v < u$, then

$$u \leq g(v, u, v, v + u, v) \leq g(u, u, u, 2u, u) \leq ug(1, 1, 1, 2, 1) = hu < u,$$

which is a contradiction. Thus $u \leq v$, which implies

$$u \leq g(v, u, v, v + u, v) \leq g(v, v, v, 2v, v) \leq vg(1, 1, 1, 2, 1) = hv. \qquad \square$$

**Proposition 1.7** *If $g \in R_2$ and $u, v \in [0, \infty)$ are such that*

$$u \leq \max\{g(v, u, v, u + v, 0), g(v, u, v, u, v)\}, \tag{10}$$

*then $u \leq hv$ where $h = g(1, 1, 1, 1, 1)$.*

*Proof* Let $\max\{g(v, u, v, u + v, 0), g(v, u, v, u, v)\} = g(v, u, v, u + v, 0)$. If $v < u$, then (d) implies

$$u \leq g(v, u, v, v + u, 0) \leq g(u, u, u, 2u, 0) \leq ug(1, 1, 1, 2, 0) = ug(1, 1, 1, 1, 1) = hu < u,$$

which is a contradiction. Thus $u \leq v$, and hence,

$$u \leq g(v, u, v, v + u, 0) \leq g(v, v, v, 2v, 0) \leq vg(1, 1, 1, 2, 0) = vg(1, 1, 1, 1, 1) = hv.$$

Let $\max\{g(v, u, v, u + v, 0), g(v, u, v, u, v)\} = g(v, u, v, u, v)$. If $v < u$, then

$$u \leq g(v, u, v, u, v) \leq g(u, u, u, u, u) \leq ug(1, 1, 1, 1, 1) = hu < u.$$

This contradicts our assumption, that is, we should have $u \leq v$. Then

$$u \leq g(v, u, v, u, v) \leq g(v, v, v, v, v) \leq vg(1, 1, 1, 1, 1) = hv,$$

which completes the proof. □

## 2 Main results

We state and proof our main results in this section.

**Lemma 2.1** *Let $(X, p)$ be a partial metric space and $T, S : X \to C(X)$ be two multifunctions. Suppose that there exist $\alpha \in (0, \infty)$ and $g \in R_1 \cup R_2$ such that $\alpha p(x, Tx) \leq p(x, y)$ or $\alpha p(y, Sy) \leq p(x, y)$ implies*

$$H_p(Tx, Sy) \leq g\big(p(x, y), p(y, Sy), p(x, Tx), p(x, Sy), p(y, Tx)\big), \tag{11}$$

*for all $x, y \in X$. Then for every $x \in F(T) \cup F(S)$ we have $p(x, x) = 0$.*

*Proof* Without loss of generality, we can suppose that $x \in Tx$. Then $p(x, Tx) = p(x, x)$ and hence

$$
\begin{aligned}
p(x, Sx) \leq H_p(Tx, Sx) &\leq g\big(p(x, x), p(x, Sx), p(x, Tx), p(x, Sx), p(x, Tx)\big) \\
&\leq g\big(p(x, x), p(x, Sx), p(x, x), p(x, Sx), p(x, x)\big).
\end{aligned}
\tag{12}
$$

By using Proposition 1.6 if $g \in R_1$ or Proposition 1.7 if $g \in R_2$, we have

$$p(x, x) \leq p(x, Sx) \leq hp(x, x).$$

However, since $h < 1$ we have $p(x, x) = 0$. □

**Lemma 2.2** *Let $(X, p)$ be a partial metric space and $T, S : X \to C(X)$ be two multifunctions. Suppose that there exist $\alpha \in (0, \infty)$ and $g \in R_1 \cup R_2$ such that $\alpha p(x, Tx) \leq p(x, y)$ or $\alpha p(y, Sy) \leq p(x, y)$ implies*

$$H_p(Tx, Sy) \leq g\big(p(x, y), p(y, Sy), p(x, Tx), p(x, Sy), p(y, Tx)\big),$$

*for all $x, y \in X$. Then $F(T) = F(S)$.*

*Proof* If $x \in Tx$, then $p(x, Tx) = p(x, x) = 0$ by Lemma 2.1. Hence,

$$
\begin{aligned}
p(x, Sx) \leq H_p(Tx, Sx) &\leq g\big(p(x, x), p(x, Sx), p(x, Tx), p(x, Sx), p(x, Tx)\big) \\
&\leq g\big(p(x, x), p(x, Sx), p(x, x), p(x, Sx), p(x, x)\big) \\
&\leq g\big(0, p(x, Sx), 0, p(x, Sx), 0\big).
\end{aligned}
\tag{13}
$$

