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Accelerated Mann and CQ algorithms for finding a fixed point of a nonexpansive mapping

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Abstract

The purpose of this paper is to present accelerations of the Mann and CQ algorithms. We first apply the Picard algorithm to the smooth convex minimization problem and point out that the Picard algorithm is the steepest descent method for solving the minimization problem. Next, we provide the accelerated Picard algorithm by using the ideas of conjugate gradient methods that accelerate the steepest descent method. Then, based on the accelerated Picard algorithm, we present accelerations of the Mann and CQ algorithms. Under certain assumptions, we show that the new algorithms converge to a fixed point of a nonexpansive mapping. Finally, we show the efficiency of the accelerated Mann algorithm by numerically comparing with the Mann algorithm. A numerical example is provided to illustrate that the acceleration of the CQ algorithm is ineffective.

Keywords: nonexpansive mapping; convex minimization problem; Picard algorithm; Mann algorithm; CQ algorithm; conjugate gradient method; steepest descent method

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Suppose that $C \subset H$ is nonempty, closed and convex. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in C$. The set of fixed points of T is defined by $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$.

In this paper, we consider the following fixed point problem.

Problem 1.1 Suppose that $T : C \to C$ is nonexpansive with $\mathrm{Fix}(T) \neq \emptyset$. Then

$$\text{find } x^* \in C \text{ such that } T(x^*) = x^*.$$

The fixed point problems for nonexpansive mappings [1–4] capture various applications in diversified areas, such as convex feasibility problems, convex optimization problems, problems of finding the zeros of monotone operators, and monotone variational inequalities (see [1, 5] and the references therein). The Picard algorithm [6], the Mann algorithm

[7, 8], and the CQ method [9] are useful fixed point algorithms to solve the fixed point problems. Meanwhile, to guarantee practical systems and networks (see, *e.g.*, [10–13]) are stable and reliable, the fixed point has to be quickly found. Recently, Sakurai and Liduka [14] accelerated the Halpern algorithm and obtained a fast algorithm with strong convergence. Inspired by their work, we focus on the Mann and the CQ algorithms and present new algorithms to accelerate the approximation of a fixed point of a nonexpansive mapping.

We first apply the Picard algorithm to the smooth convex minimization problem and illustrate that the Picard algorithm is the steepest descent method [15] for solving the minimization problem. Since conjugate gradient methods [15] have been widely seen as an efficient accelerated version of most gradient methods, we introduce an accelerated Picard algorithm by combining the conjugate gradient methods and the Picard algorithm. Finally, based on the accelerated Picard algorithm, we present accelerations of the Mann and CQ algorithms.

In this paper, we propose two accelerated algorithms for finding a fixed point of a nonexpansive mapping and prove the convergence of the algorithms. Finally, the numerical examples are presented to demonstrate the effectiveness and fast convergence of the accelerated Mann algorithm and the ineffectiveness of the accelerated CQ algorithm.

2 Mathematical preliminaries

2.1 Picard algorithm and our algorithm

The Picard algorithm generates the sequence $\{x_n\}_{n=0}^{\infty}$ as follows: given $x_0 \in H$,

$$x_{n+1} = Tx_n, \quad n \geq 0. \tag{1}$$

The Picard algorithm (1) converges to a fixed point of the mapping T if $T : C \to C$ is contractive (see, *e.g.*, [1]).

When $\mathrm{Fix}(T)$ is the set of all minimizers of a convex, continuously Fréchet differentiable functional f over H, that algorithm (1) is the steepest descent method [15] to minimize f over H. Suppose that the gradient of f, denoted by ∇f, is Lipschitz continuous with a constant $L > 0$ and define $T^f : H \to H$ by

$$T^f := I - \lambda \nabla f, \tag{2}$$

where $\lambda \in (0, 2/L)$ and $I : H \to H$ stands for the identity mapping. Accordingly, T^f satisfies the contraction condition (see, *e.g.*, [10]) and

$$\mathrm{Fix}\bigl(T^f\bigr) = \arg\min_{x \in H} f(x) := \Bigl\{x^* \in H : f\bigl(x^*\bigr) = \min_{x \in H} f(x)\Bigr\}.$$

Therefore, algorithm (1) with $T := T^f$ can be expressed as follows:

$$\begin{cases} d_{n+1}^f := -\nabla f(x_n), \\ x_{n+1} := T^f(x_n) = x_n - \lambda \nabla f(x_n) = x_n + \lambda d_{n+1}^f. \end{cases} \tag{3}$$

