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Split common fixed point problem for two quasi-pseudo-contractive operators and its algorithm construction

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Abstract
The split common fixed point problem for two quasi-pseudo-contractive operators is studied. Some properties for quasi-pseudo-contractive operators are presented. An iterative algorithm for solving the split common fixed point problem for two quasi-pseudo-contractive operators is constructed. Strong convergence theorems are proved. A unified framework for the study of this class problem and class of operators is provided.

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1 Introduction

The split common fixed point problem has recently attracted so much attention (see, *e.g.*, [1–13]) due to the fact that it is a generalization of the split feasibility problem and the convex feasibility problem. In this paper, we aim to construct iterative algorithms for solving the split common fixed point problem for the class of quasi-pseudo-contractive operators. This more general class, which properly includes the classes of quasi-nonexpansive operators, directed operators, and demicontractive operators, is more desirable for example in fixed point methods in image recovery where in many cases, it is possible to map the set of images possessing a certain property to the fixed point set of a nonlinear quasi-nonexpansive operator. Our work is related to significant real-world applications; see for instance [14–18] and [19–21], where such methods were applied to the inverse problem of intensity-modulated radiation therapy and to the dynamic emission tomographic image reconstruction. Based on the related work in the literature, we present a unified framework for the study of this class problem and class of operators and propose iterative algorithms and study their convergence.

To begin with, let us recall that the split feasibility problem is to find a point

$$x^* \in C \quad \text{such that} \quad Ax^* \in Q, \tag{1.1}$$

where C and Q are two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively and $A : H_1 \to H_2$ is a bounded linear operator.

The split feasibility problem in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [14] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. They used their simultaneous multiprojections algorithm to obtain iterative algorithms to solve the split feasibility problem. Their algorithms, as well as others, see, *e.g.*, Byrne [22], involve matrix inversion at each iterative step. Calculating the inverses of matrices is very time-consuming, particularly if the dimensions are large. Therefore, a new algorithm for solving the split feasibility problem was devised by Byrne [15], called the CQ-algorithm, with the following iterative step:

$$x_{n+1} = P_C\bigl(x_n - \gamma A^*(I - P_Q)Ax_n\bigr), \quad n \geq 0, \tag{1.2}$$

where $0 < \gamma < 2/\|A\|^2$ and P_Q denotes the nearest point projection from H_2 onto Q. The CQ-algorithm converges to a solution of the split feasibility problem, for any starting vector $x_0 \in \mathbb{R}^N$, whenever the split feasibility problem has a solution. When the split feasibility problem has no solutions, the CQ-algorithm converges to a minimizer of $\|P_Q(Ac) - Ac\|$ over all $c \in C$, whenever such a minimizer exists.

In the case where C and Q in (1.1) are the intersections of finitely many fixed point sets of nonlinear operators, problem (1.1) is called by Censor and Segal [1] the split common fixed point problem. More precisely, the split common fixed point problem requires one to seek an element $x^* \in H$ satisfying

$$x^* \in \bigcap_{i=1}^{m} \operatorname{Fix}(T_i) \quad \text{and} \quad Ax^* \in \bigcap_{j=1}^{n} \operatorname{Fix}(S_j), \tag{1.3}$$

where $\operatorname{Fix}(T_i)$ and $\operatorname{Fix}(S_j)$ denote the fixed point sets of two classes of nonlinear operators $T_i : H_1 \to H_1$ and $S_j : H_2 \to H_2$, respectively.

Remark 1.1 If we set $C = \bigcap_{i=1}^{m} \operatorname{Fix}(T_i)$ and $Q = \bigcap_{j=1}^{n} \operatorname{Fix}(S_j)$, a natural problem arises: could we use iterative algorithm (1.2) to approach the solution of the split common fixed point problem (1.3)? However, in this situation, Byrne's CQ-algorithm does not work because the metric projection onto fixed point sets is generally not easy to calculate.

Consequently, in order to solve the two-set split common fixed point problem, Censor and Segal [1] constructed the following iterative algorithm without using the projection.

