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Convergence theorems for finite family of a general class of multi-valued strictly pseudo-contractive mappings

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Abstract
Let H be a real Hilbert space, K a nonempty subset of H, and $T : K \to CB(K)$ a multi-valued mapping. Then T is called a *generalized k-strictly pseudo-contractive multi-valued mapping* if there exists $k \in [0, 1)$ such that, for all $x, y \in D(T)$, we have $D^2(Tx, Ty) \le \|x - y\|^2 + kD^2(Ax, Ay)$, where $A := I - T$, and I is the identity operator on K. A Krasnoselskii-type algorithm is constructed and proved to be an approximate fixed point sequence for a common fixed point of a finite family of this class of maps. Furthermore, assuming existence, strong convergence to a common fixed point of the family is proved under appropriate additional assumptions.

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1 Introduction

Let H be a real Hilbert space, $CB(H)$ denote the collection of nonempty, closed, and bounded subsets of H. For $A, B \in CB(H)$, the *Hausdorff metric* is defined by

$$D(A, B) = \max\Big\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\Big\}.$$

Let $T : D(T) \subseteq H \to CB(H)$ be a multi-valued mapping on H. A point $x \in D(T)$ is called a *fixed point* of T if $x \in Tx$. The fixed point set of T is denoted by $F(T) := \{x \in D(T) : x \in Tx\}$.

The study of fixed points for *multi-valued nonexpansive mappings* using Hausdorff metric was introduced by Markin [1], and studied extensively by Nadler [2]. Since then, many results have appeared in the literature (see, *e.g.*, Nadler [2] and Panyanak [3], and the references contained in them). Many of these results have found nontrivial applications in pure and applied sciences. Examples of such applications are, in control theory, convex optimization, differential inclusions, and economics (*especially in game theory and market economy*). For early results involving fixed points of multi-valued mappings and their applications see, for example, Brouwer [4], Daffer and Kaneko [5], Downing and Kirk [6], Geanakoplos [7], Kakutani [8], Nash [9, 10]. For details on the applications of this type of mappings in nonsmooth differential equations, one may consult Chang [11], Chidume *et* *al.* [12], Deimling [13], Khan *et al.* [14, 15], Reich *et al.* [16–18], Song and Wnag [19] and the references therein.

In studying the equation $Au = 0$, where A is a monotone operator defined on a real Hilbert space, Browder [20], introduced an operator T defined by $T := I - A$, where I is the identity mapping on H. He called such an operator a *pseudo-contractive mapping*. It is easily seen that the zeros of A are precisely the fixed points of the pseudo-contractive mapping T. It is well known that every nonexpansive mapping is pseudo-contractive but the converse is not true. In fact, in general, pseudo-contractive mappings are not necessarily continuous.

Moreover, Browder and Petryshyn [21] introduced the subclass of single-valued pseudo-contractive maps given below.

Definition 1.1 Let K be a nonempty subset of a real Hilbert space H. A map $T : K \to H$ is called *strictly pseudo-contractive* if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\big\|(x - Tx) - (y - Ty)\big\|^2, \quad \forall x, y \in K. \tag{1.1}$$

Since then, several extensions of this class of mappings to multi-valued cases have been defined and studied. For results on the approximation of common fixed points of families of *multi-valued* nonexpansive mappings (see, for example, Abbas *et al.* [22]), and for *multi-valued* strictly pseudo-contractive mappings see, Chidume *et al.* [12], Chidume and Ezeora [23], Ofoedu and Zegeye [24], Panyanak [3], Shahzad and Zegeye [25] and the references therein.

The class of multi-valued pseudo-contractive mappings introduced by Chidume *et al.* [12] is as follows.

Definition 1.2 Let H be a real Hilbert space and let K be a nonempty, closed, and convex subset of H. Let $T : K \to CB(K)$ be a mapping. Then T is called *a multi-valued k-strictly pseudo-contractive mapping* if there exists $k \in [0, 1)$ such that, for all $x, y \in D(T)$, we have

$$D^2(Tx, Ty) \leq \|x - y\|^2 + k\big\|(x - u) - (y - v)\big\|^2 \tag{1.2}$$

for all $u \in Tx$, $v \in Ty$.