The conjugate gradient methods [15] are popular acceleration methods of the steepest descent method. The conjugate gradient direction of f at x_n ($n \geq 0$) is $d_{n+1}^{f,CGD} := -\nabla f(x_n) +$

$\beta_n d_n^{f,CGD}$, where $d_0^{f,CGD} := -\nabla f(x_0)$ and $\{\beta_n\}_{n=0}^{\infty} \subset (0,\infty)$, which, together with (2), implies that

$$d_{n+1}^{f,CGD} = \frac{1}{\lambda}\bigl(T^f(x_n) - x_n\bigr) + \beta_n d_n^{f,CGD}. \tag{4}$$

By replacing $d_{n+1}^{f} := -\nabla f(x_n)$ in algorithm (3) with $d_{n+1}^{f,CGD}$ defined by (4), we get the accelerated Picard algorithm as follows:

$$\begin{cases} d_{n+1}^{f,CGD} := \frac{1}{\lambda}\bigl(T^f(x_n) - x_n\bigr) + \beta_n d_n^{f,CGD}, \\ x_{n+1} := x_n + \lambda d_{n+1}^{f,CGD}. \end{cases} \tag{5}$$

The convergence condition of Picard algorithm is very restrictive and it does not converge for general nonexpansive mappings (see, *e.g.*, [16]). So, in 1953 Mann [8] introduced the Mann algorithm

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{6}$$

and showed that the sequence generated by it converges to a fixed point of a nonexpansive mapping. In this paper we combine (5)-(6) and the CQ algorithm to present two novel algorithms.

2.2 Some lemmas

We will use the following notation:

(1) $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x and $x_n \to x$ means that $\{x_n\}$ converges strongly to x.

(2) $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

Lemma 2.1 *Let H be a real Hilbert space. There hold the following identities*:

(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$, $\forall x,y \in H$,

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$, $t \in [0,1]$, $\forall x,y \in H$.

Lemma 2.2 *Let K be a closed convex subset of a real Hilbert space H, and let P_K be the* (*metric or nearest point*) *projection from H onto K* (*i.e., for $x \in H$, $P_K x$ is the only point in K such that $\|x - P_K x\| = \inf\{\|x-z\| : z \in K\}$*). *Given $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if there holds the relation*

$$\langle x-z, y-z\rangle \leq 0 \quad \textit{for all } y \in K.$$

Lemma 2.3 (See [17]) *Let K be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_K u$. Suppose that $\{x_n\}$ is such that $\omega_w(x_n) \subset K$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\| \quad \textit{for all } n.$$

Then $x_n \to q$.

Lemma 2.4 (See [2]) *Let C be a closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$. *If a sequence $\{x_n\}$ in C is such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z = Tz$.*

Lemma 2.5 (See [18]) *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq a_n + u_n, \quad n \geq 0,$$

where $\{u_n\}$ *is a sequence of nonnegative real numbers such that* $\sum_{n=1}^{\infty} u_n < \infty$. *Then* $\lim_{n\to\infty} a_n$ *exists.*

3 The accelerated Mann algorithm

In this section, we present the accelerated Mann algorithm and give its convergence.

Algorithm 3.1 Choose $\mu \in (0,1]$, $\lambda > 0$, and $x_0 \in H$ arbitrarily, and set $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1)$, $\{\beta_n\}_{n=0}^{\infty} \subset [0,\infty)$. Set $d_0 := (T(x_0) - x_0)/\lambda$. Compute d_{n+1} and x_{n+1} as follows:

$$\begin{cases} d_{n+1} := \frac{1}{\lambda}\bigl(T(x_n) - x_n\bigr) + \beta_n d_n, \\ y_n := x_n + \lambda d_{n+1}, \\ x_{n+1} := \mu\alpha_n x_n + (1-\mu\alpha_n) y_n. \end{cases} \tag{7}$$

We can check that Algorithm 3.1 coincides with the Mann algorithm (6) when $\beta_n :\equiv 0$ and $\mu := 1$.

In this section we make the following assumptions.

Assumption 3.1 The sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ satisfy

(C1) $\sum_{n=0}^{\infty} \mu\alpha_n(1-\mu\alpha_n) = \infty$,

(C2) $\sum_{n=0}^{\infty} \beta_n < \infty$.