Algorithm 1.2 **Initialization:** Let $x_0 \in \mathbb{R}^N$ be arbitrary.
Iterative step: For $k \geq 0$ let

$$x_{k+1} = T\bigl(x_k + \lambda A^*(S - I)Ax_k\bigr), \quad k \geq 0, \tag{1.4}$$

where T and S are directed operators and $\lambda \in (0, 2/\gamma)$ with γ being the spectral radius of the operator A^*A.

They have shown the following convergence theorem.

Theorem 1.3 *Assume that* $T - I$ *and* $S - I$ *are demiclosed at* 0. *If* $\Gamma := \{x \in \operatorname{Fix}(T); Ax \in \operatorname{Fix}(S)\} \neq \emptyset$, *i.e., the problem is consistent, then any sequence* $\{x_k\}$, *generated by Algorithm* 1.2, *converges to a split common fixed point* $x^* \in \Gamma$.

Remark 1.4 Note that the underlying space in Theorem 1.3 is a finite-dimensional space $\mathbb{R}^N$. Hence, the strong convergence and weak convergence are consistent. Could we extend it to an infinite-dimensional space?

In [2], Moudafi demonstrated this work for us. He not only extended the space to the infinite-dimensional case but also extended the operators to a general class of operators and obtained the following algorithm and result.

Algorithm 1.5 **Initialization:** Let $x_0 \in H_1$ be arbitrary.
Iterative step: For $k \in \mathbb{N}$ set $u_k = x_k + \lambda A^*(S - I)Ax_k$ and let

$$x_{k+1} = (1 - \alpha_k)u_k + \alpha_k T(u_k), \quad k \in \mathbb{N}, \tag{1.5}$$

where $\lambda \in (0, \frac{1-\mu}{\gamma})$ with γ being the spectral radius of the operator A^*A and $\alpha_k \in (0,1)$.

Theorem 1.6 *Given a bounded linear operator $A : H_1 \to H_2$, let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be demicontractive operators (with constants β and μ, respectively) with nonempty* $\operatorname{Fix}(T) = C$ *and* $\operatorname{Fix}(S) = Q$. *Assume that $T - I$ and $S - I$ are demiclosed at* 0. *If $\Gamma \neq \emptyset$, then any sequence $\{x_k\}$ generated by the Algorithm* 1.5 *converges weakly to a split common fixed point $x^* \in \Gamma$, provided that $\alpha_k \in (\delta, 1 - \beta - \delta)$ for a small enough $\delta > 0$.*

Remark 1.7 It is clear that Algorithm 1.5 is a relaxation version of Algorithm 1.2. Theorem 1.6 extended Theorem 1.3 from directed operators to demicontractive operators and from finite-dimensional spaces to infinite-dimensional spaces.

Remark 1.8 Notice that Theorem 1.6 has only weak convergence in infinite-dimensional spaces, and it is well known that the strong convergence theorem is always more convenient to use. Could we construct an algorithm such that the strong convergence is guaranteed in the infinite-dimensional spaces?

For this purpose, He and Du [23] presented the following hybrid algorithm.

Algorithm 1.9

$$\begin{cases} x_1 \in C_1 \text{ chosen arbitrarily}, \\ y_n = (1 - \alpha)x_n + \alpha T x_n, \\ z_n = \beta x_n + (1 - \beta)T y_n, \\ w_n = P_C(z_n + \lambda A^*(S - I)Ax_n), \\ C_{n+1} = \{v \in C_n : \|w_n - v\| \le \|z_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where P is a projection operator.

Remark 1.10 Algorithm 1.9 has strong convergence under some mild assumptions. However, Algorithm 1.9 is involved with the computation of metric projection. This might seriously affect the efficiency of the method.

To overcome the above difficulty, the so-called self-adaptive method which permits step-size being selected self-adaptively was developed. Especially, in [24], Yao *et al.* presented the following algorithm.

