Karapınar et al. *Fixed Point Theory and Applications* (2015) 2015:105
DOI 10.1186/s13663-015-0352-z

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

α-(ψ,ϕ) Contractive mappings on quasi-partial metric spaces

Erdal Karapınar[1,2*], Leila Gholizadeh[3], Hamed H Alsulami[2] and Maha Noorwali[2]

*Correspondence: erdalkarapinar@yahoo.com
[1]Department of Mathematics, Atilim University, İncek, Ankara, 06836, Turkey
[2]Nonlinear Analysis and Applied Mathematics Research Group (NAAM), King Abdulaziz University, Jeddah, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
In this paper, we consider α-(ψ,ϕ) contractive mappings in the setting of quasi-partial metric spaces and verify the existence of a fixed point on such spaces. Moreover, we present some examples and applications in integral equations of our obtained results.

MSC: 46T99; 46N40; 47H10; 54H25

Keywords: quasi-partial metric space; fixed point; α-admissible; contractive mapping

1 Introduction and preliminaries

One of the most interesting extensions of distance function was reported by Matthews [1] by introducing the notion of a partial metric in which self-distance need not be zero. In this celebrated report, Matthews [1] successfully characterized the distinguished result, Banach contraction mapping, in the setting of partial metric spaces. Later, many authors have generalized some fixed point theorems on such a space, see *e.g.* [1–24] and the related references therein. Very recently, Karapınar *et al.* [13] presented quasi-partial metric spaces and investigated the existence and uniqueness of certain operators in the context of quasi-partial metric spaces.

Throughout this paper, we suppose that $\mathbb{R}_0^+ = [0,+\infty)$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ denotes the set of all positive integers. First, we recall some basic concepts and notations. For more information, see [1, 13].

Definition 1.1 A quasi-metric on a nonempty set X is a function $d : X \times X \to [0,+\infty)$ such that for all $x,y,z \in X$:

(QM1) $d(x,y) = 0 \Leftrightarrow x = y$,

(QM2) $d(x,y) \leq d(x,z) + d(z,y)$.

A quasi-metric space is a pair (X,d) such that X is a nonempty set and d is a quasi-metric on X.

Definition 1.2 A partial metric on a nonempty set X is a function $p : X \times X \to [0,+\infty)$ such that for all $x,y,z \in X$:

(PM1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

(PM2) $p(x,x) \leq p(x,y)$,

(PM3) $p(x,y) = p(y,x)$,

(PM4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Definition 1.3 (See [13]) A quasi-partial metric space on a nonempty set X is a function $q : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$:
(QPM1) if $q(x,x) = q(x,y) = q(y,y)$, then $x = y$ (equality),
(QPM2) $q(x,x) \leq q(x,y)$ (small self-distances),
(QPM3) $q(x,x) \leq q(y,x)$ (small self-distances),
(QPM4) $q(x,z) + q(y,y) \leq q(x,y) + p(y,z)$ (triangle inequality).
A quasi-partial metric space is a pair (X, q) such that X is a nonempty set and q is a partial metric on X.

If $q(x,y) = q(y,x)$ for all $x, y \in X$, then (X, q) becomes a partial metric space.

Lemma 1.1 (See [13]) *Let (X, q) be a quasi-partial metric space. Then the following holds: If $q(x,y) = 0$ then $x = y$.*

For a partial metric p on X, the function $d_p : X \times X \to [0, +\infty)$ defined by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X. For a quasi-partial metric q on X, the function $d_q : X \times X \to [0, +\infty)$ defined by

$$d_q(x,y) = q(x,y) + q(y,x) - q(x,x) - q(y,y)$$

is a metric on X and

$$p_q(x,y) = \frac{1}{2}\big[q(x,y) + q(y,x)\big]$$

is a partial metric on X. Notice also that for a quasi-partial metric q on X, the function $d_m^q : X \times X \to [0, +\infty)$ defined by

$$d_m^q(x,y) = \frac{q(x,y) + q(y,x)}{2} - \min\big\{q(x,x), q(y,y)\big\}$$

is a metric on X.

Definition 1.4 (See [13]) Let (X, q) be a quasi-partial metric space. Then:
(i) a sequence $\{x_n\} \subset X$ converges to $x \in X$ if and only if

$$q(x,x) = \lim_{n\to+\infty} q(x, x_n) = \lim_{n\to+\infty} q(x_n, x);$$

(ii) a sequence $\{x_n\} \subset X$ is called a Cauchy sequence if and only if $\lim_{n,m\to+\infty} q(x_n, x_m)$ and $\lim_{n,m\to+\infty} q(x_m, x_n)$ exist (and are finite);
(iii) the quasi-partial metric space is said to be complete if every Cauchy sequence $\{x_n\} \subset X$ converges, with respect to τ_q, to a point $x \in X$ such that

$$q(x,x) = \lim_{n,m\to+\infty} q(x_n, x_m) = \lim_{n,m\to+\infty} q(x_m, x_n);$$

(iv) a mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$.

Lemma 1.2 (See [13]) *Let (X, q) be a quasi-partial metric space. Let (X, p_q) be the corresponding partial metric space, and let (X, d_{p_q}) be the corresponding metric space. The following statements are equivalent*:

(A) *The sequence $\{x_n\}$ is Cauchy in (X, q).*
(B) *The sequence $\{x_n\}$ is Cauchy in (X, p_q).*
(C) *The sequence $\{x_n\}$ is Cauchy in (X, d_{p_q}).*

Lemma 1.3 (See [13]) *Let (X, q) be a quasi-partial metric space. Let (X, p_q) be the corresponding partial metric space, and let (X, d_{p_q}) be the corresponding metric space. The following statements are equivalent*:

(A) (X, q) *is complete.*
(B) (X, p_q) *is complete.*
(C) (X, d_{p_q}) *is complete.*

Moreover,

$$\lim_{n\to\infty} d_{p_q}(x, x_n) = 0 \quad \Leftrightarrow \quad p_q(x, x) = \lim_{n\to\infty} p_q(x, x_n) = \lim_{n,m\to\infty} p_q(x_n, x_m) \tag{1}$$

$$\Leftrightarrow \quad q(x, x) = \lim_{n,m\to\infty} q(x, x_n) = \lim_{n,m\to\infty} q(x_n, x_m) \tag{2}$$

$$= \lim_{n\to\infty} q(x_n, x) = \lim_{n,m\to\infty} q(x_m, x_n). \tag{3}$$

In this paper, we shall handle Definition 1.5 in the following way.

Definition 1.5 (See [13]) Let (X, q) be a quasi-partial metric space. Then:

(ii)$_a$ a sequence $\{x_n\}$ in X is called a left-Cauchy sequence if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $n > m > N$;
(ii)$_b$ a sequence $\{x_n\}$ in X is called a right-Cauchy sequence if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $m > n > N$;
(iii)$_a$ the quasi-partial metric space is said to be left-complete if every left-Cauchy sequence $\{x_n\}$ in X is convergent;
(iii)$_b$ the quasi-partial metric space is said to be right-complete if every left-Cauchy sequence $\{x_n\}$ in X is convergent.

