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The study of fixed points for multivalued mappings in a Menger probabilistic metric space endowed with a graph

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Abstract

We study the existence of fixed points for multivalued mappings $f : S \to S$, where (S, F, T) is a complete Menger *PM*-space with a *t*-norm of *H*-type T and S is endowed with a directed graph $G = (V(G), E(G))$ such that $V(G) = S$ and $\Delta = \{(x, x) : x \in S\} \subset E(G)$. The obtained results recover several existing fixed point theorems from the literature. As applications, we obtain a convergence result of successive approximations for certain nonlinear operators defined on a complete metric space. This last result allows us to establish a Kelisky-Rivlin type result for a class of modified q-Bernstein operators on the space $C([0, 1])$.

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Keywords: fixed point; multivalued mapping; Menger probabilistic metric, directed graph; modified q-Bernstein operator

1 Introduction

In recent years, many results related to metric fixed point theory in partially ordered sets have appeared. The first work in this direction was the 2004 paper of Ran and Reurings [1], where they established a fixed point result, which can be considered as a combination of two fundamental fixed point theorems: the Banach contraction principle and the Knaster-Tarski fixed point theorem. More precisely, Ran and Reurings considered a class of single-valued mappings $f : X \to X$, where (X, d) is a complete metric space endowed with a certain partial order $\preceq$. The considered mappings are supposed to be continuous, monotone with respect to the partial order $\preceq$, and satisfying a Banach contraction inequality for every pair $(x, y) \in X \times X$ such that $x \preceq y$. If for some $x_0 \in X$ we have $x_0 \preceq fx_0$, they proved that the Picard sequence $\{f^n x_0\}$ converges to a fixed point of f. By combining this result with the Schauder fixed point theorem, Ran and Reurings obtained some existence and uniqueness results of positive definite solutions to some nonlinear matrix equations. Nieto and Rodríguez-López [2] extended the result of Ran and Reurings to single-valued mappings that are not necessarily continuous. Under an additional assumption, that is, $x_n \preceq x$ for all n, whenever $\{x_n\}$ is an increasing sequence with respect to the partial order $\preceq$ and convergent to x, they proved that f has at least one fixed point. For other related results, we refer to [3–6] and references therein.

In [7], Jachymski presented an interesting concept in fixed point theory with some general structures by using the context of metric spaces endowed with a graph. He proved that it is possible to unify a large class of fixed point theorems including the previous cited results by considering single-valued mappings satisfying a Banach contraction inequality for every pair $(x,y) \in X \times X$ such that (x,y) is an edge of a certain directed graph G. He also presented a new proof of the Kelisky and Rivlin theorem [8] concerning Bernstein operators using a fixed point theorem for linear operators on a Banach space following from a fixed point theorem in a metric space with a graph.

Very recently, Dinevari and Frigon [9] extended some fixed point results of Jachymski [7] to multivalued mappings. They introduced the notions of multivalued G-contractions and weak G-contractions for which they established fixed point theorems. They also presented a comparison between fixed point sets obtained from Picard iterations starting from different points. For other related works, we refer to [10, 11] and references therein.

Recently, Kamran *et al.* [12] extended the results of Jachymski [7] to the setting of Menger probabilistic metric spaces. They introduced the class of probabilistic G-contraction single-valued mappings and studied the existence of fixed points for such mappings.

Our aim in this paper is to study the existence of fixed points for nonempty multivalued mappings defined on a complete Menger probabilistic metric space (S,F,T), where T is a t-norm of H-type and S is a set endowed with a directed graph G.

The paper is organized as follows. In Section 2, we recall some basic concepts on Menger probabilistic metric spaces and fix some notations. In Section 3, we introduce the class of multivalued G-contractions, and we study the existence of fixed points for such mappings. Some interesting consequences are derived from our main result in this section. In particular, we obtain existence results of fixed points for nonempty closed multivalued G-contractions, a probabilistic version of the fixed point theorem for (ε,λ)-uniformly locally contractive multivalued maps due to Nadler, and many other results including also the case of single-valued mappings. In Section 4, we introduce the class of multivalued weak G-contractions, for which we study the existence of fixed points. Finally, in Section 5, we present an application to modified q-Bernstein polynomials. More precisely, we obtain a Kelisky-Rivlin type result for a class of modified q-Bernstein operators on the space $C([0,1])$.

2 Preliminaries and notations

The introduction of the general concept of statistical metric spaces is due to Karl Menger (1942), who dealt with probabilistic geometry. The new theory of fundamental probabilistic structures was developed later on by many authors. In this section, we start by recalling some basic concepts from Menger probabilistic metric spaces. For more details on such spaces, we refer to [13–16].

A mapping $F:\mathbb{R}\to[0,1]$ is called a distribution function if it satisfies the following conditions:

(d1) F is nondecreasing;

(d2) F is left continuous;

(d3) $\inf_{t\in\mathbb{R}} F(t)=0$ and $\sup_{t\in\mathbb{R}} F(t)=1$.

If, in addition, we have

(d4) $F(0)=0$,

then F is called a distance distribution function.

Let $\mathcal{D}^+$ be the set defined by

$$\mathcal{D}^+=\Big\{F:\mathbb{R}\to[0,1]: F \text{ is distance distribution function}, \lim_{t\to+\infty} F(t)=1\Big\}.$$

The element $\delta_0 \in \mathcal{D}^+$ defined by

$$\delta_0(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0, \end{cases}$$

is the Dirac distribution function.

Definition 2.1 A mapping $T : [0,1] \times [0,1] \to [0,1]$ is said to be a triangular norm (briefly t-norm) if for every $x, y, z \in [0,1]$, we have
(t1) $T(x,y) = T(y,x)$;
(t2) $T(x, T(y,z)) = T(T(x,y),z)$;
(t3) $T(x,y) \leq T(x,z)$ if $y \leq z$;
(t4) $T(x,1) = x$.

The commutativity (t1), the monotonicity (t3), and the boundary condition (t4) imply that for each t-norm T and for each $x \in [0,1]$, we have the following boundary conditions:

$$T(x,1) = T(1,x) = x$$

and

$$T(x,0) = T(0,x) = 0.$$

Typical examples of t-norms are $T_M(x,y) = \min\{a,b\}$ and $T_P(x,y) = xy$.

Definition 2.2 A t-norm T is said to be of H-type if the family of functions $\{T^n\}_{n\in\mathbb{N}}$ is equicontinuous at $t = 1$, where $T^n : [0,1] \to [0,1]$ is recursively defined by

$$T^1(t) = T(t,t), \qquad T^{n+1}(t) = T\bigl(T^n(t),t\bigr); \quad t \in [0,1], n = 1,2,\ldots.$$

A trivial example of a t-norm of H-type is $T_M = \min$, but there exist t-norms of H-type with $T \neq T_M$ (see, *e.g.*, [13]).