Algorithm 1.11 Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $\psi : C \to H_1$ be a δ-contraction with $\delta \in [0, \frac{\sqrt{2}}{2})$. Let $A : H_1 \to H_2$ be a bounded linear operator. For given $x_0 \in C$, assume that $\{x_n\}$ has been constructed. If $\nabla f(x_n) = 0$, then stop and x_n is a solution of the (1.1). Otherwise, continue and compute x_{n+1} by the recursion

$$x_{n+1} = P_C\left[\alpha_n \psi(x_n) + (1-\alpha_n)\left(x_n - \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}\nabla f(x_n)\right)\right], \quad n \geq 0, \tag{1.7}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\rho_n\} \subset (0,2)$.

Consequently, Yao *et al.* proved the strong convergence of (1.7) under some additional conditions. Further, Zhou and Wang [25] used a new analysis technique to prove the convergence of (1.7) under some mild conditions.

The purpose of this paper is twofold. First, we will consider the split common fixed point problem for the class of quasi-pseudo-contractive operators which is more general than that the classes of quasi-nonexpansive operators, directed operators and demicontractive operators. Secondly, we will construct iterative algorithms with strong convergence without using the projection. Our results provide a unified framework for the study of this problem and this class of operators.

2 Preliminaries

In this section, we collect some tools including some definitions, some useful inequalities and lemmas which will be used to derive our main results in the next section.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be an operator. We use $\mathrm{Fix}(T)$ to denote the set of fixed points of T, that is, $\mathrm{Fix}(T) = \{x \mid x = Tx, x \in C\}$.

Definition 2.1 An operator $T : C \to C$ is said to be

(i) Nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.
(ii) Quasi-nonexpansive if $\|Tx - x^*\| \leq \|x - x^*\|$ for all $x \in C$ and $x^* \in \mathrm{Fix}(T)$.
(iii) Firmly nonexpansive if $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$ for all $x, y \in C$.
(iv) Firmly quasi-nonexpansive if $\|Tx - x^*\|^2 \leq \|x - x^*\|^2 - \|Tx - x\|^2$ for all $x \in C$ and $x^* \in \mathrm{Fix}(T)$.
(v) Strictly pseudo-contractive if $\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2$ for all $x, y \in C$, where $k \in [0,1)$.
(vi) Directed if $\langle Tx - x^*, Tx - x\rangle \leq 0$ for all $x \in C$ and $x^* \in \mathrm{Fix}(T)$.
(vii) Demicontractive if $\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|Tx - x\|^2$ for all $x \in C$ and $x^* \in \mathrm{Fix}(T)$, where $k \in [0,1)$.

Remark 2.2 The concept of directed operators was introduced and investigated by Bauschke and Combettes in [26] and by Combettes in [27]. They proved that $T : C \to C$

is directed if and only if

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 - \|Tx - x\|^2$$

for all $x \in C$ and $x^* \in \mathrm{Fix}(T)$. It can be seen easily that the class of directed operators coincides with that of firmly quasi-nonexpansive operators.

Remark 2.3 From the above definitions, we note that the class of demicontractive operators contains important operators such as the directed operators, the quasi-nonexpansive operators and the strictly pseudo-contractive operators with fixed points. Such a class of operators is fundamental because it includes many types of nonlinear operators arising in applied mathematics and optimization.

Definition 2.4 An operator $T : C \to C$ is said to be pseudo-contractive if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2$$

for all $x, y \in C$.

The interest of pseudo-contractive operators lies in their connection with monotone operators; namely, T is a pseudo-contraction if and only if the complement $I - T$ is a monotone operator. It is well known that T is pseudo-contractive if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2$$

for all $x, y \in C$.

Definition 2.5 An operator $T : C \to C$ is said to be quasi-pseudo-contractive if

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \|Tx - x\|^2 \tag{2.1}$$

for all $x \in C$ and $x^* \in \mathrm{Fix}(T)$.

It is obvious that the class of quasi-pseudo-contractive mappings includes the class of demicontractive mappings.

Definition 2.6 An operator $T : C \to C$ is said to be *L-Lipschitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|$$

for all $x, y \in C$.

Usually, the convergence of fixed point algorithms requires some additional smoothness properties of the mapping T such as demi-closedness.

Definition 2.7 An operator T is said to be demiclosed if, for any sequence $\{x_n\}$ which weakly converges to $\tilde{x}$, and if the sequence $\{T(x_n)\}$ strongly converges to z, then $T(\tilde{x}) = z$.