Remark 1 It is clear that a sequence $\{x_n\}$ in a quasi-partial metric space is Cauchy if and only if it is left-Cauchy and right-Cauchy. Analogously, a quasi-partial metric space (X, q) is complete if and only if it is left-complete and right-complete.

Very recently, Samet *et al.* [14] introduced the concept α-admissible mappings and established various fixed point theorems for such mappings in complete metric spaces. Later, in 2013, Karapınar *et al.* [15] proved the existence and uniqueness of a fixed point for triangular α-admissible mappings. For more on α-admissible and triangular α-admissible mappings, see [14, 15].

Definition 1.6 [14] Let T be a self-mapping on X and $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Definition 1.7 [15] Let T be a self-mapping on X and $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is a triangular α-admissible mapping if T is α-admissible and

$$x, y, z \in X, \quad \alpha(x, z) \geq 1 \quad \text{and} \quad \alpha(z, y) \geq 1 \quad \Rightarrow \quad \alpha(x, y) \geq 1.$$

Very recently, Popescu [16] improved the notion of α-admissible as follows.

Definition 1.8 [16] Let $T : X \to X$ be a self-mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Then T is said to be α-orbital admissible if

$$\text{(T3)} \quad \alpha(x, Tx) \geq 1 \quad \Rightarrow \quad \alpha\big(Tx, T^2x\big) \geq 1.$$

Inspired by the notion of α-admissible defined by Popescu [16], we state the following definitions.

Definition 1.9 [16] Let $T : X \to X$ be a self-mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Then T is said to be right-α-orbital admissible if

$$\text{(T3)}' \quad \alpha(x, Tx) \geq 1 \quad \Rightarrow \quad \alpha\big(Tx, T^2x\big) \geq 1,$$

and be left-α-orbital admissible if

$$\text{(T3)}'' \quad \alpha(Tx, x) \geq 1 \quad \Rightarrow \quad \alpha\big(T^2x, Tx\big) \geq 1.$$

Note that a mapping T is α-orbital admissible if it is both right-α-orbital admissible and left-α-orbital admissible.

Popescu [16] refined the notion of triangular α-admissible as follows.

Definition 1.10 [16] Let $T : X \to X$ be a self-mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Then T is said to be triangular α-orbital admissible if T is α-orbital admissible and

$$\text{(T4)}' \quad \alpha(x, y) \geq 1 \quad \text{and} \quad \alpha(y, Ty) \geq 1 \quad \Rightarrow \quad \alpha(x, Ty) \geq 1.$$

Triangular α-admissible defined by Popescu [16] imposes the following definitions.

Definition 1.11 [16] Let $T : X \to X$ be a self-mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Then T is said to be triangular α-orbital admissible if T is right-α-orbital admissible and

$$\text{(T4)}'' \quad \alpha(x, y) \geq 1 \quad \text{and} \quad \alpha(y, Ty) \geq 1 \quad \Rightarrow \quad \alpha(x, Ty) \geq 1,$$

and be triangular left-α-orbital admissible if T is α-orbital admissible and

$$\text{(T4)} \quad \alpha(Tx, x) \geq 1 \quad \text{and} \quad \alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(Tx, y) \geq 1.$$

Notice that a mapping T is triangular α-orbital admissible if it is both triangular right-α-orbital admissible and triangular left-α-orbital admissible.

It was noted in [16] that each α-admissible mapping is an α-orbital admissible mapping and each triangular α-admissible mapping is a triangular α-orbital admissible mapping. The converse is false, see *e.g.* [16], Example 7.

Definition 1.12 [16] Let (X,d) be a b-metric space, X is said α-regular if for every sequence $\{x_n\}$ in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x) \geq 1$ for all k.

Lemma 1.4 [16] *Let $T : X \to X$ be a triangular α-orbital admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for each $n \in \mathbb{N}_0$. Then we have $\alpha(x_n,x_m) \geq 1$ for all $m,n \in \mathbb{N}$ with $n < m$.*

The following result can be easily derived from Lemma 1.4.

Lemma 1.5 *Let $T : X \to X$ be a triangular α-orbital admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(Tx_0,x_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for each $n \in \mathbb{N}_0$. Then we have $\alpha(x_m,x_n) \geq 1$ for all $m,n \in \mathbb{N}$ with $n < m$.*

In this paper, we investigate and extend the existence of a fixed point of (ψ,ϕ)-contractive mappings on quasi-partial metric spaces via α-admissibility.

2 Main results

In this section, we shall present the main theorem of the paper. For our aim, we need to define the following class of auxiliary mappings: Let Λ be set of functions $\varphi : [0,+\infty) \to [0,+\infty)$ such that $\varphi^{-1}(\{0\}) = \{0\}$:

$$\Psi = \{\psi \in \Lambda \mid \psi \text{ is continuous, non-decreasing}\} \quad \text{and}$$

$$\Phi = \{\phi \in \Lambda \mid \phi \text{ is lower semi-continuous}\}.$$

Let (X,q) be a quasi-partial metric space. We consider the following expressions:

$$M(x,y) = \max\bigl\{q(x,y),q(x,Tx),q(y,Ty)\bigr\}, \tag{4}$$

$$N(x,y) = \min\bigl\{d_m^q(x,Tx),d_m^q(y,Ty),d_m^q(x,Ty),d_m^q(y,Tx)\bigr\} \tag{5}$$

for all $x,y \in X$.

Definition 2.1 Let (X,q) be a quasi-partial metric space where X is a nonempty set. We say that X is said to be α-left-regular if for every sequence $\{x_n\}$ in X such that $\alpha(x_{n+1},x_n) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x,x_{n(k)}) \geq 1$ for all k. Analogously, a quasi-partial metric space X is said to be an α-right-regular if for every sequence $\{x_n\}$ in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x) \geq 1$ for all k. We say that X is regular if it is both α-left-regular and α-right-regular.

Our first result is the following.

Theorem 2.1 *Let (X,q) be a complete quasi-partial metric space. Let $T:X\to X$ be a self-mapping. Assume that there exist $\psi\in\Psi$, $\phi\in\Phi$, $L\geq 0$ and a function $\alpha:X\times X\to[0,+\infty)$ such that for all $x,y\in X$,*

$$\alpha(x,y)\psi\bigl(q(Tx,Ty)\bigr)\leq\psi\bigl(M(x,y)\bigr)-\phi\bigl(M(x,y)\bigr)+LN(x,y). \tag{6}$$

Also, suppose that the following assertions hold:
(i) *T is triangular α-orbital admissible*;
(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$ and $\alpha(Tx_0,x_0)\geq 1$*;
(iii) *T is continuous or X is α-regular.*
Then T has a fixed point $u\in X$ and $q(u,u)=0$.

Proof We construct a sequence $\{x_n\}$ in X in the following way:

$$x_n=Tx_{n-1}\quad\text{for all } n\in\mathbb{N}.$$

If $q(x_{n_0},x_{n_0+1})=0$ for some $n_0\geq 0$, then we have $x_{n_0}=x_{n_0+1}=Tx_{n_0}$, that is, x_{n_0} is the fixed point of T. Consequently, we suppose that $q(x_n,x_{n+1})>0$ for all $n\in\mathbb{N}_0$.