Definition 2.3 A Menger probabilistic metric space (briefly, Menger PM-space) is a triple (S,F,T), where S is a nonempty set, $F : S \times S \to \mathcal{D}^+$, and $T : [0,1] \times [0,1] \to [0,1]$ is a t-norm such that for every $x, y, z \in S$, we have
(PM1) $F(x,y) = \delta_0 \Leftrightarrow x = y$;
(PM2) $F(x,y) = F(y,x)$;
(PM3) $F(x,z)(t+s) \geq T(F(x,y)(t), F(y,z)(s))$ for all $t, s \geq 0$.

Let (S,F,T) be a Menger PM-space. For $\varepsilon > 0$ and $\delta \in (0,1]$, the (ε,δ)-neighborhood of $x \in S$ is denoted by $N_x(\varepsilon,\delta)$ and is defined by

$$N_x(\varepsilon,\delta) = \bigl\{y \in S : F(x,y)(\varepsilon) > 1 - \delta\bigr\}.$$

Furthermore, if $\sup_{0<a<1} T(a,a) = 1$, then the family of neighborhoods

$$\bigl\{N_x(\varepsilon,\delta) : x \in S, \varepsilon > 0, \delta \in (0,1]\bigr\}$$

determines a Hausdorff topology for S.

Definition 2.4 Let (S,F,T) be a Menger PM-space.

(i) A sequence $\{x_n\} \subset S$ converges to an element $x \in S$ if for every $\varepsilon > 0$ and $\delta \in (0,1]$, there exists $N \in \mathbb{N}$ such that $x_n \in N_x(\varepsilon,\delta)$ for every $n \geq N$.

(ii) A sequence $\{x_n\} \subset S$ is a Cauchy sequence if for every $\varepsilon > 0$ and $\delta \in (0,1]$, there exists $N \in \mathbb{N}$ such that $F(x_n,x_m)(\varepsilon) > 1 - \lambda$, whenever $n,m \geq N$.

(iii) A Menger PM-space is complete if every Cauchy sequence in S converges to a point in S.

(iv) A subset A of S is closed if every convergent sequence in A converges to an element of A.

Let (S,F,T) be a Menger PM-space. For all $\lambda \in (0,1]$, we define the mapping $d_\lambda : S \times S \to [0,\infty)$ by

$$d_\lambda(x,y) = \inf\{t > 0 : F(x,y)(t) > 1 - \lambda\} \quad \text{for all } x,y \in S.$$

We denote

$$D(x,y) = \sup\{d_\lambda(x,y) : \lambda \in (0,1]\} \quad \text{for all } x,y \in S.$$

The following lemma (see [17, 18]) will be useful later.

Lemma 2.5 *Let (S,F,T) be a Menger PM-space. For every $\lambda \in (0,1]$, we have*

(i) *$d_\lambda(x,y) < t$ if and only if $F(x,y)(t) > 1 - \lambda$;*

(ii) *$d_\lambda(x,y) = 0$ for all $\lambda \in (0,1]$ if and only if $x = y$;*

(iii) *$d_\lambda(x,y) = d_\lambda(y,x)$ for all $x,y \in S$;*

(iv) *if T is of H-type, then for each $\lambda \in (0,1]$, there exists $\mu \in (0,\lambda]$ such that for each $m \in \mathbb{N}$,*

$$d_\lambda(x_0,x_m) \leq \sum_{i=1}^{m} d_\mu(x_{i-1},x_i) \quad \text{for all } x_0,x_1,\ldots,x_m \in S.$$

Let (S,F,T) be a Menger PM-space. Let G be a directed graph. The set of its vertices and the set of its edges are denoted by $V(G)$ and $E(G)$, respectively. We assume that $S = V(G)$, Δ the diagonal in $X \times X$ is contained in $E(G)$, and G has no parallel edges. We identify G with the pair $(V(G),E(G))$.

For $N \in \mathbb{N}$, we say that $(x^i)_{i=0}^N$ is an N-directed path from $x \in S$ to $y \in S$ if

$$x^0 = x, \qquad x^N = y, \qquad (x^{i-1},x^i) \in E(G) \quad \text{and} \quad D(x^{i-1},x^i) < \infty \quad \text{for all } i = 1,2,\ldots,N.$$

For $x \in S$ and $N \in \mathbb{N}$, we denote

$$[x]_G^N = \{y \in S : \text{there is an } N\text{-directed path from } x \text{ to } y\}$$

and

$$[x]_G = \bigcup [x]_G^N.$$

Let $x \in S$, $N \in \mathbb{N}$, $y \in [x]_G^N$ and $z \in [x]_G$. We define

$$p_N(x,y) = \inf\left\{ \sum_{i=1}^{N} D\bigl(x^{i-1},x^i\bigr) : \bigl(x^i\bigr)_{i=0}^{N} \text{ is an } N\text{-directed path from } x \text{ to } y \right\}$$

and

$$p(x,z) = \inf\left\{ \sum_{i=1}^{N} D\bigl(x^{i-1},x^i\bigr) : \bigl(x^i\bigr)_{i=0}^{N} \text{ is an } N\text{-directed path from } x \text{ to } z \text{ for some } N \in \mathbb{N} \right\}.$$

We have the following properties.

Lemma 2.6 *Let $x \in S$, $N \in \mathbb{N}$, $y \in [x]_G^N$, and $z \in [x]_G$. Then*

(i) *$p_N(x,y) \geq P_{N+m}(x,y)$ for every $m, N \in \mathbb{N}$;*

(ii) *$p(x,z) = \inf\{p_k(x,z) : k \in \mathbb{N}, z \in [x]_G^k\}$.*

Proof Let $m, N \in \mathbb{N}$. Let $(x^i)_{i=0}^N$ be an N-directed path from x to y, that is,

$$x^0 = x, \qquad x^N = y, \qquad \bigl(x^{i-1},x^i\bigr) \in E(G) \quad \text{and} \quad D\bigl(x^{i-1},x^i\bigr) < \infty \quad \text{for all } i = 1,2,\ldots,N.$$

Let

$$x^{N+i} = y \quad \text{for all } i = 1,2,\ldots,m.$$

Since $\Delta \subset E(G)$ and $D(y,y) = 0$, then $(x^i)_{i=0}^{N+m}$ is an $N+m$-directed path from x to y. Moreover, we have

$$\sum_{i=1}^{N} D\bigl(x^{i-1},x^i\bigr) = \sum_{i=1}^{N+m} D\bigl(x^{i-1},x^i\bigr) \geq P_{N+m}(x,y).$$

Thus we proved (i). Now, let $k \in \mathbb{N}$ such that $z \in [x]_k^G$. For every $(x^i)_{i=0}^k$, a k-directed path from x to z, we have

$$\sum_{i=1}^{k} D\bigl(x^{i-1},x^i\bigr) \geq p(x,z).$$

Then $p_k(x,z) \geq p(x,z)$. This implies that

$$p(x,z) \leq \inf\bigl\{p_k(x,z) : k \in \mathbb{N}, z \in [x]_G^k\bigr\}.$$

Finally, let $N \in \mathbb{N}$ and $(x^i)_{i=1}^N$ be an N-directed path from x to z. We have

$$\sum_{i=1}^{k} D\bigl(x^{i-1},x^i\bigr) \geq p_N(x,z) \geq \inf\bigl\{p_k(x,z) : k \in \mathbb{N}, z \in [x]_G^k\bigr\}.$$

Then we deduce that

$$p(x,z) \geq \inf\{p_k(x,z) : k \in \mathbb{N}, z \in [x]_G^k\},$$

which yields (ii). □

3 The study of fixed points for multivalued *G*-contractions

In this section, we establish fixed point results for a multivalued contraction with respect to the graph G in a Menger PM-space.