For all $x, y \in H$, the following conclusions hold:

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \quad t \in [0,1], \tag{2.2}$$

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \tag{2.3}$$

and

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle. \tag{2.4}$$

Lemma 2.8 ([28]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n, \quad n \in \mathbb{N},$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9 ([29]) *Let $\{w_n\}$ be a sequence of real numbers. Assume $\{w_n\}$ does not decrease at infinity, that is, there exists at least a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \le w_{n_k+1}$ for all $k \ge 0$. For every $n \ge N_0$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{i \le n : w_{n_i} < w_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \ge N_0$

$$\max\{w_{\tau(n)}, w_n\} \le w_{\tau(n)+1}.$$

3 Main results

In this section, we first show several properties for Lipschitzian operators and quasi-pseudo-contractive operators. These properties will be very useful for our main theorem. The first property is said to be commutativity in the sense of the set of fixed points of two operators.

Property 3.1 (Commutativity) Let H be a Hilbert space. Let $T : H \to H$ be an L-Lipschitzian operator with $L > 1$. Then

$$\mathrm{Fix}\big(\big((1-\zeta)I + \zeta T\big)T\big) = \mathrm{Fix}\big(T\big((1-\zeta)I + \zeta T\big)\big) = \mathrm{Fix}(T)$$

for all $\zeta \in (0, \frac{1}{L})$.

Proof We will divide our proof into two steps:

(i) $\mathrm{Fix}(((1-\zeta)I + \zeta T)T) = \mathrm{Fix}(T)$;

(ii) $\mathrm{Fix}(T((1-\zeta)I + \zeta T)) = \mathrm{Fix}(T)$.

Proof of (i). $\mathrm{Fix}(T) \subset \mathrm{Fix}(((1-\zeta)I + \zeta T)T)$ is obvious. We only need to prove that $\mathrm{Fix}(((1-\zeta)I + \zeta T)T) \subset \mathrm{Fix}(T)$. Let $x^\dagger \in \mathrm{Fix}(((1-\zeta)I + \zeta T)T)$. Thus, $((1-\zeta)I + \zeta T)Tx^\dagger = x^\dagger$. Observe that

$$\begin{aligned}\|x^\dagger - Tx^\dagger\| &= \|((1-\zeta)I + \zeta T)Tx^\dagger - Tx^\dagger\| \\ &= \zeta\|T(Tx^\dagger) - Tx^\dagger\| \\ &\leq \zeta L\|Tx^\dagger - x^\dagger\|.\end{aligned}$$

Since $\zeta L < 1$, we get $x^\dagger = Tx^\dagger$. That is, $x^\dagger \in \mathrm{Fix}(T)$. Hence, $\mathrm{Fix}(((1-\zeta)I + \zeta T)T) \subset \mathrm{Fix}(T)$.

Proof of (ii). $\mathrm{Fix}(T) \subset \mathrm{Fix}(T((1-\zeta)I + \zeta T))$ is obvious. Next, we show that $\mathrm{Fix}(T((1-\zeta)I + \zeta T)) \subset \mathrm{Fix}(T)$.

Take any $x^* \in \mathrm{Fix}(T((1-\zeta)I+\zeta T))$. We have $T((1-\zeta)I+\zeta T)x^* = x^*$. Set $U = (1-\zeta)I+\zeta T$. We have $TUx^* = x^*$. Write $Ux^* = y^*$. Then $Ty^* = x^*$. Now we show $x^* = y^*$. In fact,

$$\begin{aligned}\|x^* - y^*\| &= \|Ty^* - Ux^*\| \\ &= \|Ty^* - (1-\zeta)x^* - \zeta Tx^*\| \\ &= \zeta\|Ty^* - Tx^*\| \\ &\leq \zeta L\|y^* - x^*\|.\end{aligned}$$

Since $\zeta < \frac{1}{L}$, we deduce $y^* = x^* \in \mathrm{Fix}(U) = \mathrm{Fix}(T)$. Thus, $x^* \in \mathrm{Fix}(T)$. Hence, $\mathrm{Fix}(T((1-\zeta)I + \zeta T)) \subset \mathrm{Fix}(T)$. Therefore, $\mathrm{Fix}(T((1-\zeta)I + \zeta T)) = \mathrm{Fix}(T)$. □

The second property is the demiclosed principle for the operator $I - T((1-\zeta)I + \zeta T)$ under some mild conditions.