By (ii), we have $\alpha(x_0,Tx_0)\geq 1$ and $\alpha(Tx_0,x_0)\geq 1$. On account of (i), we derive that

$$\begin{aligned}
&\alpha(x_0,x_1)=\alpha(x_0,Tx_0)\geq 1\quad\Rightarrow\quad\alpha(x_1,x_2)=\alpha(Tx_0,Tx_1)\geq 1,\\
&\alpha(x_1,x_0)=\alpha(Tx_0,x_0)\geq 1\quad\Rightarrow\quad\alpha(x_2,x_1)=\alpha(Tx_1,Tx_0)\geq 1.
\end{aligned}$$

Recursively, we obtain that

$$\alpha(x_n,x_{n+1})\geq 1\quad\text{and}\quad\alpha(x_{n+1},x_n)\geq 1,\quad\forall n\in\mathbb{N}_0. \tag{7}$$

Regarding (6) and (7), we find that

$$\begin{aligned}
\psi\bigl(q(x_n,x_{n+1})\bigr)&=\psi\bigl(q(Tx_{n-1},Tx_n)\bigr)\\
&\leq\alpha(x_{n-1},x_n)\psi\bigl(q(Tx_{n-1},Tx_n)\bigr)\\
&\leq\psi\bigl(M(x_{n-1},x_n)\bigr)-\phi\bigl(M(x_{n-1},x_n)\bigr)+LN(x_{n-1},x_n),
\end{aligned}\tag{8}$$

where

$$\begin{aligned}
N(x_{n-1},x_n)&=\min\bigl\{d_m^q(x_{n-1},x_n),d_m^q(x_n,x_{n+1}),d_m^q(x_{n-1},x_{n+1}),d_m^q(x_n,x_n)\bigr\}\\
&=0
\end{aligned}\tag{9}$$

and

$$\begin{aligned}
M(x_{n-1},x_n)&=\max\bigl\{q(x_{n-1},x_n),q(x_{n-1},Tx_{n-1}),q(x_n,Tx_n)\bigr\}\\
&=\max\bigl\{q(x_{n-1},x_n),q(x_{n-1},x_n),q(x_n,x_{n+1})\bigr\}\\
&=\max\bigl\{q(x_{n-1},x_n),q(x_n,x_{n+1})\bigr\}.
\end{aligned}\tag{10}$$

Thus, we conclude from (8) that

$$\begin{aligned} \psi\bigl(q(x_n,x_{n+1})\bigr) &\le \psi\bigl(\max\bigl\{q(x_{n-1},x_n),q(x_n,x_{n+1})\bigr\}\bigr) \\ &\quad - \phi\bigl(\max\bigl\{q(x_{n-1},x_n),q(x_n,x_{n+1})\bigr\}\bigr) \end{aligned} \tag{11}$$

by taking (10) and (9) into account.

If for some n we have $\max\{q(x_{n-1},x_n),q(x_n,x_{n+1})\} = q(x_n,x_{n+1})$, then (11) yields that

$$\psi\bigl(q(x_n,x_{n+1})\bigr) \le \psi\bigl(q(x_n,x_{n+1})\bigr) - \phi\bigl(q(x_n,x_{n+1})\bigr).$$

Consequently, we conclude that $\phi(q(x_n,x_{n+1})) = 0$. Since $\phi^{-1}(\{0\}) = \{0\}$, we get $q(x_n,x_{n+1}) = 0$, which contradicts the assumption that $q(x_n,x_{n+1}) > 0$ for all $n \in \mathbb{N}_0$. Thus, we have

$$M(x_{n-1},x_n) = q(x_{n-1},x_n). \tag{12}$$

Hence, (8) turns into

$$\psi\bigl(q(x_n,x_{n+1})\bigr) \le \psi\bigl(q(x_{n-1},x_n)\bigr) - \phi\bigl(q(x_{n-1},x_n)\bigr) \le \psi\bigl(q(x_{n-1},x_n)\bigr) \quad \text{for all } n \in \mathbb{N}.$$

Due to the property of the auxiliary function ψ, we have

$$q(x_n,x_{n+1}) \le q(x_{n-1},x_n) \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

Eventually, we observe that the sequence $\{q(x_n,x_{n+1})\}$ is non-increasing. So, there exists $\delta \ge 0$ such that

$$\lim_{n\to+\infty} q(x_n,x_{n+1}) = \delta.$$

If $\delta > 0$, taking $\limsup_{n\to+\infty}$ in inequality (11), by keeping (10) and (12) in the mind, we obtain that

$$\limsup_{n\to+\infty} \psi\bigl(q(x_n,x_{n+1})\bigr) \le \limsup_{n\to+\infty} \psi\bigl(q(x_{n-1},x_n)\bigr) - \liminf_{n\to+\infty} \phi\bigl(q(x_{n-1},x_n)\bigr).$$

By continuity of ψ and lower semi-continuity of ϕ, we obtain $\psi(\delta) \le \psi(\delta) - \phi(\delta)$, which is a contradiction. So,

$$\lim_{n\to+\infty} q(x_n,x_{n+1}) = 0. \tag{14}$$

Analogously, we derive that

$$\lim_{n\to+\infty} q(x_{n+1},x_n) = 0. \tag{15}$$

Now, we shall show that $\{x_n\}$ is a Cauchy sequence in the quasi-partial metric space (X,q), that is, the sequence $\{x_n\}$ is left-Cauchy and right-Cauchy.

Suppose that $\{x_n\}$ is not a left-Cauchy sequence in (X,q). Then there is $\varepsilon > 0$ such that for each integer k there exist integers $n(k) > m(k) > k$ such that

$$q(x_{n(k)}, x_{m(k)}) \geq \varepsilon. \tag{16}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ so that it is the smallest integer with $n(k) > m(k)$ satisfying (16). Consequently, we have

$$q(x_{n(k)-1}, x_{m(k)}) < \varepsilon. \tag{17}$$

Due to the triangle inequality, we have

$$\begin{aligned}
\varepsilon &\leq q(x_{n(k)}, x_{m(k)}) \\
&\leq q(x_{n(k)}, x_{n(k)-1}) + q(x_{n(k)-1}, x_{m(k)}) - q(x_{n(k)-1}, x_{n(k)-1}) \\
&< q(x_{n(k)}, x_{n(k)-1}) + \varepsilon.
\end{aligned} \tag{18}$$

Letting $k \to \infty$ and taking (14) into account, we get that

$$\lim_{k\to\infty} q(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{19}$$