By using Proposition 1.6 whenever $g \in R_1$ or Proposition 1.7 in case $g \in R_2$, we have $p(x, Sx) \leq h0 = 0$, and thus $x \in F(S)$. Thus, $F(T) \subseteq F(S)$. Similarly, we can show that $F(S) \subseteq F(T)$, which completes the proof. $\square$

In what follows, we state our main existence result.

**Theorem 2.3** *Let $(X, p)$ be a complete partial metric space and $T, S : X \to C(X)$ be two multifunctions. Suppose that there exist $g \in R_1 \cup R_2$ and $\alpha \in (0, 1)$, such that $\alpha(h + 1) \leq 1$ where $h = g(1, 1, 1, 2, 1)$ if $g \in R_1$ and $h = g(1, 1, 1, 1, 1)$ if $g \in R_2$. Suppose also that $\alpha p(x, Tx) \leq p(x, y)$ or $\alpha p(y, Sy) \leq p(x, y)$ implies*

$$
H_p(Tx, Sy) \leq g\big(p(x, y), p(y, Sy), p(x, Tx), p(x, Sy), p(y, Tx)\big),
\tag{14}
$$

*for all $x, y \in X$. Then $F(T) = F(S)$ and $F(T)$ is nonempty.*

*Proof* By Lemma 2.2 we already have $F(T) = F(S)$. Fix arbitrary $1 > r > h$ and $x_0 \in X$ and choose $x_1 \in Tx_0$ such that $\alpha p(x_0, Tx_0) < p(x_0, x_1)$. Then by the hypothesis of the theorem and condition (iii) or (c) in Definition 1.2, respectively, we have

$$
\begin{aligned}
p(x_1, Sx_1) \leq H_p(Tx_0, Sx_1) &\leq g\big(p(x_0, x_1), p(x_1, Sx_1), p(x_0, Tx_0), p(x_0, Sx_1), p(x_1, Tx_0)\big) \\
&\leq g\big(p(x_0, x_1), p(x_1, Sx_1), p(x_0, x_1), p(x_0, Sx_1), p(x_1, x_1)\big) \\
&\leq g\big(p(x_0, x_1), p(x_1, Sx_1), p(x_0, x_1), p(x_0, x_1) + p(x_1, Sx_1) - p(x_1, x_1), p(x_1, x_1)\big) \\
&\leq g\big(p(x_0, x_1), p(x_1, Sx_1), p(x_0, x_1), p(x_0, x_1) + p(x_1, Sx_1), p(x_1, x_1)\big),
\end{aligned}
$$

where obviously $p(x_0, Sx_1) \leq p(x_0, x_1) + p(x_1, Sx_1) - p(x_1, x_1)$ due to triangle inequality in PMS. Suppose that $g \in R_1$. Since

$$
p(x_1, Sx_1) \leq g\big(p(x_0, x_1), p(x_1, Sx_1), p(x_0, x_1), p(x_0, x_1) + p(x_1, Sx_1), p(x_0, x_1)\big),
$$

then, by Proposition 1.6, we have

$$
p(x_1, Sx_1) \leq hp(x_0, x_1) < rp(x_0, x_1).
\tag{15}
$$

Now let $g \in R_2$. Since

$$
p(x_1, Sx_1) \leq g\big(p(x_0, x_1), p(x_1, Sx_1), p(x_0, x_1), p(x_0, x_1) + p(x_1, Sx_1) - p(x_1, x_1), p(x_1, x_1)\big),
$$

and obviously $0 \leq p(x_1, x_1) \leq p(x_0, x_1) + p(x_1, Sx_1)$, we let $a = p(x_1, x_1)$ and employ condition (d) in Definition 1.2 to get

$$
p(x_1, Sx_1) \leq g\big(p(x_0, x_1), p(x_1, Sx_1), p(x_0, x_1), p(x_0, x_1) + p(x_1, Sx_1), 0\big).
$$