Definition 3.1 Let (S,F,T) be a Menger PM-space and $f : S \to S$ be a multivalued mapping with nonempty values. We say that f is a multivalued G-contraction if there exists $\kappa \in (0,1)$ such that

$$(x,y) \in E(G), u \in fx \quad \Longrightarrow \quad \exists v \in fy\colon (u,v) \in E(G), F(u,v)(\kappa t) \geq F(x,y)(t), \forall t > 0.$$

Lemma 3.2 *Let (S,F,T) be a Menger PM-space and $f : S \to S$ be a multivalued G-contraction with respect to some $\kappa \in (0,1)$. We have*

$$(x,y) \in E(G), u \in fx \quad \Longrightarrow \quad \exists v \in fy\colon (u,v) \in E(G), d_\lambda(u,v) \leq \kappa d_\lambda(x,y), \forall \lambda \in (0,1].$$

Proof Let $(x,y) \in E(G)$ and $u \in fx$. Since f is a multivalued G-contraction with respect to $\kappa \in (0,1)$, there exists $v \in (0,1)$ such that

$$F(u,v)(\kappa t) \geq F(x,y)(t) \quad \text{for all } t > 0.$$

Let $\lambda \in (0,1]$ be fixed. Let $s > 0$ be such that $F(x,y)(s) > 1 - \lambda$. Then we have

$$F(u,v)(\kappa s) > 1 - \lambda,$$

which implies that

$$s \geq \kappa^{-1} d_\lambda(u,v).$$

By the definition of the inf, we get the desired result. □

Lemma 3.3 *Let (S,F,T) be a Menger PM-space with T a t-norm of H-type and $f : S \to S$ be a multivalued G-contraction. Let $\varepsilon > 0$ and $N \in \mathbb{N}$. Then, for every $x \in S$ and $y \in [x]_G^N$, one has*

$$\forall x_1 \in fx, \exists y_1 \in fy \cap [x_1]_G^N\colon \quad p_N(x_1,y_1) \leq \kappa\bigl(p_N(x,y) + \varepsilon\bigr) \tag{3.1}$$

and inductively, for all $k = 1, 2, \ldots,$

$$\forall x_{k+1} \in fx_k, \exists y_{k+1} \in fy_k \cap [x_{k+1}]_G^N\colon \quad p_N(x_{k+1},y_{k+1}) \leq \kappa^{k+1}\bigl(p_N(x,y) + \varepsilon\bigr). \tag{3.2}$$

Proof Let $\varepsilon > 0$ and $(x^i)_{i=0}^N$ be an N-directed path from $x \in S$ to $y \in [x]_G^N$ such that

$$\sum_{i=1}^{N} D(x^{i-1}, x^i) < p_N(x,y) + \varepsilon.$$

Let $x_1 \in fx$. Since f is a multivalued G-contraction, by Lemma 3.2, there exists $x_1^1 \in fx^1$ such that

$$(x_1, x_1^1) \in E(G) \quad \text{and} \quad D(x_1, x_1^1) \leq \kappa D(x, x^1) < \infty.$$

Again, there exists $x_1^2 \in fx^2$ such that

$$(x_1^1, x_1^2) \in E(G) \quad \text{and} \quad D(x_1^1, x_1^2) \leq \kappa D(x^1, x^2) < \infty.$$

Recursively, for $i = 3, 4, \dots, N$, there exists $x_1^i \in fx^i$ such that

$$(x_1^{i-1}, x_1^i) \in E(G) \quad \text{and} \quad D(x_1^{i-1}, x_1^i) \leq \kappa D(x^{i-1}, x^i) < \infty.$$

Now, if we take $y_1 = x_1^N$, we get $y_1 \in fy \cap [x_1]_G^N$ and

$$\begin{aligned} p_N(x_1, y_1) &\leq \sum_{i=1}^{N} D(x_1^{i-1}, x_1^i) \\ &\leq \kappa \sum_{i=1}^{N} D(x^{i-1}, x^i) \\ &\leq \kappa \bigl(p_N(x,y) + \varepsilon\bigr). \end{aligned}$$

Thus (3.1) is proved.

Let $x_2 \in fx_1$. As previously, there exists $y_2 \in fy_1 \cap [x_2]_G^N$ such that

$$p_N(x_2, y_2) \leq \kappa \sum_{i=1}^{N} D(x_1^{i-1}, x_1^i) \leq \kappa^2 \bigl(p_N(x,y) + \varepsilon\bigr).$$

Continuing this process, by induction, we obtain (3.2). □

The following concepts are adaptations of those introduced in [9] to the case of Menger PM-spaces.

Definition 3.4 Let (S, F, T) be a Menger PM-space. Let $f : S \to S$ be a multivalued mapping with nonempty values.

(i) Let $N \in \mathbb{N}$. We say that a sequence $\{x_n\} \subset S$ is a G_N-Picard trajectory from x_0 if $x_n \in [x_{n-1}]_G^N \cap fx_{n-1}$ for all $n \geq 1$. We denote by $\mathcal{T}_N(f, G, x_0)$ the set of all such G_N-Picard trajectories from x_0.

(ii) We say that a sequence $\{x_n\} \subset S$ is a G-Picard trajectory from x_0 if $x_n \in [x_{n-1}]_G \cap fx_{n-1}$ for all $n \geq 1$. We denote by $\mathcal{T}(f, G, x_0)$ the set of all such G-Picard trajectories from x_0.

Definition 3.5 Let (S,F,T) be a Menger PM-space. Let $f : S \to S$ be a multivalued mapping with nonempty values.

(i) Let $N \in \mathbb{N}$. We say that f is G_N-Picard continuous from $x_0 \in S$ if the limit of any convergent sequence $\{x_n\} \in \mathcal{T}_N(f,G,x_0)$ is a fixed point of T.

(ii) We say that f is G-Picard continuous from $x_0 \in S$ if the limit of any convergent sequence $\{x_n\} \in \mathcal{T}(f,G,x_0)$ is a fixed point of T.

Now, we are able to establish our first main result.