Moreover, $\{x_n\}_{n=0}^{\infty}$ satisfies

(C3) $\{T(x_n) - x_n\}_{n=0}^{\infty}$ is bounded.

Before doing the convergence analysis of Algorithm 3.1, we first show the two lemmas.

Lemma 3.1 *Suppose that* $T: H \to H$ *is nonexpansive with* $\mathrm{Fix}(T) \neq \emptyset$ *and that Assumption* 3.1 *holds. Then* $\{d_n\}_{n=0}^{\infty}$ *and* $\{\|x_n - p\|\}_{n=0}^{\infty}$ *are bounded for any* $p \in \mathrm{Fix}(T)$. *Furthermore,* $\lim_{n\to\infty} \|x_n - p\|$ *exists.*

Proof We have from (C2) that $\lim_{n\to\infty} \beta_n = 0$. Accordingly, there exists $n_0 \in \mathbb{N}$ such that $\beta_n \leq 1/2$ for all $n \geq n_0$. Define $M_1 := \max\{\max_{1\leq k\leq n_0} \|d_k\|, (2/\lambda)\sup_{n\in\mathbb{N}} \|T(x_n) - x_n\|\}$. Then (C3) implies that $M_1 < \infty$. Assume that $\|d_n\| \leq M_1$ for some $n \geq n_0$. The triangle inequality ensures that

$$\|d_{n+1}\| = \left\| \frac{1}{\lambda}\bigl(T(x_n) - x_n\bigr) + \beta_n d_n \right\| \leq \frac{1}{\lambda}\bigl\|T(x_n) - x_n\bigr\| + \beta_n\|d_n\| \leq M_1, \tag{8}$$

which means that $\|d_n\| \leq M_1$ for all $n \geq 0$, *i.e.*, $\{d_n\}_{n=0}^{\infty}$ is bounded.

The definition of $\{y_n\}_{n=0}^{\infty}$ implies that

$$\begin{aligned} y_n &= x_n + \lambda\left(\frac{1}{\lambda}\bigl(T(x_n) - x_n\bigr) + \beta_n d_n\right) \\ &= T(x_n) + \lambda\beta_n d_n. \end{aligned} \tag{9}$$

The nonexpansivity of T and (9) imply that, for any $p \in \mathrm{Fix}(T)$ and for all $n \geq n_0$,

$$\begin{aligned} \|y_n - p\| &= \big\| T(x_n) + \lambda\beta_n d_n - p \big\| \\ &\leq \big\| T(x_n) - T(p) \big\| + \lambda\beta_n \|d_n\| \\ &\leq \|x_n - p\| + \lambda M_1 \beta_n. \end{aligned} \tag{10}$$

Therefore, we find

$$\begin{aligned} \|x_{n+1} - p\| &= \big\| \mu\alpha_n(x_n - p) + (1 - \mu\alpha_n)(y_n - p) \big\| \\ &\leq \mu\alpha_n \|x_n - p\| + (1 - \mu\alpha_n)\|y_n - p\| \\ &\leq \mu\alpha_n \|x_n - p\| + (1 - \mu\alpha_n)\big\{ \|x_n - p\| + \lambda M_1 \beta_n \big\} \\ &\leq \|x_n - p\| + \lambda M_1 \beta_n, \end{aligned} \tag{11}$$

which implies

$$\|x_n - p\| \leq \|x_0 - p\| + \lambda M_1 \sum_{k=0}^{n-1} \beta_k < \infty.$$

So, we get that $\{x_n\}_{n=0}^{\infty}$ is bounded. From (10) it follows that $\{y_n\}_{n=0}^{\infty}$ is bounded.

In addition, using Lemma 2.5, (C2), and (11), we obtain $\lim_{n\to\infty} \|x_n - p\|$ exists. □