Now by Proposition 1.7, we have

$$p(x_1, Sx_1) \leq hp(x_0, x_1) < rp(x_0, x_1). \tag{16}$$

We choose a number $\mu$ such that $\inf_{y \in Sx_1} p(x_1, y) = p(x_1, Sx_1) < \mu < rp(x_0, x_1)$. Thus there exists $x_2 \in Sx_1$ such that $p(x_1, x_2) < \mu < rp(x_0, x_1)$. Since $\alpha p(x_1, Sx_1) < p(x_1, x_2)$, by using (14) and the properties of the function $g$ we have

$$
\begin{aligned}
p(x_2, Tx_2) &\leq H_p(Sx_1, Tx_2) \leq g\big(p(x_1, x_2), p(x_2, Tx_2), p(x_1, Sx_1), p(x_1, Tx_2), p(x_2, Sx_1)\big) \\
&\leq g\big(p(x_1, x_2), p(x_2, Tx_2), p(x_1, x_2), p(x_1, x_2) + p(x_2, Tx_2) - p(x_2, x_2), p(x_2, x_2)\big) \\
&\leq g\big(p(x_1, x_2), p(x_2, Tx_2), p(x_1, x_2), p(x_1, x_2) + p(x_2, Tx_2), p(x_1, x_2)\big).
\end{aligned}
$$

Now, if $g \in R_1$, using Proposition 1.6 and mimicking the proof of (15) we obtain

$$p(x_2, Tx_2) \leq hp(x_1, x_2) < rp(x_1, x_2). \tag{17}$$

If $g \in R_2$, letting $a = p(x_2, x_2)$ we get

$$
\begin{aligned}
p(x_2, Tx_2) &\leq g\big(p(x_1, x_2), p(x_2, Tx_2), p(x_1, x_2), p(x_1, x_2) + p(x_2, Tx_2) - p(x_2, x_2), p(x_2, x_2)\big) \\
&\leq g\big(p(x_1, x_2), p(x_2, Tx_2), p(x_1, x_2), p(x_1, x_2) + p(x_2, Tx_2), 0\big),
\end{aligned}
$$

and hence Proposition 1.7 yields

$$p(x_2, Tx_2) \leq hp(x_1, x_2) < rp(x_1, x_2). \tag{18}$$

In a similar way, we can choose $x_3 \in Tx_2$ such that

$$p(x_2, x_3) < rp(x_1, x_2) < r^2 p(x_0, x_1). \tag{19}$$

By continuing this process, we obtain a sequence $\{x_n\}_{n \geq 1}$ in $X$ such that

$$x_{2n-1} \in Tx_{2n-2}, \qquad x_{2n} \in Sx_{2n-1}, \tag{20}$$

which satisfies

$$p(x_n, x_{n+1}) \leq r^n p(x_0, x_1). \tag{21}$$

Then $p(x_{2n}, Tx_{2n}) \leq hp(x_{2n-1}, x_{2n})$ and $p(x_{2n-1}, Sx_{2n-1}) \leq hp(x_{2n-2}, x_{2n-1})$.

If $x_m = x_{m+1}$ for some $m \geq 1$ where $m = 2k$, then

$$p(x_{2k}, x_{2k}) \leq p(x_{2k}, Tx_{2k}) \leq p(x_{2k}, x_{2k+1}) = p(x_{2k}, x_{2k}),$$

so $p(x_{2k}, Tx_{2k}) = p(x_{2k}, x_{2k})$, and hence $x_{2k} \in Tx_{2k}$. Thus $T$ and $S$ have a fixed point. If $m = 2k + 1$ in a similar way we find that $T$ and $S$ have a fixed point.