Theorem 3.6 *Let (S,F,T) be a complete Menger PM-space with T a t-norm of H-type and $f : S \to S$ be a multivalued G-contraction. Suppose that for some $N \in \mathbb{N}$, we have*

(i) *there exists $x_0 \in S$ such that $[x_0]_G^N \cap fx_0 \neq \emptyset$;*

(ii) *f is G_N-Picard continuous from x_0.*

Then there exists a sequence $\{x_n\} \in \mathcal{T}_N(f,G,x_0)$ converging to $x^* \in S$, a fixed point of f.

Proof Since $[x_0]_G^N \cap fx_0 \neq 0$, we can take an element $x_1 \in [x_0]_G^N \cap fx_0$. Let $\varepsilon > 0$. By Lemma 2.5(iv) and Lemma 3.3, there exists $x_2 \in fx_1 \cap [x_1]_G^N$ such that

$$D(x_1,x_2) \le p_N(x_1,x_2) \le \kappa\bigl(p_N(x_0,x_1)+\varepsilon\bigr).$$

Again, since $x_2 \in [x_1]_G^N \cap fx_1$, there exists $x_3 \in [x_2]_G^N \cap fx_2$ such that

$$D(x_2,x_3) \le p_N(x_2,x_3) \le \kappa^2\bigl(p_N(x_0,x_1)+\varepsilon\bigr).$$

More generally, for $n \ge 2$, there exists $x_{n+1} \in [x_n]_G^N \cap fx_n$ such that

$$D(x_n,x_{n+1}) \le p_N(x_n,x_{n+1}) \le \kappa^n\bigl(p_N(x_0,x_1)+\varepsilon\bigr).$$

Then $\{x_n\} \in \mathcal{T}_N(f,G,x_0)$ and for $m \ge 1$,

$$\begin{aligned} D(x_n,x_{n+m}) &\le \sum_{i=n}^{n+m-1} D(x_i,x_{i+1}) \\ &\le \sum_{i=n}^{n+m-1} \kappa^i\bigl(P_N(x_0,x_1)+\varepsilon\bigr) \\ &\le \frac{k^n}{1-k}\bigl(P_N(x_0,x_1)+\varepsilon\bigr). \end{aligned}$$

Let us prove now that $\{x_n\}$ is a Cauchy sequence in the Menger PM-space (S,F,T). Let $t > 0$ and $\delta \in (0,1]$. From the above inequality, since $\kappa \in (0,1)$, there exists some $p \in \mathbb{N}$ such that

$$d_\delta(x_n,x_{n+m}) \le D(x_n,x_{n+m}) < t \quad \text{for all } n,m \ge p.$$

Using Lemma 2.5(i), we obtain

$$F(x_n,x_{n+m})(t) > 1-\delta \quad \text{for all } n,m \ge p,$$

which proves that $\{x_n\}$ is Cauchy. Since (S,F,T) is complete and f is G_N-Picard continuous from x_0, there exists some $x^* \in S$ such that $\{x_n\}$ converges to x^*, a fixed point of f. □

Suppose now that $f : S \to S$ is a multivalued G-contraction with closed values, and let us consider the assumption (i) of Theorem 3.6 with the following assumption: (ii)′ If $\{x_n\} \subset S$ is a sequence in $\mathcal{T}_N(f,G,x_0)$ converging to some $x \in S$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k},x) \in E(G)$ for every $k \in \mathbb{N}$.

From Theorem 3.6, we know that there exists $\{x_n\} \in \mathcal{T}_N(f,G,x_0)$ that converges to some $x* \in S$. At first, observe that $D(x_n,x^*) \to 0$, as $n \to \infty$. In fact, since $\{x_n\}$ converges to x^* with respect to the Menger PM-space (S,F,T) for $\varepsilon > 0$ and $\delta \in (0,1]$, there exists some $p \in \mathbb{N}$ such that $F(x_n,x^*)(\varepsilon) > 1-\delta$ for every $n \geq p$. By Lemma 2.5(i), we have $d_\delta(x_n,x^*) < \varepsilon$ for every $n \geq p$. Then we have $D(x_n,x^*) \leq \varepsilon$ for every $n \geq p$. This proves that $D(x_n,x^*) \to 0$, as $n \to \infty$. Now, since f is a multivalued G-contraction, by Lemma 3.2, for every $k \in \mathbb{N}$, there exists $y_{n_k+1} \in fx^*$ such that $(x_{n_k+1},y_{n_k+1}) \in E(G)$ and $D(x_{n_k+1},y_{n_k+1}) \leq \kappa D(x_{n_k},x^*)$. By using the triangular inequality and the above expression, for all $k \in \mathbb{N}$, we have

$$\begin{aligned} D\big(y_{n_k+1},x^*\big) &\leq D(y_{n_k+1},x_{n_k+1}) + D\big(x_{n_k+1},x^*\big) \\ &\leq \kappa D\big(x_{n_k},x^*\big) + D\big(x_{n_k+1},x^*\big). \end{aligned}$$

Letting $k \to \infty$ in the above inequality, we obtain $D(y_{n_k+1},x^*) \to 0$, as $k \to \infty$. As previously, by Lemma 2.5(i), we have $\{y_{n_k+1}\}$ converges to x^* with respect to the Menger PM-space (S,F,T). Since fx^* is closed, we get $x^* \in fx^*$, that is, x^* is a fixed point of f. Thus we proved the following result.

Corollary 3.7 *Let (S,F,T) be a complete Menger PM-space with T a t-norm of H-type and $f : S \to S$ be a multivalued G-contraction with nonempty closed values. Suppose that for some $N \in \mathbb{N}$, we have*

(i) *there exists $x_0 \in S$ such that $[x_0]_G^N \cap fx_0 \neq \emptyset$;*
(ii)′ *if $\{x_n\} \subset S$ is a sequence in $\mathcal{T}_N(f,G,x_0)$ converging to some $x \in S$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k},x) \in E(G)$ for every $k \in \mathbb{N}$.*

Then there exists a sequence $\{x_n\} \in \mathcal{T}_N(f,G,x_0)$ converging to $x^* \in S$, a fixed point of f.

Remark 3.8 Condition (ii)′ in Corollary 3.7 can be replaced by: if $\{x_n\} \subset S$ is a sequence in $\mathcal{T}_N(f,G,x_0)$ converging to some $x \in S$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k},x) \in E(G)$ for every k large enough.

Taking $N = 1$ and $G = S \times S$ in Corollary 3.7, we obtain the following Nadler fixed point theorem in Menger PM-spaces.

Corollary 3.9 *Let (S,F,T) be a complete Menger PM-space with T a t-norm of H-type and $f : S \to S$ be a multivalued mapping with nonempty closed values. Suppose that*

(i) *there exists $(x_0,x_1) \in S \times S$ such that $x_1 \in fx_0$ and $D(x_0,x_1) < \infty$;*
(ii) *there exists $\kappa \in (0,1)$ such that*

$$(x,y) \in S \times S, u \in fx \quad \Longrightarrow \quad \exists v \in fy\colon F(u,v)(\kappa t) \geq F(x,y)(t), \forall t > 0.$$

Then f has a fixed point.