Property 3.2 (Demiclosedness) Let H be a Hilbert space. Let $T : H \to H$ be an L-Lipschitzian operator with $L > 1$. If $I - T$ is demiclosed at 0, then $I - T((1-\zeta)I + \zeta T)$ is also demiclosed at 0 when $\zeta \in (0, \frac{1}{L})$.

Proof Let the sequence $\{u_n\} \subset H$ satisfying $u_n \rightharpoonup \tilde{x}$ and $u_n - T((1-\zeta)I+\zeta T)u_n \to 0$. Next, we will show that $\tilde{x} \in \mathrm{Fix}(T((1-\zeta)I + \zeta T))$.

From Property 3.1, we only need to prove that $\tilde{x} \in \mathrm{Fix}(T)$. As a matter of fact, since T is L-Lipschizian, we have

$$\begin{aligned}\|u_n - Tu_n\| &\leq \|u_n - T((1-\zeta)I + \zeta T)u_n\| + \|T((1-\zeta)I + \zeta T)u_n - Tu_n\| \\ &\leq \|u_n - T((1-\zeta)I + \zeta T)u_n\| + \zeta L\|u_n - Tu_n\|.\end{aligned}$$

It follows that

$$\|u_n - Tu_n\| \leq \frac{1}{1-\zeta L}\|u_n - T((1-\zeta)I + \zeta T)u_n\|.$$

Hence,

$$\lim_{n\to\infty}\|u_n - Tu_n\| = 0.$$

By the demi-closedness of $I - T$, we immediately deduce $\tilde{x} \in \mathrm{Fix}(T)$. □

The third property is the quasi-nonexpansivity of the composite quasi-pseudo-contractive operator under some mild assumptions.

Property 3.3 (Quasi-nonexpansivity) Let H be a Hilbert space. Let $T : H \to H$ be an L-Lipschitz quasi-pseudo-contractive operator. Then the operator $(1-\xi)I + \xi T((1-\eta)I + \eta T)$ is quasi-nonexpansive when $0 < \xi < \eta < \frac{1}{\sqrt{1+L^2}+1}$. That is,

$$\left\|(1-\xi)x + \xi T\big((1-\eta)x + \eta Tx\big) - u^\dagger\right\| \le \left\|x - u^\dagger\right\|,$$

for all $x \in H$ and $u^\dagger \in \mathrm{Fix}(T)$.

Proof Since $u^\dagger \in \mathrm{Fix}(T)$, we have from (2.1)

$$\begin{aligned}\left\|T\big((1-\eta)I + \eta T\big)x - u^\dagger\right\|^2 \le{}& \left\|(1-\eta)\big(x - u^\dagger\big) + \eta\big(Tx - u^\dagger\big)\right\|^2 \\ &+ \left\|(1-\eta)x + \eta Tx - T\big((1-\eta)x + \eta Tx\big)\right\|^2\end{aligned} \tag{3.1}$$

and

$$\left\|Tx - u^\dagger\right\|^2 \le \left\|x - u^\dagger\right\|^2 + \|Tx - x\|^2, \tag{3.2}$$

for all $x \in H$.

Since T is L-Lipschitzian and $x - ((1-\eta)x + \eta Tx) = \eta(x - Tx)$, we have

$$\left\|Tx - T\big((1-\eta)x + \eta Tx\big)\right\| \le \eta L\|x - Tx\|. \tag{3.3}$$