Remark 1.3 It is easy to see that inequality (1.2) is equivalent to

$$D^2(Tx, Ty) \leq \|x - y\|^2 + k \inf_{(u,v)}\big\|(x - u) - (y - v)\big\|^2.$$

Using the above definition, Chidume *et al.* [12] proved the following theorem, which extends the result of Browder and Petryshyn [21].

Theorem 1.4 ([12]) *Let K be a nonempty, closed, and convex subset of a real Hilbert space H. Suppose that $T : K \to CB(K)$ is a multi-valued k-strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. Assume that $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n,$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1 - k)$. Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Chidume and Ezeora [23] extended the theorems obtained in Chidume *et al.* [12] to a finite family $\{T_i, i = 1, 2, \dots, m\}$ of multi-valued k_i-strictly pseudo-contractive mappings using a Krasnoselskii-type algorithm.

More precisely, they obtained the following theorems.

Theorem 1.5 ([23]) *Let K be a nonempty, closed, and convex subset of a real Hilbert space H and $T_i : K \to CB(K)$ be a finite family of multi-valued k_i-strictly pseudo-contractive mappings, $k_i \in (0,1)$, $i = 1, \dots, m$, such that $\bigcap_{i=1}^m F(T_i) \neq \emptyset$. Assume that, for $p \in \bigcap_{i=1}^m F(T_i)$, $T_i p = \{p\}$.*

Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,

$$x_{n+1} = \lambda_0 x_n + \lambda_1 y_n^1 + \cdots + \lambda_m y_n^m,$$

where $y_n^i \in T_i x_n$, $n \geq 1$ and $\lambda_i \in (k, 1)$, $i = 0, 1, \dots, m$, such that $\sum_{i=0}^m \lambda_i = 1$ and $k := \max\{k_i, i = 1, \dots, m\}$. Then $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$, $\forall i = 1, \dots, m$.

Theorem 1.6 ([23]) *Let K be a nonempty, closed, and convex subset of a real Hilbert space H and $T_i : K \to CB(K)$ be a finite family of multi-valued k_i-strictly pseudo-contractive mappings, $k_i \in (0,1)$, $i = 1, \dots, m$, such that $\bigcap_{i=1}^m F(T_i) \neq \emptyset$. Assume that, for $p \in \bigcap_{i=1}^m F(T_i)$, $T_i p = \{p\}$ and T_i, $i = 1, \dots, m$, is hemicompact and continuous. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = \lambda_0 x_n + \lambda_1 y_n^1 + \cdots + \lambda_m y_n^m,$$

where $y_n^i \in T_i x_n$, $n \geq 1$ and $\lambda_i \in (k, 1)$, $i = 0, 1, \dots, m$ such that $\sum_{i=0}^m \lambda_i = 1$ with $k := \max\{k_i, i = 1, \dots, m\}$. Then the sequence $\{x_n\}$ converges strongly to an element of $\bigcap_{i=1}^m F(T_i)$.

Chidume and Okpala [26] introduced the following definition for multi-valued k-strictly pseudo-contractive mappings.

Definition 1.7 ([26]) Let H be a real Hilbert space and let K be a nonempty subset of H. Let $T : K \to CB(K)$ be a mapping. Then T is called a *generalized k-strictly pseudo-contractive multi-valued mapping* if there exists $k \in [0,1)$ such that, for all $x, y \in D(T)$, we have

$$D^2(Tx, Ty) \leq \|x - y\|^2 + kD^2(Ax, Ay), \quad \text{where } A := I - T, \tag{1.3}$$

and I is the identity operator on K.