Corollary 3.10 *Let (S,F,T) be a complete Menger PM-space with T a t-norm of H-type and $f: S \to S$ be a multivalued G-contraction. Suppose that*

(i) *there exists $x_0 \in S$ such that $[x_0]_G \cap fx_0 \neq \emptyset$;*

(ii) *f is G-Picard continuous from x_0.*

Then there exist $N \in \mathbb{N}$ and a sequence $\{x_n\} \in \mathcal{T}_N(f,G,x_0)$ converging to $x^* \in S$, a fixed point of f.

Proof From (i), there exists some $N \in \mathbb{N}$ such that $[x_0]_G^N \cap fx_0 \neq \emptyset$. Since from (ii) f is G-Picard continuous from x_0, then it is G_N-Picard continuous from x_0. Now, the result follows from Theorem 3.6. □

Similarly, from Corollary 3.7, we have the following result.

Corollary 3.11 *Let (S,F,T) be a complete Menger PM-space with T a t-norm of H-type and $f: S \to S$ be a multivalued G-contraction with nonempty closed values. Suppose that*

(i) *there exists $x_0 \in S$ such that $[x_0]_G \cap fx_0 \neq \emptyset$;*

(ii) *if $\{x_n\} \subset S$ is a sequence in $\mathcal{T}(f,G,x_0)$ converging to some $x \in S$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k},x) \in E(G)$ for every $k \in \mathbb{N}$.*

Then there exist $N \in \mathbb{N}$ and a sequence $\{x_n\} \in \mathcal{T}_N(f,G,x_0)$ converging to $x^* \in S$, a fixed point of f.

From Corollary 3.11, we can obtain a probabilistic version of the fixed point theorem for (ε,λ)-uniformly locally contractive multivalued maps due to Nadler [19]. At first, let us introduce some concepts.

Definition 3.12 Let (S,F,T) be a Menger PM-space, $\varepsilon > 0$ and $x,y \in S$. We say that $\{x,y\}$ is ε-chainable in S if there exists a finite set of points $\{x_0 = x, x_1, \ldots, x_{p-1}, x_p = y\} \subset S$ $(p \geq 1)$ such that for all $\lambda \in (0,1]$,

$$F(x_{i-1},x_i)(\varepsilon) > 1-\lambda \quad \text{for all } i = 1,2,\ldots,p.$$

Definition 3.13 Let (S,F,T) be a Menger PM-space, $\varepsilon > 0$, $\kappa \in (0,1)$ and $f: S \to S$ be a multivalued mapping. We say that f is (ε,κ)-uniformly locally contractive if

$$\begin{aligned}&(x,y) \in S \times S, u \in fx, F(x,y)(\varepsilon) > 1-\lambda, \forall\lambda \in (0,1]\\ &\quad \Longrightarrow \quad \exists v \in fy: F(u,v)(\kappa t) \geq F(x,y)(t), \forall t > 0.\end{aligned}$$

We have the following result.

Corollary 3.14 *Let (S,F,T) be a complete Menger PM-space with T a t-norm of H-type and $f: S \to S$ be a multivalued mapping with closed nonempty values. Assume that there exist $\varepsilon > 0$ and $\kappa \in (0,1)$ such that*

(i) *f is (ε,κ)-uniformly locally contractive;*

(ii) *there exist $x_0 \in S$ and $\tilde{x} \in fx_0$ such that $\{x_0,\tilde{x}\}$ is ε-chainable in S.*

Then f has a fixed point.

Proof Let us consider the graph G with

$$E(G) = \bigl\{(x,y) \in S \times S : F(x,y)(\varepsilon) > 1-\lambda, \forall\lambda \in (0,1]\bigr\}.$$

From (i), f is a multivalued G-contraction with respect to the graph G. From (ii), we have $\tilde{x} \in fx_0 \cap [x_0]_G$. Then the result follows from Corollary 3.11. □

Let us consider now the case of a single-valued mapping. From Definition 3.1, a single-valued mapping $f : S \to S$ is a G-contraction if there exists $\kappa \in (0,1)$ such that

$$(x,y) \in E(G) \quad \Longrightarrow \quad (fx,fy) \in E(G), F(fx,fy)(\kappa t) \geq F(x,y)(t), \forall t > 0.$$

In this case, for a given $N \in \mathbb{N}$ and a given $x_0 \in S$, the set of G_N-Picard trajectories from x_0 is given by

$$\mathcal{T}_N(f,G,x_0) = \left\{ \{x_n\}_{n\in\mathbb{N}} : x_n = f^n x_0 \right\}.$$

From Theorem 3.6, we obtain the following result concerning single-valued mappings.

Corollary 3.15 *Let (S,F,T) be a complete Menger PM-space with T a t-norm of H-type and $f : S \to S$ be a single-valued G-contraction. Suppose that there exists $x_0 \in S$ such that $fx_0 \in [x_0]_G$. Suppose that one of the following conditions is satisfied*:
(i) *if $\{f^n x_0\}$ converges to some $x \in S$, then $x = fx$*;
(ii) *if $\{f^n x_0\}$ converges to some $x \in S$, then there exists a subsequence $\{f^{n_k} x_0\}$ of $\{f^n x_0\}$ such that $(f^{n_k} x_0, x) \in E(G)$ for every $k \in \mathbb{N}$.*
Then f has a fixed point.

From Corollary 3.15, we can obtain a probabilistic version of Kirk, Srinivasan and Veeramani's fixed point theorem for cyclic mappings [20].

Corollary 3.16 *Let (S,F,T) be a complete Menger PM-space with T a t-norm of H-type. Let $(A_i)_{i=1}^p$ be a family of nonempty closed subsets. Suppose that $f : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ is a single-valued mapping satisfying the following conditions*:
(i) *$f(A_i) \subseteq A_{i+1}$ for every $i = 1,2,\ldots,p$, with $A_{p+1} = A_1$*;
(ii) *there exists $\kappa \in (0,1)$ such that for all $i = 1,2,\ldots,p$,*

$$(x,y) \in A_i \times A_{i+1} \quad \Longrightarrow \quad F(fx,fy)(\kappa t) \geq F(x,y)(t), \forall t > 0;$$

(iii) *there exists $x_0 \in A_1$ such that $D(x_0,fx_0) < \infty$.*
Then f has a fixed point.

Proof Let G be the graph such that

$$E(G) = \bigcup_{i=1}^{p} A_i \times_{i=1}^{p} A_{i+1}.$$

From conditions (i) and (iii), we have $(x_0,fx_0) \in E(G)$ and $D(x_0,fx_0) < \infty$. This implies that $fx_0 \in [x_0]_G^1 \subset [x_0]_G$. Conditions (i) and (ii) imply that f is a single-valued G-contraction. Suppose now that $\{f^n x_0\}$ converges to some $x \in S$. Since A_i is closed for every $i = 1,2,\ldots,p$, it is easy to observe that $x \in \bigcap_{i=1}^p A_i$. On the other hand, from condition (i), we can take a subsequence $\{f^{n_k} x_0\}$ of $\{f^n x_0\}$ such that $\{f^{n_k} x_0\} \subset A_1$ for every $k \in \mathbb{N}$. Thus $(f^{n_k} x_0, x) \in A_1 \times \bigcap_{i=1}^p A_i \subset A_1 \times A_2$ for every $k \in \mathbb{N}$. The conclusion follows from Corollary 3.15. □

4 The study of fixed points for multivalued weak *G*-contractions

In this section, we study the existence of fixed points for multivalued weak G-contractions.