On the other hand, again by the triangle inequality, we find that

$$\begin{aligned}
q(x_{n(k)}, x_{m(k)}) &\leq q(x_{n(k)}, x_{n(k)-1}) + q(x_{n(k)-1}, x_{m(k)-1}) + q(x_{m(k)-1}, x_{m(k)}) \\
&\quad - q(x_{n(k)-1}, x_{n(k)-1}) - q(x_{m(k)-1}, x_{m(k)-1}) \\
&\leq q(x_{n(k)}, x_{n(k)-1}) + q(x_{n(k)-1}, x_{m(k)-1}) + q(x_{m(k)-1}, x_{m(k)})
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
q(x_{n(k)-1}, x_{m(k)-1}) &\leq q(x_{n(k)-1}, x_{n(k)}) + q(x_{n(k)}, x_{m(k)}) + q(x_{m(k)}, x_{m(k)-1}) \\
&\quad - q(x_{n(k)}, x_{n(k)}) - q(x_{m(k)}, x_{m(k)}) \\
&\leq q(x_{n(k)-1}, x_{n(k)}) + q(x_{n(k)}, x_{m(k)}) + q(x_{m(k)}, x_{m(k)-1}).
\end{aligned} \tag{21}$$

Letting $k \to \infty$ and taking (14), (15), (19), (20), (21) into account, we derive that

$$\lim_{k\to\infty} q(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon, \tag{22}$$

$$\begin{aligned}
q(x_{n(k)-1}, x_{m(k)}) &\leq q(x_{n(k)-1}, x_{n(k)}) + q(x_{n(k)}, x_{m(k)}) - q(x_{n(k)}, x_{n(k)}) \\
&\leq q(x_{n(k)-1}, x_{n(k)}) + q(x_{n(k)}, x_{m(k)}).
\end{aligned} \tag{23}$$

Letting $k \to \infty$ and taking (14), (18), (20), (23) into account, we derive that

$$\lim_{k\to\infty} q(x_{n(k)-1}, x_{m(k)}) = \varepsilon. \tag{24}$$

Since T is triangular α-orbital admissible, from Lemma 1.4 and Lemma 1.5 we derive that

$$\alpha(x_n, x_m) \geq 1 \quad \text{and} \quad \alpha(x_m, x_n) \geq 1, \quad \forall n > m \in \mathbb{N}_0. \tag{25}$$

Regarding (6) and (25), we find that

$$\begin{aligned} \psi\bigl(q(x_{n(k)}, x_{m(k)})\bigr) &= \psi\bigl(q(Tx_{n(k)-1}, Tx_{m(k)-1})\bigr) \\ &\leq \alpha(x_{n(k)-1}, x_{m(k)-1})\psi\bigl(q(Tx_{n(k)-1}, Tx_{m(k)-1})\bigr) \\ &\leq \psi\bigl(M(x_{n(k)-1}, x_{m(k)-1})\bigr) - \phi\bigl(M(x_{n(k)-1}, x_{m(k)-1})\bigr) \\ &\quad + LN(x_{n(k)-1}, x_{m(k)-1}), \end{aligned} \tag{26}$$

where

$$\begin{aligned} &N(x_{n(k)-1}, x_{m(k)-1}) \\ &\quad = \min\bigl\{d_m^q(x_{n(k)-1}, x_{n(k)}), d_m^q(x_{m(k)-1}, x_{m(k)}), d_m^q(x_{n(k)-1}, x_{m(k)}), d_m^q(x_{m(k)-1}, x_{n(k)})\bigr\} \end{aligned} \tag{27}$$

and

$$M(x_{n(k)-1}, x_{m(k)-1}) = \max\bigl\{q(x_{n(k)-1}, x_{m(k)-1}), q(x_{n(k)-1}, x_{n(k)}), q(x_{m(k)-1}, x_{m(k)})\bigr\}. \tag{28}$$

Regarding (14) and (15), we note that

$$\lim_{k\to\infty} N(x_{n(k)-1}, x_{m(k)-1}) = 0. \tag{29}$$

On the other hand, we get that

$$\lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon \tag{30}$$

due to the limits (14), (15), (19).

From the above observation, letting $k \to \infty$ in (26), we obtain

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon).$$

So, $\phi(\varepsilon) = 0$, which is a contradiction with respect to the fact that $\varepsilon > 0$. Thus $\{x_n\}$ is a left-Cauchy sequence in the metric space (X, q). Analogously, we derive that $\{x_n\}$ is a right-Cauchy sequence in the metric space (X, q). Since (X, q) is complete, then from Lemma 1.3 (X, d_{p_q}) is a complete metric space. Therefore, the sequence $\{x_n\}$ converges to a point $u \in X$ in (X, d_{p_q}), that is,

$$\lim_{n\to+\infty} d_{p_q}(x_n, u) = 0.$$

Again, from Lemma 1.3,

$$p_q(u, u) = \lim_{n\to+\infty} p_q(x_n, u) = \lim_{n\to+\infty} p_q(x_n, x_n).$$

On the other hand, by (14) and the condition (QPM2) from Definition 1.3,

$$\lim_{n\to+\infty} q(x_n, x_n) = 0. \tag{31}$$

So, it follows that

$$q(u,u) = \lim_{n\to+\infty} \frac{1}{2}\big[q(x_n,u) + q(u,x_n)\big] = \lim_{n\to+\infty} q(x_n,x_n) = 0. \tag{32}$$

Now, for proving fixed point of T, first we suppose that T is continuous, then we have

$$Tu = \lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} x_{n+1} = u.$$

So, u is a fixed point of T.

As the last step, suppose that X is α-regular. Hence it is α-right-regular, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Now, we show that $q(u, Tu) = 0$. Assume this is not true, then from (6) we obtain

$$\begin{aligned}
\psi\big(q(x_{n(k)+1}, Tu)\big) &= \psi\big(q(Tx_{n(k)}, Tu)\big) \\
&\leq \alpha(x_{n(k)}, u)\psi\big(q(Tx_{n(k)}, Tu)\big) \\
&\leq \psi\big(M(x_{n(k)}, u)\big) - \phi\big(M(x_{n(k)}, u)\big) \\
&\quad + L\min\big\{d_m^q(x_{n(k)}, Tx_{n(k)}), d_m^q(u, Tu), d_m^q(x_{n(k)}, Tu), d_m^q(u, Tx_{n(k)})\big\},
\end{aligned}$$

where

$$M(x_{n(k)}, u) = \max\big\{q(x_{n(k)}, u), q(x_{n(k)} Tx_{n(k)}), q(u, Tu)\big\} \tag{33}$$

$$\qquad = \max\big\{q(x_{n(k)}, u), q(x_{n(k)} x_{n(k)+1}), q(u, Tu)\big\}. \tag{34}$$

It is obvious that $\lim_{k\to+\infty} q(x_{n(k)}, Tu) = q(u, Tu)$. Therefore, using (14) and (32), we deduce that

$$\lim_{K\to+\infty} M(x_{n(k)}, u) \leq \max\big\{0, 0, q(u, Tu)\big\} = q(u, Tu).$$

Also,

$$\lim_{K\to+\infty} N(x_{n(k)}, u) = 0$$

because (14), (15) and (31) give $\lim_{n\to+\infty} d_m^q(x_{n(k)}, Tx_{n(k)}) = 0$. Now, by using the properties of ψ and ϕ and taking the upper limit as $n \to +\infty$, we obtain

$$\psi\big(q(u, Tu)\big) \leq \psi\big(q(u, Tu)\big) - \phi\big(q(u, Tu)\big).$$

Then $\phi(q(u, Tu)) = 0$, *i.e.*, $q(u, Tu) = 0$, and so $Tu = u$. Now, we conclude that T has a fixed point $u \in X$ and $q(u, u) = 0$. □

As a consequence of Theorem 2.1, we may state the following corollaries.