Remark 1.8 It is shown in [26] that every multi-valued k-strictly pseudo-contractive map as defined in [12] is a generalized k-strictly pseudo-contractive multi-valued mapping. An example is given in [26] of a generalized k-strictly pseudo-contractive mapping that is not a multi-valued k-strictly pseudo-copseudo-contractive mapping. The definition given here appears to be more natural than that given in [12].

It is our purpose in this paper to prove that a Krasnoselskii-type algorithm, under appropriate conditions, converges strongly to a common fixed point of a *finite family* $\{T_i, i = 1, 2, \dots, m\}$ of *generalized k_i*-strictly pseudo-contractive multi-valued mappings in a real Hilbert space. In the setting where the algorithms agree, our theorems generalize the results of Chidume *et al.* [12], and Chidume and Ezeora [23]. Moreover, they improve and extend to a finite family the results of Chidume and Okpala [26]. Also, several assumptions in the results of Chidume and Ezeora [23] (*e.g.*, $\lambda_i \in (k,1)$, for all i and T_i is continuous (see, *e.g.*, [23], Theorem 2.4) and hemicompact for each i), are significantly weakened.

The rest of this paper is organized as follows. Some known results and useful lemmas are listed in Section 2. In Section 3, we state and prove our main theorem and the corollaries that follow from the theorem. In the last section, we show an illustrative example where our theorem is applicable.

2 Preliminaries

We first recall some definitions, notations, and results which will be needed in proving our main results:

(i) $x_n \to x$: $\{x_n\}$ converges strongly to x as $n \to \infty$.
(ii) H: a real Hilbert space with an induced norm $\|\cdot\|$.
(iii) $F(T) := \{x \in K : x \in Tx\}$.
(iv) $CB(K)$, is the collection of nonempty, closed, and bounded subsets of K.

To simplify notation, we shall denote $(D(A,B))^2$ by $D^2(A,B)$ for all A, B elements of $CB(H)$.

Definition 2.1 A multi-valued mapping $T : K \subseteq H \to CB(K)$ is called Lipschitzian if there exists $L > 0$ such that

$$D(Tx, Ty) \le L\|x - y\| \tag{2.1}$$

for each $x, y \in K$. If $L < 1$ in inequality (2.1), the mapping T is called a *contraction*, and if $L = 1$, it is called *nonexpansive*.

We recall the following proposition.

Proposition 2.2 ([26]) *Let K be a nonempty subset of a real Hilbert space H and $T : K \to CB(K)$ be a generalized k-strictly pseudo-contractive multi-valued mapping. Then T is Lipschitzian.*

Remark 2.3 Since every Lipschitz map is continuous, we would not make any continuity assumption on our mapping T throughout this paper.

Definition 2.4 A map $T : K \to CB(K)$ is said to be *hemicompact* if, for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence, say, $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.

Remark 2.5 If K is compact, then every multi-valued mapping $T : K \to CB(K)$ is hemicompact.

The following lemma will also be used in the sequel.

Lemma 2.6 ([27]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following condition*:

$$a_{n+1} \le a_n + \sigma_n, \quad n \ge 0,$$

such that $\sum_{n=1}^{\infty}\sigma_n < \infty$. *Then* $\lim a_n$ *exists. If, in addition,* $\{a_n\}$ *has a subsequence that converges to* 0, *then* a_n *converges to* 0 *as* $n \to \infty$.

Lemma 2.7 ([23]) *Let* H *be a real Hilbert space and let* $\{x_i, i = 1, 2, \dots, m\} \subseteq H$. *For* $\alpha_i \in (0,1)$, $i = 1, 2, \dots, m$ *such that* $\sum_{i=1}^{m}\alpha_i = 1$, *the following identity holds*:

$$\left\|\sum_{i=1}^{m}\alpha_i x_i\right\|^2 = \sum_{i=1}^{m}\alpha_i\|x_i\|^2 - \sum_{1\leq i<j\leq m}\alpha_i\alpha_j\|x_i - x_j\|^2.$$

The following properties of the Hausdorf distance were established in [26].