Definition 4.1 Let (S, F, T) be a Menger PM-space and $f : S \to S$ be a multivalued mapping with nonempty values. We say that f is a multivalued weak G-contraction if there exists $\kappa \in (0,1)$ such that

$$(x, y) \in S \times S, y \in [x]_G, u \in fx \quad \Longrightarrow \quad \exists v \in fy \cap [u]_G: p(u, v) \le \kappa p(x, y).$$

From the next result, we observe that the class of multivalued weak G-contractions is larger than the class of multivalued G-contractions.

Lemma 4.2 *Let (S, F, T) be a Menger PM-space and $f : S \to S$ be a multivalued mapping with nonempty values. If f is a G-contraction, then f is a weak G-contraction.*

Proof Suppose that $f : S \to S$ is a multivalued G-contraction. Let $(x, y) \in S \times S$ such that $u \in fx$ and $y \in [x]_G$. Let an arbitrary $\varepsilon > 0$. From Lemma 2.6(ii), there exists $N = N(\varepsilon)$ such that $y \in [x]_G^N$ and

$$p_N(x, y) \le p(x, y) + \varepsilon.$$

On the other hand, by Lemma 3.3, there exists $v \in fy \cap [u]_G^N$ such that

$$\begin{aligned} p(u, v) &\le p_N(u, v) \\ &\le \kappa \bigl(p_N(x, y) + \varepsilon\bigr) \\ &\le \kappa \bigl(p(x, y) + 2\varepsilon\bigr). \end{aligned}$$

Letting $\varepsilon \to 0$ in the above inequality, we get $p(u, v) \le \kappa p(x, y)$, which completes the proof. □

We have the following fixed point result for the class of weak G-contractions.

Theorem 4.3 *Let (S, F, T) be a complete Menger PM-space with T a t-norm of H-type and $f : S \to S$ be a multivalued weak G-contraction. Suppose that the following conditions hold:*

(i) *there exists $x_0 \in S$ such that $[x_0]_G \cap fx_0 \neq \emptyset$;*
(ii) *f is G-Picard continuous from x_0.*

Then there exists a sequence $\{x_n\} \in \mathcal{T}(f, G, x_0)$ converging to x^*, a fixed point of f.

Proof Let $x_1 \in [x_0]_G \cap fx_0$. Since f is a weak G-contraction, there exists $x_2 \in [x_1]_G \cap fx_1$ such that

$$p(x_1, x_2) \le \kappa p(x_0, x_1).$$

Again, there exists $x_3 \in [x_2]_G \cap fx_2$ such that

$$p(x_2, x_3) \le \kappa p(x_1, x_2).$$

Continuing this process, we obtain a sequence $\{x_n\} \in \mathcal{T}(f, G, x_0)$ such that

$$D(x_{n+1}, x_n) \leq p(x_{n+1}, x_n) \leq \kappa^n p(x_0, x_1) \quad \text{for all } n \in \mathbb{N}.$$

This implies that $\{x_n\}$ is a Cauchy sequence in the complete Menger PM-space (S, F, T). Then by (ii), $\{x_n\}$ converges to a fixed point of f. □

The next result concerns weak G-contraction multivalued mappings with nonempty closed values.

Theorem 4.4 *Let (S, F, T) be a complete Menger PM-space with T a t-norm of H-type and $f : S \to S$ be a multivalued weak G-contraction with nonempty closed values. Suppose that the following conditions hold*:

(i) *there exists $x_0 \in S$ such that $[x_0]_G \cap fx_0 \neq \emptyset$;*

(ii) *for every sequence $\{x_n\} \in \mathcal{T}(f, G, x_0)$ converging to some $x \in S$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x \in [x_{n_k}]_G$ for every $k \in \mathbb{N}$, and $p(x_{n_k}, x^*) \to 0$, as $k \to \infty$.*

Then there exists a sequence $\{x_n\} \in \mathcal{T}(f, G, x_0)$ converging to x^*, a fixed point of f.

Proof From the proof of Theorem 4.3, we know that there exists a sequence $\{x_n\} \in \mathcal{T}(f, G, x_0)$ converging to some $x^* \in S$. By condition (ii), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x \in [x_{n_k}]_G$ for every $k \in \mathbb{N}$, and $p(x_{n_k}, x^*) \to 0$, as $k \to \infty$. Since f is a weak G-contraction, there exists a sequence $\{y_{n_k}\} \subset S$ such that $y_{n_k} \in fx^* \cap [x_{n_k}]_G$ and

$$p(x_{n_k+1}, y_{n_k+1}) \leq \kappa p\bigl(x_{n_k}, x^*\bigr) \quad \text{for all } k \in \mathbb{N}.$$

On the other hand, we have

$$\begin{aligned} D\bigl(y_{n_k+1}, x^*\bigr) &\leq D\bigl(x^*, x_{n_k+1}\bigr) + D(x_{n_k+1}, y_{n_k+1}) \\ &\leq D\bigl(x^*, x_{n_k+1}\bigr) + p(x_{n_k+1}, y_{n_k+1}) \\ &\leq D\bigl(x^*, x_{n_k+1}\bigr) + \kappa p\bigl(x_{n_k}, x^*\bigr). \end{aligned}$$

Letting $k \to \infty$ in the above inequality, we obtain that $\{y_{n_k+1}\}$ converges to x^*. Since fx^* is closed, we have $x^* \in fx^*$, that is, x^* is a fixed point of f. □

Finally, from Theorem 4.3 and Theorem 4.4, we obtain the following fixed point theorem for single-valued mappings.

Theorem 4.5 *Let (S, F, T) be a complete Menger PM-space with T a t-norm of H-type and $f : S \to S$ be a single weak G-contraction mapping, that is, there exists some $\kappa \in (0, 1)$ such that*

$$(x, y) \in S \times S, y \in [x]_G \quad \Longrightarrow \quad fy \in [fx]_G, p(fx, fy) \leq \kappa p(x, y).$$

Suppose that there exists $x_0 \in S$ such that $fx_0 \in [x_0]_G$. Suppose that one of the following conditions is satisfied:

(i) *if* $\{f^n x_0\}$ *converges to some* $x \in S$, *then* $x = fx$;
(ii) *if* $\{f^n x_0\}$ *converges to some* $x \in S$, *then there exists a subsequence* $\{f^{n_k} x_0\}$ *of* $\{f^n x_0\}$ *such that* $p(f^{n_k} x_0, x) \to 0$, *as* $k \to \infty$, *and* $f^{n_k} x_0 \in [x]_G$ *for every* $k \in \mathbb{N}$.