Lemma 3.2 *Suppose that* $T : H \to H$ *is nonexpansive with* $\mathrm{Fix}(T) \neq \emptyset$ *and that Assumption* 3.1 *holds. Then*

$$\lim_{n\to\infty} \big\| x_n - T(x_n) \big\| = 0.$$

Proof By (7)-(9) and the nonexpansivity of T, we have

$$\begin{aligned} &\big\| x_{n+1} - T(x_{n+1}) \big\| \\ &\quad = \big\| \big[\mu\alpha_n x_n + (1 - \mu\alpha_n)\big(T(x_n) + \lambda\beta_n d_n\big) \big] - T\big(\mu\alpha_n x_n + (1 - \mu\alpha_n)\big(T(x_n) + \lambda\beta_n d_n\big)\big) \big\| \\ &\quad = \big\| \big[\mu\alpha_n\big(x_n - T(x_n)\big) + (1 - \mu\alpha_n)\lambda\beta_n d_n \big] \\ &\qquad + \big[T(x_n) - T\big(\mu\alpha_n x_n + (1 - \mu\alpha_n)\big(T(x_n) + \lambda\beta_n d_n\big)\big) \big] \big\| \\ &\quad \leq \mu\alpha_n \big\| x_n - T(x_n) \big\| + (1 - \mu\alpha_n)\lambda\beta_n \|d_n\| \\ &\qquad + \big\| x_n - \big[\mu\alpha_n x_n + (1 - \mu\alpha_n)\big(T(x_n) + \lambda\beta_n d_n\big) \big] \big\| \\ &\quad \leq \mu\alpha_n \big\| x_n - T(x_n) \big\| + (1 - \mu\alpha_n)\lambda\beta_n \|d_n\| \\ &\qquad + (1 - \mu\alpha_n)\big\| x_n - T(x_n) \big\| + (1 - \mu\alpha_n)\lambda\beta_n \|d_n\| \\ &\quad \leq \big\| x_n - T(x_n) \big\| + 2\lambda\beta_n \|d_n\| \\ &\quad \leq \big\| x_n - T(x_n) \big\| + 2\lambda\beta_n M_1, \end{aligned}$$

which, with (C2) and Lemma 2.5, yields that the limit of $\|x_n - T(x_n)\|$ exists.

On the other hand, for any $p \in \operatorname{Fix}(T)$ and for all $n \geq n_0$, using the triangle inequality, the Cauchy-Schwarz inequality, and Lemma 2.1(ii), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\| \mu\alpha_n x_n + (1-\mu\alpha_n) y_n - p \right\|^2 \\
&= \left\| \mu\alpha_n x_n + (1-\mu\alpha_n)\bigl(T(x_n) + \lambda\beta_n d_n\bigr) - p \right\|^2 \\
&= \left\| \mu\alpha_n (x_n - p) + (1-\mu\alpha_n)\bigl(T(x_n) - p\bigr) + (1-\mu\alpha_n)\lambda\beta_n d_n \right\|^2 \\
&\leq \left\| \mu\alpha_n (x_n - p) + (1-\mu\alpha_n)\bigl(T(x_n) - p\bigr) \right\|^2 + \left\| (1-\mu\alpha_n)\lambda\beta_n d_n \right\|^2 \\
&\quad + 2\left\| \mu\alpha_n (x_n - p) + (1-\mu\alpha_n)\bigl(T(x_n) - p\bigr) \right\| \left\| (1-\mu\alpha_n)\lambda\beta_n d_n \right\| \\
&\leq \mu\alpha_n \|x_n - p\|^2 + (1-\mu\alpha_n)\left\| T(x_n) - p \right\|^2 - \mu\alpha_n(1-\mu\alpha_n)\left\| T(x_n) - x_n \right\|^2 \\
&\quad + (1-\mu\alpha_n)^2\lambda^2\beta_n^2 M_1^2 + 2\bigl[\mu\alpha_n\|x_n - p\| + (1-\mu\alpha_n)\left\| T(x_n) - p \right\|\bigr] \\
&\quad \times (1-\mu\alpha_n)\lambda\beta_n M_1 \\
&\leq \|x_n - p\|^2 - \mu\alpha_n(1-\mu\alpha_n)\left\| T(x_n) - x_n \right\|^2 \\
&\quad + \beta_n(1-\mu\alpha_n)\lambda\bigl\{2M_1\|x_n - p\| + (1-\mu\alpha_n)\lambda\beta_n M_1^2\bigr\} \\
&\leq \|x_n - p\|^2 - \mu\alpha_n(1-\mu\alpha_n)\left\| T(x_n) - x_n \right\|^2 + \beta_n M_2.
\end{aligned}$$