First, taking $L = 0$ in Theorem 2.1, we have the following.

Corollary 2.1 *Let (X,q) be a complete quasi-partial metric space. Let $T : X \to X$ be a self-mapping. Suppose that there exist $\psi \in \Psi$, $\phi \in \Phi$ and a function $\alpha : X \times X \to [0,+\infty)$ such that for all $x, y \in X$,*

$$\alpha(x,y)\psi\bigl(q(Tx,Ty)\bigr) \leq \psi\bigl(M(x,y)\bigr) - \phi\bigl(M(x,y)\bigr). \tag{35}$$

Also, suppose that the following assertions hold:

(i) *T is triangular α-orbital admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$*;
(iii) *T is continuous or X is α-regular.*

Then T has a fixed point $u \in X$ and $q(u,u) = 0$.

Corollary 2.2 *Let (X,q) be a complete quasi-partial metric space. Let $T : X \to X$ be a self-mapping. Suppose that there exist $k \in [0,1)$, $L \geq 0$ and a function $\alpha : X \times X \to [0,+\infty)$ such that for all $x, y \in X$,*

$$\alpha(x,y)q(Tx,Ty) \leq kM(x,y) + LN(x,y). \tag{36}$$

Also, suppose that the following assertions hold:

(i) *T is triangular α-orbital admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$*;
(iii) *T is continuous or X is α-regular.*

Then T has a fixed point $u \in X$ and $q(u,u) = 0$.

Proof It follows by taking $\psi(t) = t$ and $\phi(t) = (1-k)t$ in Theorem 2.1. □

Corollary 2.3 *Let (X,q) be a complete quasi-partial metric space. Let $T : X \to X$ be a self-mapping. Suppose that there exist $k \in [0,1)$, $L \geq 0$ and a function $\alpha : X \times X \to [0,+\infty)$ such that for all $x, y \in X$,*

$$\alpha(x,y)q(Tx,Ty) \leq kM(x,y). \tag{37}$$

Also, suppose that the following assertions hold:

(i) *T is triangular α-orbital admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$*;
(iii) *T is continuous or X is α-regular.*

Then T has a fixed point $u \in X$ and $q(u,u) = 0$.

Proof It is sufficient to take $L = 0$ in Corollary 2.2. □

Corollary 2.4 *Let (X,q) be a complete quasi-partial metric space. Let $T : X \to X$ be a self-mapping. Suppose that there exist $k \in [0,1)$, $L \geq 0$ and a function $\alpha : X \times X \to [0,+\infty)$ such that for all $x, y \in X$,*

$$\alpha(x,y)q(Tx,Ty) \leq kq(x,y). \tag{38}$$

Also, suppose that the following assertions hold:

(i) *T is triangular α-orbital admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$*;
(iii) *T is continuous or X is α-regular.*

Then T has a fixed point $u \in X$ and $q(u,u) = 0$.

Proof By following the lines in the proof of Theorem 2.1, we derive the desired result. We skip the details to avoid repetition. □

Denote by Λ' the set of functions $\lambda : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:
(1) λ is a Lebesgue-integrable mapping on each compact subset of $[0, +\infty)$,
(2) for every $\epsilon > 0$, we have $\int_0^\epsilon \lambda(s)\, ds > 0$.

We have the following result.

Corollary 2.5 *Let (X, q) be a complete quasi-partial metric space. Let $T : X \to X$ be a self-mapping. Suppose that there exist $\lambda, \beta \in \Lambda'$, $L \geq 0$ and a function $\alpha : X \times X \to [0, +\infty)$ such that for all $x, y \in X$,*

$$\int_0^{\alpha(x,y)q(Tx,Ty)} \lambda(s)\, ds \leq \int_0^{M(x,y)} \lambda(s)\, ds - \int_0^{M(x,y)} \beta(s)\, ds + LN(x,y). \tag{39}$$

Also, suppose that the following assertions hold:
(i) *T is triangular α-orbital admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$*;
(iii) *T is continuous or X is α-regular.*

Then T has a fixed point $u \in X$ and $q(u,u) = 0$.

Proof It follows from Theorem 2.1 by taking

$$\psi(t) = \int_0^t \lambda(s)\, ds$$

and

$$\phi(t) = \int_0^t \beta(s)\, ds.$$

□

Taking $L = 0$ in Corollary 2.5, we obtain the following result.

Corollary 2.6 *Let (X, q) be a complete quasi-partial metric space. Let $T : X \to X$ be a self-mapping. Suppose that there exist $\lambda, \beta \in \Lambda'$ and a function $\alpha : X \times X \to [0, +\infty)$ such that for all $x, y \in X$,*

$$\int_0^{\alpha(x,y)q(Tx,Ty)} \lambda(s)\, ds \leq \int_0^{M(x,y)} \lambda(s)\, ds - \int_0^{M(x,y)} \beta(s)\, ds. \tag{40}$$

Also, suppose that the following assertions hold:
(i) *T is triangular α-orbital admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$*;

(iii) *T is continuous or X is α-regular.*
Then T has a fixed point $u \in X$ and $q(u,u) = 0$.

Now, let $\mathcal{F}$ be the set of functions $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying the following hypotheses:

(φ_1) φ is non-decreasing,
(φ_2) $\sum_{n=0}^{+\infty} \varphi^n(t)$ converges for all $t > 0$.

Note that if $\varphi \in \mathcal{F}$, φ is said to be a (C)-comparison function. It is easily proved that if φ is a (C)-comparison function, then $\varphi(t) < t$ for any $t > 0$. Our second main result is as follows.

Theorem 2.2 *Let (X,q) be a complete quasi-partial metric space. Let $T : X \to X$ be a mapping such that there exist $\varphi \in \mathcal{F}$, $L \geq 0$ and a function $\alpha : X \times X \to [0,+\infty)$ such that for all $x,y \in X$,*

$$\alpha(x,y)q(Tx,Ty) \leq \varphi\bigl(M(x,y)\bigr) + LN(x,y). \tag{41}$$

Also, suppose that the following assertions hold:
(i) *T is triangular α-orbital admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(Tx_0,x_0) \geq 1$;*
(iii) *T is continuous or X is α-regular.*
Then T has a fixed point $u \in X$ and $q(u,u) = 0$.