Suppose that $x_n \neq x_{n+1}$, for all $n \geq 1$. Repeated application of the triangle inequality implies

$$p(x_n, x_{n+m}) \leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{n+m-1}, x_{n+m})$$

$$\leq r^n p(x_0, x_1) + r^{n+1} p(x_0, x_1) + \cdots + r^{n+m-1} p(x_0, x_1)$$

$$\leq r^n p(x_0, x_1)\left(1 + r + r^2 + \cdots + r^{m-1}\right)$$

$$\leq \frac{r^n}{1-r} p(x_0, x_1).$$

Then we get

$$\lim_{n \to \infty} p(x_n, x_{n+m}) \to 0,$$

and hence $\{x_n\}_{n \geq 1}$ is a Cauchy sequence in $(X, p)$. Regarding Lemma 1.1, $\{x_n\}_{n \geq 1}$ is also a Cauchy sequence in $(X, d_p)$. Since $(X, p)$ is a complete partial metric space, by Lemma 1.1, $(X, d_p)$ is also complete. Thus $\{x_n\}_{n \geq 1}$ converges to a limit, say, $x \in X$, that is,

$$\lim_{n \to \infty} d_p(x_n, x) = 0. \tag{22}$$

Notice that Lemma 1.1 yields

$$p(x, x) = \lim_{n \to \infty} p(x_n, x) = \lim_{n,m \to \infty} p(x_n, x_m) = 0. \tag{23}$$

Now, we claim that for each $n \geq 1$ one of the relations

$$\alpha p(x_{2n}, Tx_{2n}) \leq p(x_{2n}, x) \quad \text{or} \quad \alpha p(x_{2n+1}, Sx_{2n+1}) \leq p(x_{2n+1}, x) \tag{24}$$

holds. If for some $n \geq 1$ we have $\alpha p(x_{2n}, Tx_{2n}) > p(x_{2n}, x)$ and $\alpha p(x_{2n+1}, Sx_{2n+1}) > p(x_{2n+1}, x)$ then

$$p(x_{2n}, x_{2n+1}) \leq p(x_{2n}, x) + p(x, x_{2n+1})$$

$$< \alpha p(x_{2n}, Tx_{2n}) + \alpha p(x_{2n+1}, Sx_{2n+1})$$

$$\leq \alpha p(x_{2n}, x_{2n+1}) + \alpha h p(x_{2n}, x_{2n+1}).$$

This results in $\alpha(h+1) > 1$, which contradicts the initial assumption. Hence, our claim is proved. Observe that by the assumption of the theorem, for each $n \geq 1$ we have either

$$H_p(Tx_{2n}, Sx) \leq g\big(p(x_{2n}, x), p(x, Sx), p(x_{2n}, Tx_{2n}), p(x_{2n}, Sx), p(x, Tx_{2n})\big)$$

or

$$H_p(Sx_{2n+1}, Tx) \leq g\big(p(x_{2n+1}, x), p(x, Tx), p(x_{2n+1}, Sx_{2n+1}), p(x_{2n+1}, Tx), p(x, Sx_{2n+1})\big).$$

Therefore, one of the following cases holds.

Case (i). There exists an infinite subset $I \subseteq \mathbb{N}$ such that

$$p(x_{2n+1}, Sx) \leq H_p(Tx_{2n}, Sx)$$
$$\leq g\big(p(x_{2n}, x), p(x, Sx), p(x_{2n}, Tx_{2n}), p(x_{2n}, Sx), p(x, Tx_{2n})\big),$$

for all $n \in I$.

Case (ii). There exists an infinite subset $J \subseteq \mathbb{N}$ such that

$$p(x_{2n+2}, Tx) \leq H_p(Sx_{2n+1}, Tx)$$
$$\leq g\big(p(x_{2n+1}, x), p(x, Tx), p(x_{2n+1}, Sx_{2n+1}), p(x_{2n+1}, Tx), p(x, Sx_{2n+1})\big),$$

for all $n \in J$.

In Case (i), we get

$$p(x, Sx) \leq p(x, x_{2n+1}) + p(x_{2n+1}, Sx)$$
$$\leq p(x, x_{2n+1}) + g\big(p(x_{2n}, x), p(x, Sx), p(x_{2n}, Tx_{2n}), p(x_{2n}, Sx), p(x, Tx_{2n})\big)$$
$$\leq p(x, x_{2n+1})$$
$$\quad + g\big(p(x_{2n}, x), p(x, Sx), p(x_{2n}, x_{2n+1}), p(x_{2n}, x) + p(x, Sx) - p(x, x), p(x, x_{2n+1})\big),$$

for all $n \in I$. Continuity of $g$ implies

$$p(x, Sx) \leq g\big(0, p(x, Sx), 0, 0 + p(x, Sx) - 0, 0\big). \tag{25}$$

Now by using Propositions 1.6 and 1.7, we have $p(x, Sx) = 0$, and thus $x \in Sx$.