Lemma 2.8 ([26]) *Let* E *be a normed linear space,* $B_1, B_2 \in CB(E)$, *and* $x, y \in E$ *arbitrary. The following hold*:
(a) $D(B_1, B_2) = D(x + B_1, x + B_2)$,
(b) $D(B_1, B_2) = D(-B_1, -B_2)$,
(c) $D(x + B_1, y + B_2) \leq \|x - y\| + D(B_1, B_2)$,
(d) $D(\{x\}, B_1) = \sup_{b_1\in B_1}\|x - b_1\|$,
(e) $D(\{x\}, B_1) = D(0, x - B_1)$.

3 Main results

In this section, we prove strong convergence theorems for a common fixed point of a finite family of *generalized* k-strictly pseudo-contractive multi-valued mappings in a real Hilbert space.

Henceforth, for any given finite family $\{T_i, i = 1, \dots, m\}$ of generalized k_i-strictly pseudo-contractive multi-valued mappings and arbitrary sequence $\{x_n\} \subseteq K$, let

$$S_n^i := \left\{y_n^i \in T_i x_n : D^2\big(\{x_n\}, T_i x_n\big) \leq \left\|x_n - y_n^i\right\|^2 + \frac{1}{n^2}\right\}.$$

Certainly, S_n^i is not empty for each $n \geq 1$ since by Lemma 2.8(d), we have

$$D\big(\{x_n\}, T_i x_n\big) = \sup_{y_n\in T_i x_n}\|x_n - y_n\|.$$

We now state and prove our main theorem.

Theorem 3.1 *Let* K *be a nonempty, closed, and convex subset of a real Hilbert space* H. *For* $i = 1, 2, \dots, m$, *let* $T_i : K \to CB(K)$ *be a family of generalized* k_i*-strictly pseudo-contractive multi-valued mappings with* $k_i \in (0,1)$. *Suppose that* $\bigcap_{i=1}^{m}F(T_i) \neq \emptyset$ *and assume that, for* $p \in \bigcap_{i=1}^{m}F(T_i)$, $T_i p = \{p\}$. *Define a sequence* $\{x_n\}$ *by* $x_0 \in K$ *arbitrary and*

$$x_{n+1} = (\lambda_0)x_n + \sum_{i=1}^{m}\lambda_i y_n^i, \tag{3.1}$$

where $y_n^i \in S_n^i$, $\lambda_0 \in (k,1)$, $\sum_{i=0}^{m}\lambda_i = 1$, *and* $k := \max\{k_i, i = 1, 2, \dots, m\}$. *Then, for each* $i = 1, 2, \dots, m$, $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$.

Proof Let $p \in \bigcap_{i=1}^{m} F(T_i)$. Then, using Lemma 2.7 together with Lemma 2.8(d) and (e), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \left\| \lambda_0(x_n-p) + \sum_{i=1}^{m} \lambda_i\big(y_n^i - p\big) \right\|^2 \\
&= \lambda_0\|x_n-p\|^2 + \sum_{i=1}^{m}\lambda_i\big\|y_n^i-p\big\|^2 - \sum_{i=1}^{m}\lambda_0\lambda_i\big\|x_n-y_n^i\big\|^2 \\
&\quad - \sum_{1\le i\le j\le m}\lambda_i\lambda_j\big\|y_n^i-y_n^j\big\|^2 \\
&\le \lambda_0\|x_n-p\|^2 + \sum_{i=1}^{m}\lambda_i D^2(T_ix_n, T_ip) - \sum_{i=1}^{m}\lambda_0\lambda_i\big\|x_n-y_n^i\big\|^2 \\
&\le \lambda_0\|x_n-p\|^2 + \sum_{i=1}^{m}\lambda_i\big(\|x_n-p\|^2 + k_iD^2\big(x_n - T_ix_n, \{0\}\big)\big) \\
&\quad - \sum_{i=1}^{m}\lambda_0\lambda_i\big\|x_n-y_n^i\big\|^2 \\
&= \sum_{i=0}^{m}\lambda_i\|x_n-p\|^2 + \sum_{i=1}^{m}\lambda_ik_iD^2\big(\{x_n\}, T_ix_n\big) - \sum_{i=1}^{m}\lambda_0\lambda_i\big\|x_n-y_n^i\big\|^2 \\
&\le \sum_{i=0}^{m}\lambda_i\|x_n-p\|^2 + \sum_{i=1}^{m}\lambda_ik\left(\big\|x_n-y_n^i\big\|^2 + \frac{1}{n^2}\right) \\
&\quad - \sum_{i=1}^{m}\lambda_0\lambda_i\big\|x_n-y_n^i\big\|^2, \quad \text{since } y_n^i \in S_n^i \\
&\le \|x_n-p\|^2 + \frac{k}{n^2} - \sum_{i=1}^{m}\lambda_i(\lambda_0-k)\big\|x_n-y_n^i\big\|^2.
\end{aligned}$$