Proof By following the lines in the proof of Theorem 2.1, we derive that

$$\alpha(x_n,x_{n+1}) \geq 1 \quad \text{and} \quad \alpha(x_{n+1},x_n) \geq 1, \quad \forall n \geq 1. \tag{42}$$

By combining (41) and (42), we find that

$$\begin{aligned} q(x_n,x_{n+1}) &\leq \alpha(x_{n-1},x_n)q(x_n,x_{n+1}) \\ &\leq \varphi\bigl(M(x_{n-1},x_n)\bigr) + LN(x_{n-1},x_n). \end{aligned}$$

So,

$$\begin{aligned} q(x_n,x_{n+1}) &= q(Tx_{n-1},Tx_n) \\ &\leq \varphi\bigl(M(x_{n-1},x_n)\bigr) \\ &\quad + L\min\bigl\{d_m^q(x_{n-1},Tx_{n-1}), d_m^q(x_n,Tx_n), d_m^q(x_{n-1},Tx_n), d_m^q(x_n,Tx_{n-1})\bigr\}. \end{aligned}$$

Now, similar to proving Theorem 2.1, we obtain

$$\min\bigl\{d_m^q(x_{n-1},Tx_{n-1}), d_m^q(x_n,Tx_n), d_m^q(x_{n-1},Tx_n), d_m^q(x_n,Tx_{n-1})\bigr\} = 0$$

and

$$M(x_{n-1},x_n) = \max\bigl\{q(x_{n-1},x_n), q(x_n,x_{n+1})\bigr\}.$$

Therefore

$$q(x_n, x_{n+1}) \leq \varphi\bigl(\max\bigl\{q(x_{n-1}, x_n), q(x_n, x_{n+1})\bigr\}\bigr). \tag{43}$$

If, for some $n \geq 1$, we have $q(x_{n-1}, x_n) \leq q(x_n, x_{n+1})$. So, from (43), we obtain that

$$q(x_n, x_{n+1}) \leq \varphi\bigl(q(x_n, x_{n+1})\bigr) < q(x_n, x_{n+1}),$$

which is a contradiction. Thus, for all $n \geq 1$, we have

$$M(x_{n-1}, x_n) = \max\bigl\{q(x_{n-1}, x_n), q(x_n, x_{n+1})\bigr\} = q(x_{n-1}, x_n). \tag{44}$$

Using (43) and (44), we get that

$$q(x_n, x_{n+1}) \leq \varphi\bigl(q(x_{n-1}, x_n)\bigr) \quad \text{for all } n \geq 1.$$

Iteratively, we find that

$$q(x_n, x_{n+1}) \leq \varphi^n\bigl(q(x_0, x_1)\bigr)$$

for all $n \geq 0$. By the triangle inequality, we get, for $m > n$,

$$\begin{aligned}
q(x_n, x_m) &\leq \sum_{k=n}^{k=m-1} q(x_k, x_{k+1}) - \sum_{k=n+1}^{k=m-1} q(x_k, x_k) &(45)\\
&\leq \sum_{k=n}^{k=m-1} q(x_k, x_{k+1}) &(46)\\
&\leq \sum_{k=n}^{k=+\infty} q(x_k, x_{k+1}) &(47)\\
&\leq \sum_{k=n}^{k=+\infty} \varphi^k\bigl(q(x_0, x_1)\bigr) &(48)\\
&= \varphi^n\bigl(q(x_0, x_1)\bigr) \sum_{k=0}^{k=+\infty} \varphi^n\bigl(q(x_0, x_1)\bigr). &(49)
\end{aligned}$$

Since $\sum_{k=0}^{+\infty} \varphi^k(q(x_0, x_1)) < \infty$, then $\lim_{n,m\to+\infty} q(x_n, x_m) = 0$. So, $\{x_n\}$ is a right-Cauchy sequence in (X, q). Similarly, since $\alpha(Tx_0, x_0) \geq 1$, we get

$$\lim_{n\to+\infty} q(x_m, x_n) = 0. \tag{50}$$

So, $\{x_n\}$ is a left-Cauchy sequence in (X, q). By part (ii) of Definition 1.5, $\{x_n\}$ is a Cauchy sequence in a complete quasi-partial metric space (X, q), then $\{x_n\}$ converges, with respect to τ_q, to a point $u \in X$ such that

$$\begin{aligned}
q(u, u) &= \lim_{n\to+\infty} q(x_n, u) = \lim_{n\to+\infty} q(u, x_n)\\
&= \lim_{n,m\to+\infty} q(x_n, x_m) = \lim_{n,m\to+\infty} q(x_m, x_n) = 0. &(51)
\end{aligned}$$

Suppose that T is continuous, then we have

$$Tu = \lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} x_{n+1} = u.$$

So, u is a fixed point of T.

Now, suppose that X is α-regular. Hence it is both α-left-regular and α-right-regular. Thus, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Now, we claim that $q(u, Tu) = 0$. Suppose the contrary, then $q(u, Tu) > 0$. By (41), we have

$$\begin{aligned} q(u, Tu) &\leq q(u, x_{n(k)+1}) + q(Tx_{n(k)}, Tu) - q(x_{n(k)+1}, Tx_{n(k)}) && (52)\\ &\leq q(u, x_{n(k)+1}) + q(Tx_{n(k)}, Tu) && (53)\\ &\leq q(u, x_{n(k)+1}) + \alpha(x_{n(k)}, u)\varphi\bigl(M(x_{n(k)}, u)\bigr) && (54)\\ &\quad + L\min\bigl\{d_m^q(x_{n(k)}, Tx_{n(k)}), d_m^q(u, Tu), d_m^q(u, Tx_{n(k)}), d_m^q(x_{n(k)}, Tu)\bigr\}, && (55)\end{aligned}$$

where

$$\begin{aligned} M(x_{n(k)}, u) &= \max\bigl\{q(x_{n(k)}, u), q(x_{n(k)}, Tx_{n(k)}), q(u, Tu)\bigr\} && (56)\\ &= \max\bigl\{q(x_{n(k)}, u), q(x_{n(k)}, x_{n(k)+1}), q(u, Tu)\bigr\}. && (57)\end{aligned}$$

By (51), we have

$$\lim_{k\to+\infty} \min\bigl\{d_m^q(x_{n(k)}, Tx_{n(k)}), d_m^q(u, Tu), d_m^q(u, Tx_{n(k)}), d_m^q(x_{n(k)}, Tu)\bigr\} = 0$$

and

$$\lim_{k\to+\infty} M(x_{n(k)}, u) = q(u, Tu).$$

Therefore

$$q(u, Tu) \leq \varphi\bigl(q(u, Tu)\bigr) < q(u, Tu),$$

which is a contradiction. That is, $q(u, Tu) = 0$. Thus, we obtained that u is a fixed point for T and $q(u, u) = 0$. □

The following corollary is a generalization of Theorem 17 in [22].

Corollary 2.7 *Let (X, q) be a complete quasi-partial metric space. Let $n : X \to X$ be a mapping such that there exist $\varphi \in \mathcal{F}$ and a function $\alpha : X \times X \to [0, +\infty)$ such that for all $x, y \in X$,*

$$\alpha(x, y)q(Tx, Ty) \leq \varphi\bigl(M(x, y)\bigr). \tag{58}$$

Also, suppose that the following assertions hold:

(i) *T is triangular α-orbital admissible*;

(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(Tx_0, x_0) \geq 1$;
(iii) T *is continuous or* X *is* α*-regular.*

Then T *has a fixed point* $u \in X$ *and* $q(u,u) = 0$.

We give the following two examples making effective our obtained results.