We have from Lemma 3.1 that $M_2 := \sup_{k\geq 0}(1-\mu\alpha_k)\lambda\{2M_1\|x_k - p\| + (1-\mu\alpha_k)\lambda\beta_k M_1^2\}$ is bounded. Therefore, using (C2), we obtain

$$\sum_{k=0}^{n} \mu\alpha_k(1-\mu\alpha_k)\left\| T(x_k) - x_k \right\|^2 \leq \|x_0 - p\|^2 - \|x_{n+1} - p\|^2 + M_2 \sum_{k=0}^{n} \beta_k < \infty,$$

which, with (C1), implies that

$$\liminf_{n\to\infty} \left\| T(x_n) - x_n \right\| = 0.$$

Due to existence of the limit of $\|T(x_n) - x_n\|$, we have

$$\lim_{n\to\infty} \left\| T(x_n) - x_n \right\| = 0,$$

which with Lemma 2.4 implies that $\omega_w(x_n) \subset \operatorname{Fix}(T)$. □

Theorem 3.1 *Suppose that* $T : H \to H$ *is nonexpansive with* $\operatorname{Fix}(T) \neq \emptyset$ *and that Assumption* 3.1 *holds. Then the sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *weakly converges to a fixed point of* T.

Proof To see that $\{x_n\}$ is actually weakly convergent, we need to show that $\omega_\omega(x_n)$ consists of exactly one point. Take $p, q \in \omega_w(x_n)$ and let $\{x_{n_i}\}$ and $\{x_{m_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$ and $x_{m_j} \rightharpoonup q$, respectively. Using Lemma 2.7 of [19] and Lemma 3.1, we have $p = q$. Hence, the proof is complete. □

4 The accelerated CQ algorithm

In general, the Mann algorithm (6) has only weak convergence (see [20] for an example). However, strong convergence is often much more desirable than weak convergence in many problems that arise in infinite dimensional spaces (see [21] and the references therein). In 2003, Nakajo and Takahashi [9] introduced the following modification of the Mann algorithm:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n := \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{12}$$

where C is a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ is a nonexpansive mapping, and P_K denotes the metric projection from H onto a closed convex subset K of H.

Here, we introduce an acceleration of CQ algorithm based on Algorithm 3.1 and show its strong convergence.

Theorem 4.1 *Let C be a bounded, closed and convex subset of a Hilbert space H and $T : C \to C$ be nonexpansive with* $\mathrm{Fix}(T) \neq \emptyset$. *Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $(0, a]$ for some $0 < a < 1$ and $\{\beta_n\}_{n=0}^{\infty} \subset [0,\infty)$ such that $\lim_{n\to\infty} \beta_n = 0$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the following algorithm*:

$$\begin{cases} x_0 \in C \textit{ chosen arbitrarily}, \\ d_{n+1} := \frac{1}{\lambda}\bigl(T(x_n) - x_n\bigr) + \beta_n d_n, \\ y_n := x_n + \lambda d_{n+1}, \\ z_n := \alpha_n x_n + (1-\alpha_n)y_n, \\ C_n = \bigl\{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\bigr\}, \\ Q_n = \bigl\{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\bigr\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{13}$$

where

$$\theta_n = \lambda\beta_n M_4[\lambda\beta_n M_4 + 2M_3] \to 0 \quad \textit{as } n \to \infty,$$

$M_3 = \operatorname{diam} C$ *and* $M_4 := \max\{\max_{1\le k\le n_0} \|d_k\|, (2/\lambda)M_3\}$, *$n_0$ is chosen such that $\beta_n \le 1/2$ for all $n \ge n_0$. Then $\{x_n\}_{n=0}^{\infty}$ converges in norm to $P_{\mathrm{Fix}(T)}(x_0)$.*

Proof First observe that C_n is convex. Indeed, the inequality defined in C_n can be rewritten as

$$\bigl\langle 2(x_n - z_n), z\bigr\rangle \le \|x_n\|^2 - \|z_n\|^2 + \theta_n$$

which is affine (and hence convex) in z. Next we show that $\mathrm{Fix}(T) \subset C_n$ for all n. Similar to the proof of (8), we get $\|d_n\| \leq M_4$. Due to $x_n \in C$, we have, for all $p \in \mathrm{Fix}(T) \subset C$,

$$\|x_n - p\| \leq M_3, \tag{14}$$

and

$$\|y_n - p\| \leq \|x_n - p\| + \lambda\beta_n M_4,$$

which comes from (10). Thus we get

$$\begin{aligned}\|z_n - p\| &\leq \alpha_n\|x_n - p\| + (1 - \alpha_n)\|y_n - p\| \\ &\leq \|x_n - p\| + \lambda\beta_n M_4,\end{aligned}$$