In Case (ii), we have

$$p(x, Tx) \leq p(x, x_{2n+2}) + p(x_{2n+2}, Tx)$$
$$\leq p(x, x_{2n+2}) + g\big(p(x_{2n+1}, x), p(x, Tx), p(x_{2n+1}, Sx_{2n+1}), p(x_{2n+1}, Tx), p(x, Sx_{2n+1})\big)$$
$$\leq p(x, x_{2n+2})$$
$$\quad + g\big(p(x_{2n+1}, x), p(x, Tx), p(x_{2n+1}, x_{2n+2}),$$
$$p(x_{2n+1}, x) + p(x, Tx) - p(x, x), p(x, x_{2n+2})\big),$$

for all $n \in J$. Since $g$ is continuous, we obtain

$$p(x, Tx) \leq g\big(0, p(x, Tx), 0, 0 + p(x, Tx) - 0, 0\big). \tag{26}$$

Again, by using Propositions 1.6 and 1.7, we have $p(x, Tx) = 0$, which gives $x \in Tx$. This completes the proof. □

The following results are consequences of Theorem 2.3.

**Theorem 2.4** *Let $(X, p)$ be a complete partial metric space and $T : X \to C(X)$ be a multifunction. Suppose that there exist $\alpha \in (0,1)$ and $g \in R$ with $h = g(1,1,1,2,0)$ such that*

$\alpha(h+1) \leq 1$ *and* $\alpha p(x, Tx) \leq p(x,y)$ *implies*

$$H_p(Tx, Ty) \leq g\big(p(x,y), p(y, Ty), p(x, Tx), p(x, Ty), p(y, Tx)\big), \qquad (27)$$

*for all* $x, y \in X$. *Then* $T$ *has a fixed point.*

**Corollary 2.5** *Theorem* 1.3 *introduced in* [20] *is a special case of Theorem* 2.4.

*Proof* Define $g \in R_1$ by $g(x_1, x_2, x_3, x_4, x_5) = kx_1$. □

Now we provide the partial metric versions of Theorems 1.4 and 1.5.

**Theorem 2.6** *Let* $(X, p)$ *be a complete partial metric space and* $T : X \to CB(X)$ *be a multifunction. Assume that there exists* $r \in [0, 1)$ *such that*

$$\frac{1}{1+r} p(x, Tx) \leq p(x,y) \quad \textit{implies} \quad H_p(Tx, Ty) \leq rp(x,y), \qquad (28)$$

*for all* $x, y \in X$. *Then* $T$ *has a fixed point.*

*Proof* Define $g \in R_1$ by $g(x_1, x_2, x_3, x_4, x_5) = rx_1$. Let $\alpha = \frac{1}{1+r}$. Since $h = r$ and $\alpha(1+h) \leq 1$, by using Theorem 2.3, $T$ has a fixed point. □

**Theorem 2.7** *Let* $(X, p)$ *be a complete partial metric space and* $T : X \to C(X)$ *be a multifunction. Assume that there exist* $a, b, c \in [0, 1)$ *such that* $a + b + c < 1$ *and*

$$\frac{(1-b-c)}{1+a} p(x, Tx) \leq p(x,y) \quad \textit{implies}$$
$$H_p(Tx, Ty) \leq ap(x,y) + bp(x, Tx) + cp(y, Ty). \qquad (29)$$

*Then* $T$ *has a fixed point.*

*Proof* Define $g \in R_1$ by $g(x_1, x_2, x_3, x_4, x_5) = ax_1 + cx_2 + bx_3$. Let $\alpha = \frac{1-b-c}{1+a}$. Since $h = a + b + c$ and $\alpha(1+h) \leq 1$, by Theorem 2.3, $T$ has a fixed point. □

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