Therefore,

$$\|x_{n+1}-p\|^2 \le \|x_n-p\|^2 + \frac{k}{n^2} - \sum_{i=1}^{m}\lambda_i(\lambda_0-k)\big\|x_n-y_n^i\big\|^2, \tag{3.2}$$

and then

$$\|x_{n+1}-p\|^2 \le \|x_n-p\|^2 + \frac{k}{n^2}. \tag{3.3}$$

Inequality (3.3) and Lemma 2.6 then show that the sequence $\{\|x_n-p\|\}$ has a limit and, therefore, $\{x_n\}$ is bounded. Moreover, we have from inequality (3.2) that

$$\sum_{i=1}^{m}\lambda_i(\lambda_0-k)\big\|x_n-y_n^i\big\|^2 \le \frac{k}{n^2} + \|x_n-p\|^2 - \|x_{n+1}-p\|^2,$$

and so

$$\lambda_i(\lambda_0-k)\big\|x_n-y_n^i\big\|^2 \le \frac{k}{n^2} + \|x_n-p\|^2 - \|x_{n+1}-p\|^2 \to 0 \quad (\text{as } n\to\infty)$$

for each $i = 1, 2, \ldots, m$. Thus, for each $i = 1, 2, \ldots, m$, $\lim_{n\to\infty} \|x_n - y_n^i\| = 0$ and using the fact $d(x_n, T_i x_n) \leq \|x_n - y_n^i\|$, it follows that $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$. □

Corollary 3.2 *Let K be nonempty, closed, and convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, m$, let $T_i : K \to CB(K)$ be a family of generalized k_i-strictly pseudo-contractive multi-valued mapping with $\bigcap_{i=1}^m F(T_i) \neq \emptyset$. Assume that, for $p \in \bigcap_{i=1}^m F(T_i)$, $T_i p = \{p\}$, and that T_{i_0} is hemicompact for some i_0. Then the sequence $\{x_n\}$ defined by* (3.1) *converges strongly to a common fixed point of $\{T_i, i = 1, 2, \ldots, m\}$.*

Proof By Theorem 3.1, $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for each i and in particular $\lim_{n\to\infty} d(x_n, T_{i_0} x_n) = 0$. Since T_{i_0} is hemicompact, let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_j} \to q$ as $j \to \infty$. For each $i = 1, 2, \ldots, m$, choose $y_{n_j}^i \in T_i x_{n_j}$ such that $\|x_{n_j} - y_{n_j}^i\| \leq d(x_{n_j}, T_i x_{n_j}) + \frac{1}{j}$. Then

$$
\begin{aligned}
d(q, T_i q) &\leq \|q - x_{n_j}\| + \left\|x_{n_j} - y_{n_j}^i\right\| + d\left(y_{n_j}^i, T_i q\right) \\
&\leq \|q - x_{n_j}\| + d(x_{n_j}, T_i x_{n_j}) + \frac{1}{j} + D(T_i x_{n_j}, T_i q) \\
&\leq \|q - x_{n_j}\| + d(x_{n_j}, T_i x_{n_j}) + \frac{1}{j} + \frac{1 + \sqrt{k}}{1 - \sqrt{k}} \|x_{n_j} - q\|.
\end{aligned}
$$