and consequently,

$$\begin{aligned}\|z_n - p\|^2 &\leq \|x_n - p\|^2 + 2\lambda\beta_n M_4\|x_n - p\| + (\lambda\beta_n M_4)^2 \\ &\leq \|x_n - p\|^2 + \theta_n,\end{aligned}$$

where $\theta_n = \lambda\beta_n M_4[\lambda\beta_n M_4 + 2M_3]$. So, $p \in C_n$ for all n. Next we show that

$$\mathrm{Fix}(T) \subset Q_n \quad \text{for all } n \geq 0. \tag{15}$$

We prove this by induction. For $n = 0$, we have $\mathrm{Fix}(T) \subset C = Q_0$. Assume that $\mathrm{Fix}(T) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, we have

$$\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \geq 0, \quad \forall z \in C_n \cap Q_n.$$

As $\mathrm{Fix}(T) \subset C_n \cap Q_n$, the last inequality holds, in particular, for all $z \in \mathrm{Fix}(T)$. This together with the definition of Q_{n+1} implies that $\mathrm{Fix}(T) \subset Q_{n+1}$. Hence (15) holds for all $n \geq 0$.

Notice that the definition of Q_n actually implies $x_n = P_{Q_n}(x_0)$. This together with the fact that $\mathrm{Fix}(T) \subset Q_n$ further implies

$$\|x_n - x_0\| \leq \|p - x_0\|, \quad p \in \mathrm{Fix}(T).$$

Due to $q = P_{\mathrm{Fix}(T)}(x_0) \in \mathrm{Fix}(T)$, we have

$$\|x_n - x_0\| \leq \|q - x_0\|. \tag{16}$$

The fact that $x_{n+1} \in Q_n$ asserts that $\langle x_{n+1} - x_n, x_n - x_0\rangle \geq 0$. This together with Lemma 2.1(i) implies

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \big\|(x_{n+1} - x_0) - (x_n - x_0)\big\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.\end{aligned} \tag{17}$$

This implies that the sequence $\{\|x_n - x_0\|\}_{n=0}^{\infty}$ is increasing. Recall (14), we get that $\lim_{n\to\infty}\|x_n - x_0\|$ exists. It turns out from (17) that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

By the fact $x_{n+1} \in C_n$ we get

$$\|z_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n,$$

and thus

$$\|z_n - x_{n+1}\| \le \|x_n - x_{n+1}\| + \sqrt{\theta_n}. \tag{18}$$

On the other hand, since $z_n = \alpha_n x_n + (1-\alpha_n)T(x_n) + (1-\alpha_n)\lambda\beta_n d_n$ and $\alpha_n \le a$, we have

$$\begin{aligned}\big\|T(x_n) - x_n\big\| &= \frac{1}{1-\alpha_n}\big\|z_n - x_n - (1-\alpha_n)\lambda\beta_n d_n\big\| \\ &\le \frac{1}{1-a}\|z_n - x_n\| + \lambda\beta_n\|d_n\| \\ &\le \frac{1}{1-a}\big(\|z_n - x_{n+1}\| + \|x_{n+1} - x_n\|\big) + \lambda\beta_n M_4 \\ &\le \frac{1}{1-a}\big(2\|x_{n+1} - x_n\| + \sqrt{\theta_n}\big) + \lambda\beta_n M_4 \to 0, \end{aligned} \tag{19}$$

where the last inequality comes from (18).

Lemma 2.4 and (19) then guarantee that every weak limit point of $\{x_n\}_{n=0}^{\infty}$ is a fixed point of T. That is, $\omega_w(x_n) \subset \mathrm{Fix}(T)$. This fact, with inequality (16) and Lemma 2.3, ensures the strong convergence of $\{x_n\}_{n=0}^{\infty}$ to $q = P_{\mathrm{Fix}(T)}x_0$. □

5 Numerical examples and conclusion

In this section, we compare the original algorithms and the accelerated algorithms. The codes were written in Matlab 7.0 and run on personal computer.

Firstly, we apply the Mann algorithm (6) and Algorithm 3.1 to the following convex feasibility problem (see [1, 14]).