Example 2.1 Let $X = [0, +\infty)$ and $q(x,y) = |x-y| + x$ for all $x, y \in X$. Then (X, q) is a complete quasi-partial metric space. Consider $T : X \to X$ defined by

$$Tx = \frac{x}{2}.$$

Define $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{2}{3} & \text{otherwise.} \end{cases}$$

Take $\psi(t) = \frac{2t}{3}$ and $\phi(t) = \frac{t}{3}$ for all $t \geq 0$. Note that $\psi \in \Psi$ and $\phi \in \Phi$. Take $x \leq y$, then

$$\begin{aligned} \alpha(x,y)\psi\bigl(q(Tx,Ty)\bigr) &= \alpha(x,y)\psi\bigl(|Tx-Ty| + Tx\bigr) \\ &= \psi\left(\left|\frac{x}{2} - \frac{y}{2}\right| + \frac{x}{2}\right) = \frac{y}{3} \leq \frac{y}{2} = y - \frac{y}{2} \\ &= \psi\bigl(M(x,y)\bigr) - \phi\bigl(M(x,y)\bigr) \quad \left(\text{since } M(x,y) = \frac{3y}{2}\right) \\ &\leq \psi\bigl(M(x,y)\bigr) - \phi\bigl(M(x,y)\bigr) + LN(x,y) \end{aligned}$$

for all $L \geq 0$. Now, let $y < x$, then

$$\begin{aligned} \alpha(x,y)\psi\bigl(q(Tx,Ty)\bigr) &= \alpha(x,y)\psi\bigl(|Tx-Ty| + Tx\bigr) \\ &= \frac{2}{3}\psi\left(\left|\frac{x}{2} - \frac{y}{2}\right| + \frac{x}{2}\right) = \frac{4}{9}x - \frac{2}{9}y. \end{aligned}$$

We have two possibilities for $M(x,y)$.

Case 1: if $M(x,y) = 2x - y$ then

$$\begin{aligned} \alpha(x,y)\psi\bigl(q(Tx,Ty)\bigr) &= \frac{4}{9}x - \frac{2}{9}y \leq \frac{2}{3}x - \frac{y}{3} = \psi\bigl(M(x,y)\bigr) - \phi\bigl(M(x,y)\bigr) \\ &\leq \psi\bigl(M(x,y)\bigr) - \phi\bigl(M(x,y)\bigr) + LN(x,y) \end{aligned}$$

for all $L \geq 0$.

Case 2: if $M(x,y) = \frac{3x}{2}$ then

$$\begin{aligned} \alpha(x,y)\psi\bigl(q(Tx,Ty)\bigr) &= \frac{4}{9}x - \frac{2}{9}y \leq \frac{x}{2} = \psi\bigl(M(x,y)\bigr) - \phi\bigl(M(x,y)\bigr) \\ &\leq \psi\bigl(M(x,y)\bigr) - \phi\bigl(M(x,y)\bigr) + LN(x,y) \end{aligned}$$

for all $L \geq 0$.

Moreover, T is triangular α-orbital admissible, $\alpha(0, T0) \geq 1$ and $\alpha(T0, 0) \geq 1$. Thus, by applying Theorem 2.1, n has a fixed point, which is $u = 0$.

Example 2.2 Let $X = [0, +\infty)$ and $q(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, q) is a complete quasi-partial metric space. Consider $n : X \to X$ defined by

$$Tx = \frac{x}{1+x}.$$

Define $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Take $\psi(t) = 1$ and $\phi(t) = \frac{1}{1+t}$ for all $t \geq 0$. Note that $\psi \in \Psi$ and $\phi \in \Phi$. Take $x \leq y$, then

$$\begin{aligned} \alpha(x,y)\psi\big(q(Tx, Ty)\big) &= \alpha(x,y)\left(\frac{y}{1+y}\right) \\ &\leq \frac{y}{1+y} = 1 - \frac{1}{1+y} \\ &= \psi\big(M(x,y)\big) - \phi\big(M(x,y)\big) \quad \big(\text{since } M(x,y) = y\big) \\ &\leq \psi\big(M(x,y)\big) - \phi\big(M(x,y)\big) + LN(x,y) \end{aligned}$$

for all $L \geq 0$. Moreover, T is triangular α-orbital admissible, $\alpha(0, T0) \geq 1$ and $\alpha(T0, 0) \geq 1$. Thus, by applying Theorem 2.1, n has a fixed point, which is $u = 0$.

Example 2.3 Let $X = [0, 2]$ and $q : X \times X \to \mathbb{R}^+$ be defined by $q(x, y) = \max\{x, y\}$. Then (X, q) is a complete quasi-partial metric space. Define $T : X \to X$ by

$$T(x) = \begin{cases} \frac{x^2}{x+1} & \text{if } x \in [0,1), \\ 0 & \text{if } x \in [1,2), \\ \frac{4}{3} & \text{if } x = 2 \end{cases}$$

and $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in [0,1), \\ \frac{1}{2} & \text{if } x, y \in [1,2], \\ 0 & \text{otherwise.} \end{cases}$$

Take $\varphi : [0, +\infty) \to [0, +\infty)$ defined by

$$\varphi(t) = \frac{t^2}{t+1}.$$

By induction, we have $\varphi^n(t) \leq t(\frac{t}{1+t})^n$ for all $n \geq 1$, so it is clear that φ is a (C)-comparison function. Now, we show that (41) is satisfied for all $x, y \in X$. It suffices to prove it for $x \leq y$. Consider the following six cases.

Case 1. Let $x, y \in [0,1)$, then

$$\begin{aligned} \alpha(x,y)q(Tx,Ty) &= \alpha(x,y)\left(\frac{y^2}{y+1}\right) && (59)\\ &= \frac{y^2}{y+1} = \varphi\bigl(q(x,y)\bigr) && (60)\\ &\leq \varphi\bigl(M(x,y)\bigr). && (61) \end{aligned}$$

Case 2. Let $x, y \in [1,2)$, then

$$\begin{aligned} \alpha(x,y)q(nx,ny) &= \frac{1}{2}q(0,0) = 0 && (62)\\ &\leq \varphi\bigl(M(x,y)\bigr). && (63) \end{aligned}$$

Case 3. Let $x = y = 2$, then

$$\begin{aligned} \alpha(x,y)q(Tx,Ty) &= \frac{1}{2}q\left(\frac{4}{3},\frac{4}{3}\right) = \frac{4}{6} && (64)\\ &\leq \frac{4}{3} = \varphi(2) && (65)\\ &\leq \varphi\bigl(M(x,y)\bigr). && (66) \end{aligned}$$

Case 4. Let $x \in [0,1)$ and $y \in [1,2)$, then

$$\begin{aligned} \alpha(x,y)q(nx,ny) &= \alpha(x,y)q\left(\frac{x^2}{x+1},0\right) = 0 && (67)\\ &\leq \varphi\bigl(M(x,y)\bigr). && (68) \end{aligned}$$

Case 5. Let $x \in [0,1)$ and $y = 2$, then

$$\begin{aligned} \alpha(x,y)q(Tx,Ty) &= \alpha(x,y)q\left(\frac{x^2}{x+1},\frac{4}{3}\right) = 0 && (69)\\ &\leq \varphi\bigl(M(x,y)\bigr). && (70) \end{aligned}$$

Case 6. Let $x \in [1,2)$ and $y = 2$, then

$$\begin{aligned} \alpha(x,y)q(nx,ny) &= \frac{1}{2}q\left(0,\frac{4}{3}\right) = \frac{4}{6} && (71)\\ &\leq \frac{4}{3} = \varphi(2) && (72)\\ &= \varphi\bigl(q(x,y)\bigr) && (73)\\ &\leq \varphi\bigl(M(x,y)\bigr). && (74) \end{aligned}$$

Since, for all $x, y \in X$, $LN(x,y) \geq 0$, so we have

$$\alpha(x,y)q(Tx,Ty) \leq \varphi\bigl(M(x,y)\bigr) + LN(x,y).$$

Moreover, T is triangular α-orbital admissible, $\alpha(0, T0) \geq 1$ and $\alpha(n0, 0) \geq 1$. Then all the required hypotheses of Theorem 2.2 are satisfied. So, T has a fixed point, which is $u = 0$.