Thus, taking the limits on the right-hand side as $j \to \infty$, we have $d(q, T_i q) = 0$. Since $T_i q$ is closed, $q \in T_i q$ for each i and therefore $q \in \bigcap_{i=1}^m T_i q$. Moreover, $x_{n_j} \to q$ as $j \to \infty$ gives $\|x_{n_j} - q\| \to 0$ as $j \to \infty$. Thus, using inequality (3.3) and Lemma 2.6, $\lim_{n\to\infty} \|x_n - q\| = 0$. Therefore $\{x_n\}$ converges strongly to a common fixed point q of the maps T_i, as claimed. □

Remark 3.3 Corollary 3.2 is a significant improvement and generalization of Theorem 2.4 of [23] in the following sense:

(i) The theorem is proved for the much larger class of generalized k-strictly pseudo-contractive multi-valued mappings.
(ii) No continuity assumption is imposed on our maps.
(iii) Only one arbitrary map is required to be hemicompact.
(iv) The condition $\lambda_i \in (k, 1)$, for all i is replaced by the weaker condition $\lambda_0 \in (k, 1)$.

Furthermore, the condition $y_n^i \in S_n^i$ is more readily applicable than requiring that Tx is proximinal and weakly closed for each x, and then finding $y_n \in Tx_n$ such that $\|y_n - x_n\| = d(x_n, Tx_n)$ at each iterative step, as it is in [3] and in many other results.

Corollary 3.4 *Let K be nonempty, compact and convex subset of a real Hilbert space H, and for $i = 1, \ldots, m$, let $T_i : K \to CB(K)$ be a family of generalized k_i-strictly pseudo-contractive multi-valued mappings with $\bigcap_{i=1}^m F(T_i) \neq \emptyset$. Assume that, for $p \in \bigcap_{i=1}^m F(T_i)$, $T_i p = \{p\}$. Then the sequence $\{x_n\}$ defined by* (3.1) *converges strongly to a common fixed point of the maps T_i.*

Proof Since every multi-valued map defined on a compact set is necessarily hemicompact, $T_i : K \to CB(K)$ is hemicompact for each i. Thus, by Corollary 3.2, $\{x_n\}$ converges strongly to some $p \in \bigcap_{i=1}^m F(T_i)$. □

Remark 3.5 Our theorem and corollaries improve convergence theorems for multi-valued nonexpansive mappings in [3, 12, 14, 19, 22, 23, 26, 28], in the following sense:

(i) The class of mappings considered in this paper contains the class of multi-valued k-strictly pseudo-contractive mappings as special case, which itself properly contains the class of multi-valued nonexpansive maps.

(ii) The algorithm here is of Krasnoselskii-type, which is well known to have a geometric order of convergence.

(iii) The method of proof used here is of independent interest as it does not assume that Tx is weakly closed for each $x \in K$, or proximinal subset of K, as imposed in [12] and [23].

(iv) In the case where we have only one map, $m = 1$, we recover all the results of Chidume and Okpala [26].

4 An example

We finally give an example where the theorems are applicable. For the example, we shall need the following lemma.

Lemma 4.1 *Let a, b, c be real numbers such that $0 \leq a \leq bc$, $c > 0$. Then*

$$(a - b)^2 \leq b^2 + \left(\frac{c-2}{c}\right)a^2. \tag{4.1}$$

Proof The proof is trivially established as follows:

$$\begin{aligned} &0 \leq a \leq bc, \quad c > 0 \\ &\quad \Rightarrow \quad a^2 \leq abc \\ &\quad \Rightarrow \quad \frac{a^2}{c} \leq ab \\ &\quad \Rightarrow \quad -2ab \leq -\frac{2a^2}{c} \\ &\quad \Rightarrow \quad a^2 - 2ab + b^2 \leq a^2 - \frac{2a^2}{c} + b^2 \\ &\quad \Rightarrow \quad (a-b)^2 \leq b^2 + \left(\frac{c-2}{c}\right)a^2. \end{aligned}$$

□

Remark 4.2 If we take $c = 4$ in this lemma, we recover Lemma 3.5 of [26].