From (2.2) and (3.2), we have

$$\begin{aligned}&\left\|(1-\eta)\big(x - u^\dagger\big) + \eta\big(Tx - u^\dagger\big)\right\|^2 \\ &\quad= (1-\eta)\left\|x - u^\dagger\right\|^2 + \eta\left\|Tx - u^\dagger\right\|^2 - \eta(1-\eta)\|x - Tx\|^2 \\ &\quad\le (1-\eta)\left\|x - u^\dagger\right\|^2 + \eta\big(\left\|x - u^\dagger\right\|^2 + \|Tx - x\|^2\big) \\ &\qquad- \eta(1-\eta)\|x - Tx\|^2 \\ &\quad= \left\|x - u^\dagger\right\|^2 + \eta^2\|Tx - x\|^2.\end{aligned} \tag{3.4}$$

From (2.2) and (3.3), we get

$$\begin{aligned}&\left\|(1-\eta)x + \eta Tx - T\big((1-\eta)x + \eta Tx\big)\right\|^2 \\ &\quad= \left\|(1-\eta)\big(x - T\big((1-\eta)x + \eta Tx\big)\big) + \eta\big(Tx - T\big((1-\eta)x + \eta Tx\big)\big)\right\|^2 \\ &\quad= (1-\eta)\left\|x - T\big((1-\eta)x + \eta Tx\big)\right\|^2 + \eta\left\|Tx - T\big((1-\eta)x + \eta Tx\big)\right\|^2 \\ &\qquad- \eta(1-\eta)\|x - Tx\|^2 \\ &\quad\le (1-\eta)\left\|x - T\big((1-\eta)x + \eta Tx\big)\right\|^2 - \eta\big(1 - \eta - \eta^2L^2\big)\|x - Tx\|^2.\end{aligned} \tag{3.5}$$

By (3.1), (3.4), and (3.5), we obtain

$$\begin{aligned}\big\|T\big((1-\eta)I+\eta T\big)x-u^\dagger\big\|^2 &\le \big\|x-u^\dagger\big\|^2+\eta^2\|x-Tx\|^2\\&\quad+(1-\eta)\big\|x-T\big((1-\eta)x+\eta Tx\big)\big\|^2\\&\quad-\eta\big(1-\eta-\eta^2L^2\big)\|x-Tx\|^2\\&=\big\|x-u^\dagger\big\|^2+(1-\eta)\big\|x-T\big((1-\eta)I+\eta T\big)x\big\|^2\\&\quad-\eta\big(1-2\eta-\eta^2L^2\big)\|x-Tx\|^2.\end{aligned}\tag{3.6}$$

Since $\eta<\frac{1}{\sqrt{1+L^2}+1}$, we deduce

$$1-2\eta-\eta^2L^2>0.$$

From (3.6), we deduce

$$\big\|T\big((1-\eta)x+\eta Tx\big)-u^\dagger\big\|^2\le\big\|x-u^\dagger\big\|^2+(1-\eta)\big\|x-T\big((1-\eta)x+\eta Tx\big)\big\|^2\tag{3.7}$$

for all $x\in H$ and $u^\dagger\in\operatorname{Fix}(T)$.

Combine (2.2) and (3.7) to get

$$\begin{aligned}&\big\|(1-\xi)x+\xi T\big((1-\eta)x+\eta Tx\big)-u^\dagger\big\|^2\\&\quad=\big\|(1-\xi)\big(x-u^\dagger\big)+\xi\big(T\big((1-\eta)x+\eta Tx\big)-u^\dagger\big)\big\|^2\\&\quad=(1-\xi)\big\|x-u^\dagger\big\|^2+\xi\big\|T\big((1-\eta)x+\eta Tx\big)-u^\dagger\big\|^2\\&\qquad-\xi(1-\xi)\big\|T\big((1-\eta)x+\eta Tx\big)-x\big\|^2\\&\quad\le\xi\big[\big\|x-u^\dagger\big\|^2+(1-\eta)\big\|x-T\big((1-\eta)x+\eta Tx\big)\big\|^2\big]\\&\qquad+(1-\xi)\big\|x-u^\dagger\big\|^2-\xi(1-\xi)\big\|T\big((1-\eta)x+\eta Tx\big)-x\big\|^2\\&\quad=\big\|x-u^\dagger\big\|^2+\xi(\xi-\eta)\big\|T\big((1-\eta)x+\eta Tx\big)-x\big\|^2.\end{aligned}$$

This together with $\xi<\eta$ implies that

$$\big\|(1-\xi)x+\xi T\big((1-\eta)x+\eta Tx\big)-u^\dagger\big\|\le\big\|x-u^\dagger\big\|.$$

This completes the proof. □

In the sequel, we introduce our algorithm and prove its strong convergence.