Problem 5.1 (From [14]) Given a nonempty, closed convex set $C_i \subset \mathbb{R}^N$ ($i = 0,1,\ldots,m$),

$$\text{find } x^* \in C := \bigcap_{i=0}^{m} C_i,$$

where one assumes that $C \neq \emptyset$. Define a mapping $T : \mathbb{R}^N \to \mathbb{R}^N$ by

$$T := P_0\left(\frac{1}{m}\sum_{i=1}^{m} P_i\right), \tag{20}$$

where $P_i = P_{C_i}$ ($i = 0,1,\ldots,m$) stands for the metric projection onto C_i. Since P_i ($i = 0,1,\ldots,m$) is nonexpansive, T defined by (20) is also nonexpansive. Moreover, we find

Table 1 Computational results for Problem 5.1 with different dimensions

	Initial point		rand(*N*, 1)	200 × rand(*N*, 1)	5e	5,000e
$N = 5$	Algorithm 3.1	Iter.	2	62	5	9
$m = 5$		Sec.	0	0	0	0
	Mann algorithm	Iter.	64	68	73	74
		Sec.	0	0	0	0
$N = 1{,}000$	Algorithm 3.1	Iter.	4	448	7	425
$m = 50$		Sec.	0.0156	1.0608	0.0312	1.0608
	Mann algorithm	Iter.	505	515	478	429
		Sec.	1.2948	1.2480	1.2012	1.1076
$N = 50$	Algorithm 3.1	Iter.	6,814	7	4	9
$m = 1{,}000$		Sec.	38.9846	0.1092	0.0312	0.1248
	Mann algorithm	Iter.	8,438	7,771	8,692	6,913
		Sec.	53.6799	43.4463	49.7799	38.7818

that

$$\mathrm{Fix}(T) = \mathrm{Fix}(P_0) \bigcap_{i=1}^{m} \mathrm{Fix}(P_i) = C_0 \bigcap_{i=1}^{m} C_i = C.$$

Set $\lambda := 2$, $\mu := 0.05$, $\alpha_n := 1/(n+1)$ $(n \geq 0)$, and $\beta_n := 1/(n+1)$ in Algorithm 3.1 and $\alpha_n := \mu/(n+1)$ in the Mann algorithm (6). In the experiment, we set C_i $(i = 0, 1, \dots, m)$ as a closed ball with center $c_i \in \mathbb{R}^N$ and radius $r_i > 0$. Thus, P_i $(i = 0, 1, \dots, m)$ can be computed with

$$P_i(x) := \begin{cases} c_i + \frac{r_i}{\|c_i - x\|}(x - c_i) & \text{if } \|c_i - x\| > r_i, \\ x & \text{if } \|c_i - x\| \leq r_i. \end{cases}$$

We set $r_i := 1$ $(i = 0, 1, \dots, m)$, $c_0 := 0$ and $c_i \in (-1/\sqrt{N}, 1/\sqrt{N})^N$ $(i = 1, \dots, m)$ were randomly chosen. Set $e := (1, 1, \dots, 1)$. In Table 1, 'Iter.' and 'Sec.' denote the number of iterations and the cpu time in seconds, respectively. We took $\|T(x_n) - x_n\| < \varepsilon = 10^{-6}$ as the stopping criterion.

Table 1 illustrates that, with a few exceptions, Algorithm 3.1 significantly reduces the running time and iterations needed to find a fixed point compared with the Mann algorithm. The advantage is more obvious, as the parameters N and m become larger. It is worth further research on the reason of emergence of a few exceptions.

Next, we apply the CQ algorithm (12) and the accelerated CQ algorithm (13) to the following problem.

Problem 5.2 (From [22]) Let C be the unit closed ball $S(0,1) = \{x \in \mathbb{R}^3 \mid \|x\| \leq 1\}$ and $T : S(0,1) \to S(0,1)$ be defined by $T : (x_1, x_2, x_3)^T \mapsto (\frac{1}{\sqrt{3}} \sin(x+z), \frac{1}{\sqrt{3}} \sin(x+z), \frac{1}{\sqrt{3}}(x+y))^T$. Then

$$\text{find } x^* \in S(0,1) \text{ such that } T\big(x^*\big) = x^*.$$

He and Yang [22] showed that T is nonexpansive and has at least a fixed point in $S(0,1)$.

Take the sequence $\alpha_n = \frac{1}{n}$ in (12) and (13), and $\beta_n = \frac{1}{66 \times n^3}$, $\lambda = 1.2.$ in (13). We tested four different initial points and the numerical results are listed in Table 2.