Then f *has a fixed point.*

5 Applications: Kelisky-Rivlin type result for modified q-Bernstein polynomials

As applications, we establish in this section a Kelisky-Rivlin type result, a certain class of modified q-Bernstein polynomials.

At first, we have the following result concerning the convergence of successive approximations for a certain family of operators.

Theorem 5.1 *Let* E *be a group with respect to a certain operation* $+$. *Let* X *be a subset of* E *endowed with a certain metric* d *such that* (X, d) *is complete. Let* $X_0 \subseteq X$ *be a closed subset of* X *such that* X_0 *is a subgroup of* E. *Let us consider a single mapping* $f : X \to X$ *such that*

$$(x, y) \in X \times X, x - y \in X_0 \quad \Longrightarrow \quad d(fx, fy) \le \kappa d(x, y),$$

where $\kappa \in (0, 1)$ *is a constant. Suppose that*

$$x - fx \in X_0 \quad \text{for all } x \in X. \tag{5.1}$$

Then we have

(i) *for every* $x \in X$, *the Picard sequence* $\{f^n x\}$ *converges to a fixed point of* f;
(ii) *for every* $x \in X$, $(x + X_0) \cap \operatorname{Fix} f = \{\lim_{n\to\infty} f^n x\}$, *where* $\operatorname{Fix} f$ *denotes the set of fixed points of* f.

Proof Let us consider the mapping $F : X \times X \to \mathcal{D}^+$ defined by

$$F(x, y)(t) = \delta_0\bigl(t - d(x, y)\bigr) \quad \text{for all } x, y \in X, t > 0,$$

where δ_0 is the Dirac distribution function. Consider the graph $G = (V(G), E(G))$, where $V(G) = X$ and

$$E(G) = \bigl\{(x, y) \in X \times X : x - y \in X_0\bigr\}.$$

Observe that by (5.1), we have

$$\begin{aligned}(x, y) \in E(G) \quad &\Longrightarrow \quad x - y \in X_0 \quad \Longrightarrow \quad fx - fy = (fx - x) + (y - fy) + (x - y) \in X_0 \\ &\Longrightarrow \quad (fx, fy) \in E(G).\end{aligned}$$

Then by the definition of δ_0, we have

$$(x, y) \in E(G) \quad \Longrightarrow \quad (fx, fy) \in E(G), F(fx, fy)(\kappa t) \ge F(x, y)(t), \forall t > 0,$$

which implies that f is a single-valued G-contraction. Recall that (X, F, T_M), where $T_M = \min$, is a complete metric space (see [21]). Moreover, a sequence $\{u_n\} \subset X$ converges to some $u \in X$ with respect to d if and only if $\{u_n\}$ converges to u with respect to the Menger PM-space (X, F, T_M). Let $x_0 \in X$ be an arbitrary point. By (5.1), we have $x_0 - fx_0 \in X_0$, that is, $(x_0, fx_0) \in E(G)$, which implies that $fx_0 \in [x_0]_G$. Suppose now that $\{f^n x_0\}$ converges to some $x \in X$ with respect to (X, F, T_M). Then $\{f^n x_0\}$ converges to x with respect to the metric d. On the other hand, we have $fx_0 = (fx_0 - x_0) + x_0 \in X_0$. Again, we have $f^2x_0 = (f^2x_0 - fx_0) + fx_0 \in X_0$. Continuing in this manner, we get $f^n x_0 \in X_0$ for every $n \geq 1$. Since X_0 is closed, then $x \in X_0$. As a consequence, we have $(f^n x_0, x) \in E(G)$ for every $n \geq 1$. Finally, by Theorem 3.6 (or Corollary 3.15), we finish the proof of (i).

Now, let us prove (ii). Let $x \in X$ be an arbitrary point. From (i), we know that $\{f^n x\}$ converges with respect to the metric d to some $x^* \in X_0$, a fixed point of f. Moreover, from the proof of (i), we have $f^n x_0 - x \in X_0$ for all $n \geq 1$. Since X_0 is closed, we get $x^* - x \in X_0$, that is, $x^* \in x + X_0$. On the other hand, suppose that $u_1, u_2 \in (x + X_0) \cap \operatorname{Fix} f$, with $u_1 \neq u_2$. Since $u_1 - x, u_2 - x \in X_0$, then

$$d(u_1, u_1) = d(fu_1, fu_2) \leq \kappa d(u_1, u_2),$$

which is a contradiction. Thus we proved (ii). □

Remark 5.2 Theorem 5.1 recovers Theorem 4.1 in [7], where X was supposed to be a Banach space and f was supposed to be a linear operator.

The Bernstein operator on $\varphi \in C([0,1])$, the space of all continuous real functions on the interval $[0,1]$, is defined by

$$(B_n\varphi)(t) = \sum_{k=0}^{N} \varphi\left(\frac{k}{n}\right) \binom{n}{k} t^k (1-t)^{n-k}, \quad \varphi \in C\bigl([0,1]\bigr), t \in [0,1], n = 1, 2, \ldots.$$

Kelisky and Rivlin [8] proved that each Bernstein operator B_n is a weak operator. Moreover, for any n and $\varphi \in C([0,1])$,

$$\lim_{j\to\infty} \bigl(B_n^j \varphi\bigr)(t) = \varphi(0) + \bigl(\varphi(1) - \varphi(0)\bigr)t, \quad t \in [0,1].$$

The proof given by Kelisky and Rivlin is based on linear algebra tools, it involves the Stirling numbers of the second kind, and eigenvalues and eigenvectors of some matrices. In [7], a more easy and elegant proof based on a fixed point theorem for linear operators on a Banach space (see Theorem 4.1 in [7]) was presented.

In this section, we are interested in establishing Kelisky and Rivlin type results for a class of modified q-Bernstein polynomials. To formulate our results, we need the following definitions.

Let $q > 0$. For any $n = 0, 1, 2, \ldots$, the q-integer $[n]_q$ is defined by

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1} \quad (n = 1, 2, \ldots), \qquad [0]_q = 0.$$

The q-factorial $[n]_q!$ is defined by

$$[n]_q! = [1]_q [2]_q \cdots [n]_q \quad (n = 1, 2, \ldots), \qquad [0]_q! = 1.$$

For integers $0 \leq k \leq n$, the q-binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}.$$

It is clear that for $q = 1$, we have

$$[n]_1 = n, \qquad [n]_1! = n!, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}.$$

Definition 5.3 (Phillips [22]) For $\varphi \in C([0,1])$, the q-Bernstein polynomial of φ is defined by

$$B_n(q,\varphi)(t) = \sum_{k=0}^{N} \varphi\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q t^k \prod_{s=0}^{n-1-k} \left(1 - q^s t\right), \quad t \in [0,1], n = 1,2,\ldots.$$

(From here on an empty product is taken to be equal to 1.)