Example 4.3 Let $T_i : \mathbb{R} \to CB(\mathbb{R})$ be defined by

$$T_i x := \begin{cases} [(-1-\alpha_i)x, (-1+\alpha_i)x], & x > 0, \\ \{0\}, & x = 0, \\ [(-1+\alpha_i)x, (-1-\alpha_i)x], & x < 0, \end{cases} \tag{4.2}$$

where $\alpha_i = \frac{i}{5}$, $i = 1, 2, 3, 4, 5$. We have

$$x - T_i x := \begin{cases} [(2-\alpha_i)x, (2+\alpha_i)x], & x > 0, \\ \{0\}, & x = 0, \\ [(2+\alpha_i)x, (2-\alpha_i)x], & x < 0. \end{cases}$$

Then, for nonzero $x, y \in \mathbb{R}$,

$$D(T_i x, T_i y) = |x - y| + \alpha_i \big| |x| - |y| \big|$$

and

$$D(x - T_i x, y - T_i y) = 2|x - y| + \alpha_i \big| |x| - |y| \big|.$$

Now, set

$$a := D(x - T_i x, y - T_i y); \qquad b := |x - y|.$$

Then $a - b = D(T_i x, T_i y)$ and

$$\begin{aligned} a &= 2|x - y| + \alpha_i \big| |x| - |y| \big| \\ &\leq (2 + \alpha_i)|x - y|. \end{aligned}$$

Therefore, taking $c = c_i := 2 + \alpha_i$, $i = 1, 2, 3, 4, 5$, we have $\frac{c_i - 2}{c_i} = \frac{\alpha_i}{2 + \alpha_i}$, and by Lemma 4.1, we obtain

$$D^2(T_i x, T_i y) \leq |x - y|^2 + \frac{\alpha_i}{2 + \alpha_i} D(x - T_i x, y - T_i y).$$

Thus, each T_i, $i = 1, \ldots, 5$, is a generalized k_i-strictly pseudo-contractive multi-valued mapping with $k_i = \frac{\alpha_i}{2 + \alpha_i} \in (0, 1)$. Moreover, $p \in T_i p$ if and only if $p = 0$. Thus, for $p \in \bigcap_{i=1}^5 F(T_i p)$, $T_i p = \{p\}$.

It is also interesting to note that this family of mappings does not belong to the class discussed in [23]. We prove this by contradiction. For $i = 5$, $\alpha_5 = 1$ we have

$$T_5 x := \begin{cases} [-2x, 0], & x > 0, \\ \{0\}, & x = 0, \\ [0, -2x], & x < 0. \end{cases} \tag{4.3}$$

Observe that $0 \in T_5 x$ for all $x \in \mathbb{R}$. Now, suppose for contradiction that T_5 is a multi-valued k-strictly pseudo-contractive mapping in the sense of Definition 1.2, *i.e.*, T_5 satisfies inequality (1.2) for some $k \in (0, 1)$. Let $x = 1$, $y = 2$, $u = v = 0$. Then $u \in [-1, 0] = T_5 x$ and $v \in [-2, 0] = T_5 y$, and

$$\begin{aligned} &|x - y| = 1 = \big| |x| - |y| \big|, \qquad D(T_5 1, T_5 2) = 2, \\ &\big| (x - u) - (y - v) \big| = |x - y| = 1. \end{aligned}$$

Thus, we have

$$4 = D^2(Tx, Ty) \leq |x - y|^2 + k \big| (x - u) - (y - v) \big| \leq 2.$$

Since this is impossible, T_5 is not a k-strictly pseudo-contractive multi-valued mapping, in the sense of Definition 1.2, for any $k \in (0, 1)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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