3 Consequences and final remarks

Now, we will show that many existing results in the literature can be deduced easily from our Corollary 2.7.

3.1 Standard fixed point theorems

Taking in Corollary 2.7 $\alpha(x, y) = 1$ for all $x, y \in X$, we derive immediately the following fixed point theorem.

Corollary 3.1 *Let (X, q) be a complete quasi-partial metric space and $T : X \to X$ be a given mapping. Suppose that there exists a function $\varphi \in \mathcal{F}$ such that*

$$q(Tx, Ty) \leq \varphi\bigl(M(x, y)\bigr)$$

for all $x, y \in X$. Then T has a unique fixed point.

Corollary 3.2 *Let (X, q) be a complete quasi-partial metric space and $T : X \to X$ be a given mapping. Suppose that there exists $k \in [0, 1)$ such that*

$$q(Tx, Ty) \leq kM(x, y)$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof It is sufficient to take $\varphi(t) = kt$, where $k \in [0, 1)$, in the above corollary. □

Corollary 3.3 *Let (X, q) be a complete quasi-partial metric space and $T : X \to X$ be a given mapping. Suppose that there exists $k \in [0, 1)$ such that*

$$q(Tx, Ty) \leq kq(x, y)$$

for all $x, y \in X$. Then T has a unique fixed point.

Remark 2 As we show in Corollaries 3.1-3.3, we can list some more results as a consequence of our main theorems by choosing the auxiliary function $\alpha : X \times X \to [0, \infty)$ as it was done in *e.g.* [14, 17].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details

[1]Department of Mathematics, Atilim University, İncek, Ankara, 06836, Turkey. [2]Nonlinear Analysis and Applied Mathematics Research Group (NAAM), King Abdulaziz University, Jeddah, Saudi Arabia. [3]Young Researchers and Elite Club, Roudehen Branch, Islamic Azad University, Roudehen, Iran.

Received: 8 April 2015 Accepted: 14 June 2015 Published online: 03 July 2015

References

1. Matthews, SG: Partial metric topology. In: Proceedings of the 8th Summer Conference on General Topology and Applications. Annals of the New York Academy of Sciences, vol. 728, pp. 183-197 (1994)
2. Abbas, M, Ilić, D: Common fixed points of generalized almost nonexpansive mappings. Filomat **24**(3), 11-18 (2010)
3. Abedeljawad, T, Karapınar, E, Taş, K: Existence and uniqueness of common fixed point on partial metric spaces. Appl. Math. Lett. **24**, 1894-1899 (2011)
4. Abedeljawad, T, Karapınar, E, Taş, K: A generalized contraction principle with control functions on partial metric spaces. Comput. Math. Appl. **63**, 716-719 (2012)
5. Altun, I, Erduran, A: Fixed point theorems for monotone mappings on partial metric spaces. Fixed Point Theory Appl. **2011**, Article ID 508730 (2011)
6. Aydi, H: Fixed point theorems for generalized weakly contractive condition in ordered partial metric spaces. J. Nonlinear Anal. Optim., Theory Appl. **2**(2), 33-48 (2011)
7. Aydi, H, Karapınar, E, Shatanawi, W: Coupled fixed point results for (ψ,φ)-weakly contractive condition in ordered partial metric spaces. Comput. Math. Appl. **62**, 4449-4460 (2011)
8. Ćirić, L, Samet, B, Aydi, H, Vetro, C: Common fixed points of generalized contractions on partial metric spaces and an application. Appl. Math. Comput. **218**, 2398-2406 (2011)
9. Di Bari, C, Milojević, M, Radenović, S, Vetro, P: Common fixed points for self-mappings on partial metric spaces. Fixed Point Theory Appl. **2012**, 140 (2012)
10. Ilić, D, Pavlović, V, Rakocecić, V: Some new extensions of Banach's contraction principle to partial metric space. Appl. Math. Lett. **24**, 1326-1330 (2011)
11. Romaguera, S: Fixed point theorems for generalized contractions on partial metric spaces. Topol. Appl. **159**, 194-199 (2012)
12. Karapınar, E: Weak ϕ-contraction on partial metric spaces. J. Comput. Anal. Appl. **14**, 206-210 (2012)
13. Karapınar, E, Erhan, IM, Öztürk, A: Fixed point theorems on quasi-partial metric spaces. Math. Comput. Model. **57**, 2442-2448 (2013)
14. Samet, B, Vetro, C, Vetro, P: α-ψ-Contractive type mappings. Nonlinear Anal. **75**, 2154-2165 (2012)
15. Karapınar, E, Kuman, P, Salimi, P: On α-ψ-Meir-Keeler contractive mappings. Fixed Point Theory Appl. **2013**, 94 (2013)
16. Popescu, O: Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces. Fixed Point Theory Appl. **2014**, 190 (2014)
17. Karapınar, E, Samet, B: Generalized α-ψ contractive type mappings and related fixed point theorems with applications. Abstr. Appl. Anal. **2012**, Article ID 793486 (2012)
18. Karapınar, E: Some fixed point theorems on the class of comparable partial metric spaces. Appl. Gen. Topol. **12**, 187-192 (2011)
19. Karapınar, E: A note on common fixed point theorems in partial metric spaces. Miskolc Math. Notes **12**, 185-191 (2011)
20. Karapınar, E, Erhan, IM: Fixed point theorems for operators on partial metric spaces. Appl. Math. Lett. **24**, 1900-1904 (2011)
21. Karapınar, E: Generalizations of Caristi Kirk's theorem on partial metric spaces. Fixed Point Theory Appl. **2011**, 4 (2011)
22. Bilgili, N, Karapınar, E, Samet, B: Generalized α-ψ contractive mappings in quasi-metric spaces and related fixed-point theorems. J. Inequal. Appl. **2014**, 36 (2014)
23. Karapınar, E: Ćirić types nonunique fixed point theorems on partial metric spaces. J. Nonlinear Sci. Appl. **5**, 74-83 (2012)
24. Karapınar, E, Shobkolaei, N, Sedghi, S, Vaezpour, SM: A common fixed point theorem for cyclic operators on partial metric spaces. Filomat **26**, 407-414 (2012)