Note that for $q = 1$, the polynomials $B_n(1,\varphi)(t)$ are classical Bernstein polynomials.

We introduce the following class of modified q-Bernstein polynomials.

Definition 5.4 For $\varphi \in C([0,1])$, the modified q-Bernstein polynomial of φ is defined by

$$\mathcal{B}_n(q,\varphi)(t) = \sum_{k=0}^{N} \left|\varphi\left(\frac{[k]_q}{[n]_q}\right)\right| \begin{bmatrix} n \\ k \end{bmatrix}_q t^k \prod_{s=0}^{n-1-k} \left(1 - q^s t\right), \quad t \in [0,1], n = 1,2,\ldots.$$

(From here on an empty product is taken to be equal to 1.)

Observe that $\mathcal{B}_n(q,\cdot)$ is a nonlinear operator on $C([0,1])$. Let

$$X = \left\{\phi \in C\left([0,1]\right) : \phi(0) \geq 0, \phi(1) \geq 0\right\}.$$

Clearly, $\mathcal{B}_n(q,\cdot) : X \to X$ is well defined.

We have the following result.

Theorem 5.5 *Let $n \in \mathbb{N}$ and $0 < q \leq 1$. Then, for every $\varphi \in X$, the Picard sequence $\{\mathcal{B}_n^j(q,\varphi)\}_{j\in\mathbb{N}}$ converges to a fixed point of $\mathcal{B}_n(q,\cdot)$. Moreover, for every $\varphi \in X$, we have*

$$\lim_{j\to\infty} \max_{t\in[0,1]} \left|\mathcal{B}_n^j(q,\varphi)(t) - \omega(t)\right| = 0,$$

where $\omega(t) = \varphi(0)(1-t) + \varphi(1)t$, $t \in [0,1]$.

Proof Let $\mathbb{E} = C([0,1])$. We endow X with the metric defined by

$$d(U,V) = \max_{t\in[0,1]} \left|U(t) - V(t)\right|, \quad U, V \in X.$$

Clearly, (X,d) is a complete metric space. Let

$$X_0 = \left\{U \in \mathbb{E} : U(0) = U(1) = 0\right\}.$$

Then $X_0 \subset X$ is a closed subgroup of $\mathbb{E}$. Let $\psi, \varphi \in X$ such that $\psi - \varphi \in X_0$. Let $t \in [0,1]$, then we have

$$\begin{aligned}
\left|\mathcal{B}_n(q,\varphi)(t) - \mathcal{B}_n(q,\psi)(t)\right| &= \left|\sum_{k=0}^{N}\left(\left|\varphi\left(\frac{[k]_q}{[n]_q}\right)\right| - \left|\psi\left(\frac{[k]_q}{[n]_q}\right)\right|\right)\begin{bmatrix} n \\ k \end{bmatrix}_q t^k \prod_{s=0}^{n-1-k}\left(1-q^s t\right)\right| \\
&\leq \sum_{k=0}^{N}\left|(\varphi - \psi)\left(\frac{[k]_q}{[n]_q}\right)\right|\begin{bmatrix} n \\ k \end{bmatrix}_q t^k \prod_{s=0}^{n-1-k}\left(1-q^s t\right) \\
&\leq \sum_{k=1}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}_q t^k \prod_{s=0}^{n-1-k}\left(1-q^s t\right)d(\psi,\varphi).
\end{aligned}$$

Note that (see [23])

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q t^k \prod_{s=0}^{n-1-k}\left(1-q^s t\right) = 1.$$

Then, for $q \leq 1$, it is easy to observe that

$$\begin{aligned}
\sum_{k=1}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}_q t^k \prod_{s=0}^{n-1-k}\left(1-q^s t\right) &\leq 1 - t^n - (1-t)^n \\
&\leq 1 - \frac{1}{2^{n-1}}.
\end{aligned}$$

As a consequence, we have

$$\psi,\varphi \in X, \psi - \varphi \in X_0 \quad \Longrightarrow \quad d\left(\mathcal{B}_n(q,\varphi),\mathcal{B}_n(q,\psi)\right) \leq \left(1 - \frac{1}{2^{n-1}}\right)d(\psi,\varphi).$$

Now, let $\varphi \in X$. For any $t \in [0,1]$, we have

$$\varphi(t) - \mathcal{B}_n(q,\varphi) = \sum_{k=0}^{n}\left(\varphi(t) - \left|\varphi\left(\frac{[k]_q}{[n]_q}\right)\right|\right)\begin{bmatrix} n \\ k \end{bmatrix}_q t^k \prod_{s=0}^{n-1-k}\left(1-q^s t\right).$$

Observe that $\varphi(0) - \mathcal{B}_n(q,\varphi)(0) = \varphi(1) - \mathcal{B}_n(q,\varphi)(1) = 0$. Then, for every $\varphi \in X$, we have

$$\varphi - \mathcal{B}_n(q,\varphi) \in X_0.$$

By Theorem 5.1, we deduce that for every $\varphi \in X$, the Picard sequence $\{\mathcal{B}_n^j(q,\varphi)\}_{j\in\mathbb{N}}$ converges to a fixed point of $B_n(q,\cdot)$ and

$$(\varphi + X_0) \cap \operatorname{Fix}\mathcal{B}_n(q,\cdot) = \left\{\lim_{j\to\infty} B_n^j(q,\varphi)\right\}.$$

Let $\varphi \in X$. It is not difficult to observe that $\omega(t) = \varphi(0)(1-t) + \varphi(1)t \in \operatorname{Fix} \mathcal{B}_n(q,\cdot)$. We have also

$$\omega(t) = \varphi(t) + \theta(t),$$

where

$$\theta(t) = \varphi(0)(1-t) + \varphi(1)t - \varphi(t).$$

Observe that $\theta(0) = \theta(1) = 0$, which implies that $\theta \in X_0$. This ends the proof of Theorem 5.5. □

Remark 5.6 Note that Theorem 4.1 in [7] cannot be applied in the case of modified q-Bernstein operators since it requires linear operators defined on a certain Banach space X. Observe that in our case, X is not a linear space.

Remark 5.7 The case of modified 1-Bernstein operator was considered recently in [11]. The authors claimed that if $n \in \mathbb{N}$ for every $\varphi \in X = C([0,1])$, the Picard sequence $\{\mathcal{B}_n^j(1,\varphi)\}$ converges uniformly to a fixed point of $\mathcal{B}_n(1,\cdot)$ (see Corollary 4 in [11]). For the proof of this claim, the authors used that $\varphi - \mathcal{B}_n(1,\varphi) \in X_0$ for every $\varphi \in X$, where X_0 is the set of functions $\phi \in X$ such that $\phi(0) = \phi(1) = 0$. Unfortunately, the above property is not true. To observe this fact, we have just to consider a function $\varphi \in X$ such that $\varphi(0) < 0$ or $\varphi(1) < 0$. Our Theorem 5.5 for the case $q = 1$ is a corrected version of Corollary 4 in [11].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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