Some assumptions on the underlying spaces and involved operators are listed below.

(R1) H_1 and H_2 are two real Hilbert spaces.

(R2) $A:H_1\to H_2$ is a bounded linear operator with its adjoint A^* and $B:H_1\to H_1$ is a strong positive linear bounded operator with coefficient $\xi>\rho$.

(R3) $f:H_1\to H_1$ is a ρ-contraction, $S:H_2\to H_2$ is an L_1-Lipschitzian quasi-pseudo-contractive operator with $L_1>1$ and $T:H_1\to H_1$ is an L_2-Lipschitzian quasi-pseudo-contractive operator with $L_2>1$.

Our object is to solve the following two-set split common fixed point problem:

$$\text{find } x^* \in \operatorname{Fix}(T) \quad \text{such that} \quad Ax^* \in \operatorname{Fix}(S). \tag{3.8}$$

We use Γ to denote the set of solutions of (3.8), that is,

$$\Gamma = \{x^* \mid x^* \in \operatorname{Fix}(T), Ax^* \in \operatorname{Fix}(S)\}.$$

In the sequel, we assume $\Gamma \neq \emptyset$.

Now, we present our algorithm for finding $x^* \in \Gamma$.

Algorithm 3.4 **Initialization:** Let $x_0 \in H_1$ be arbitrary.

Iterative step: For $n \geq 0$ let

$$\begin{cases} v_n = x_n + \delta A^*[(1-\zeta_n)I + \zeta_n S((1-\eta_n)I + \eta_n S) - I]Ax_n, \\ u_n = \alpha_n f(x_n) + (I - \alpha_n B)v_n, \\ x_{n+1} = (1-\beta_n)u_n + \beta_n T((1-\gamma_n)u_n + \gamma_n Tu_n), \quad n \in \mathbb{N}, \end{cases} \tag{3.9}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$, $\{\gamma_n\}_{n\in\mathbb{N}}$, $\{\zeta_n\}_{n\in\mathbb{N}}$, and $\{\eta_n\}_{n\in\mathbb{N}}$ are five real number sequences in $(0,1)$ and δ is a constant in $(0, \frac{1}{\|A\|^2})$.

Theorem 3.5 *Suppose $T - I$ and $S - I$ are demiclosed at* 0. *Assume that the following conditions are satisfied*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $0 < a_1 < \beta_n < c_1 < \gamma_n < b_1 < \frac{1}{\sqrt{1+L_2^2}+1}$;

(C4) $0 < a_2 < \zeta_n < c_2 < \eta_n < b_2 < \frac{1}{\sqrt{1+L_1^2}+1}$.

Then the sequence $\{x_n\}$ generated by algorithm (3.9) *converges strongly to $x^* = P_\Gamma(f + I - B)x^*$.*

Proof Let $x^* = P_\Gamma(f + I - B)x^*$. Then we have $x^* \in \operatorname{Fix}(T)$ and $Ax^* \in \operatorname{Fix}(S)$. From Property 3.1 and Property 3.2, we get

$$\begin{aligned} &\big\|\big[(1-\zeta_n)I + \zeta_n S\big((1-\eta_n)I + \eta_n S\big)\big]Ax_n - Ax^*\big\|^2 \\ &\quad = \big\|\big[(1-\zeta_n)I + \zeta_n S\big((1-\eta_n)I + \eta_n S\big)\big]Ax_n \\ &\qquad - \big[(1-\zeta_n)I + \zeta_n S\big((1-\eta_n)I + \eta_n S\big)\big]Ax^*\big\|^2 \\ &\quad \leq \|Ax_n - Ax^*\|^2. \end{aligned} \tag{3.10}$$