Table 2 Computational results for Problem 5.2 with different initial points

Initial point		**(1, 0, 0)**	**(0, 1, 0)**	**(0, 0.5, 0.5)**	**(0.5, 0.5, 0.5)**
CQ algorithm	Iter.	652	167	1,441	38
	Sec.	0.1092	0.0312	0.2652	0
Accelerated CQ algorithm	Iter.	577	273	1,430	84
	Sec.	0.0936	0.0468	0.2496	0.0156

Table 2 shows that the acceleration of the CQ algorithm is ineffective, that is, the accelerated CQ algorithm does not in fact accelerate the CQ algorithm from running time or the number of iterations. The acceleration may be eliminated by the projection onto the sets C_n and Q_n.

6 Concluding remarks

In this paper, we accelerate the Mann and CQ algorithms to obtain the accelerated Mann and CQ algorithms, respectively. Then we present the weak convergence of the accelerated Mann algorithm and the strong convergence of the accelerated CQ algorithm under some conditions. The numerical examples illustrate that the acceleration of the Mann algorithm is effective, however, the acceleration of the CQ algorithm is ineffective.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

1. Bauschke, HH, Combettes, PL: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer, Berlin (2011)
2. Goebel, K, Kirk, WA: Topics in Metric Fixed Point Theory. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (1990)
3. Goebel, K, Reich, S: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Dekker, New York (1984)
4. Takahashi, W: Nonlinear Functional Analysis. Yokohama Publishers, Yokohama (2000)
5. Bauschke, HH, Borwein, JM: On projection algorithms for solving convex feasibility problems. SIAM Rev. **38**(3), 367-426 (1996)
6. Picard, E: Mémoire sur la théorie des équations aux dérivées partielles et la méthode des approximations successives. J. Math. Pures Appl. **6**, 145-210 (1890)
7. Krasnosel'skii, MA: Two remarks on the method of successive approximations. Usp. Mat. Nauk **10**, 123-127 (1955)
8. Mann, WR: Mean value methods in iteration. Proc. Am. Math. Soc. **4**, 506-510 (1953)
9. Nakajo, K, Takahashi, W: Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups. J. Math. Anal. Appl. **279**, 372-379 (2003)
10. Iiduka, H: Iterative algorithm for solving triple-hierarchical constrained optimization problem. J. Optim. Theory Appl. **148**, 580-592 (2011)
11. Iiduka, H: Fixed point optimization algorithm and its application to power control in CDMA data networks. Math. Program. **133**, 227-242 (2012)
12. Iiduka, H: Iterative algorithm for triple-hierarchical constrained nonconvex optimization problem and its application to network bandwidth allocation. SIAM J. Optim. **22**(3), 862-878 (2012)
13. Iiduka, H: Fixed point optimization algorithms for distributed optimization in networked systems. SIAM J. Optim. **23**, 1-26 (2013)
14. Sakurai, K, Liduka, H: Acceleration of the Halpern algorithm to search for a fixed point of a nonexpansive mapping. Fixed Point Theory Appl. **2014**, 202 (2014)
15. Nocedal, J, Wright, SJ: Numerical Optimization, 2nd edn. Springer Series in Operations Research and Financial Engineering. Springer, Berlin (2006)
16. Yao, Y, Marino, G, Muglia, L: A modified Korpelevich's method convergent to the minimum-norm solution of a variational inequality. Optimization **63**(4), 1-11 (2012)

17. Matinez-Yanes, C, Xu, HK: Strong convergence of the CQ method for fixed point processes. Nonlinear Anal. **64**, 2400-2411 (2006)
18. Tan, KK, Xu, HK: Approximating fixed points of nonexpansive mapping by the Ishikawa iteration process. J. Math. Anal. Appl. **178**, 301-308 (1993)
19. Suantai, S: Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings. J. Math. Anal. Appl. **311**, 506-517 (2005)
20. Genel, A, Lindenstrauss, J: An example concerning fixed points. Isr. J. Math. **22**, 81-86 (1975)
21. Bauschke, HH, Combettes, PL: A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces. Math. Oper. Res. **26**(2), 248-264 (2001)
22. He, S, Yang, Z: Realization-based method of successive projection for nonexpansive mappings and nonexpansive semigroups (submitted)