From (2.1), we deduce

$$\begin{aligned} &\big\|T\big((1-\gamma_n)u_n + \gamma_n Tu_n\big) - x^*\big\|^2 \\ &\quad \leq \|u_n - x^*\|^2 + (1-\gamma_n)\big\|u_n - T\big((1-\gamma_n)u_n + \gamma_n Tu_n\big)\big\|^2. \end{aligned}$$

This together with (3.9) and (2.2) implies that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|(1-\beta_n)u_n+\beta_n T((1-\gamma_n)u_n+\gamma_n Tu_n)-x^*\|^2\\ &= (1-\beta_n)\|u_n-x^*\|^2+\beta_n\|T((1-\gamma_n)u_n+\gamma_n Tu_n)-x^*\|^2\\ &\quad -\beta_n(1-\beta_n)\|u_n-T((1-\gamma_n)u_n+\gamma_n Tu_n)\|^2\\ &\le \|u_n-x^*\|^2-\beta_n(\gamma_n-\beta_n)\|T((1-\gamma_n)u_n+\gamma_n Tu_n)-x^*\|^2\\ &\le \|u_n-x^*\|^2. \end{aligned} \tag{3.11}$$

Note that

$$\begin{aligned}\|u_n-x^*\| &= \|\alpha_n(f(x_n)-Bx^*)+(I-\alpha_n B)(v_n-x^*)\|\\ &\le \alpha_n\|f(x_n)-Bx^*\|+\|I-\alpha_n B\|\|v_n-x^*\|\\ &\le \alpha_n\|f(x_n)-f(x^*)\|+\alpha_n\|f(x^*)-Bx^*\|+(1-\alpha_n\xi)\|v_n-x^*\|\\ &\le \alpha_n\rho\|x_n-x^*\|+\alpha_n\|f(x^*)-Bx^*\|+(1-\alpha_n\xi)\|v_n-x^*\|. \end{aligned} \tag{3.12}$$

By (2.3), we have

$$\begin{aligned}\|v_n-x^*\| &= \|x_n-x^*+\delta A^*[(1-\zeta_n)I+\zeta_n S((1-\eta_n)I+\eta_n S)-I]Ax_n\|^2\\ &= \|x_n-x^*\|^2+\delta^2\|A^*[(1-\zeta_n)I+\zeta_n S((1-\eta_n)I+\eta_n S)-I]Ax_n\|^2\\ &\quad +2\delta\langle x_n-x^*,A^*[(1-\zeta_n)I+\zeta_n S((1-\eta_n)I+\eta_n S)-I]Ax_n\rangle. \end{aligned} \tag{3.13}$$

Since A is a linear operator, with adjoint A^*, we have

$$\begin{aligned}&\langle x_n-x^*,A^*[(1-\zeta_n)I+\zeta_n S((1-\eta_n)I+\eta_n S)-I]Ax_n\rangle\\ &\quad=\langle A(x_n-x^*),[(1-\zeta_n)I+\zeta_n S((1-\eta_n)I+\eta_n S)-I]Ax_n\rangle\\ &\quad=\langle[(1-\zeta_n)I+\zeta_n S((1-\eta_n)I+\eta_n S)]Ax_n-Ax^*,[(1-\zeta_n)I\\ &\qquad+\zeta_n S((1-\eta_n)I+\eta_n S)-I]Ax_n\rangle\\ &\qquad-\|[(1-\zeta_n)I+\zeta_n S((1-\eta_n)I+\eta_n S)-I]Ax_n\|^2. \end{aligned} \tag{3.14}$$

Again using (2.3), we obtain

$$\begin{aligned}&\langle[(1-\zeta_n)I+\zeta_n S((1-\eta_n)I+\eta_n S)]Ax_n-Ax^*,[(1-\zeta_n)I+\zeta_n S((1-\eta_n)I+\eta_n S)-I]Ax_n\rangle\\ &\quad=\frac{1}{2}(\|[(1-\zeta_n)I+\zeta_n S((1-\eta_n)I+\eta_n S)]Ax_n-Ax^*\|^2\\ &\qquad+\|[(1-\zeta_n)I+\zeta_n S((1-\eta_n)I+\eta_n S)-I]Ax_n\|^2-\|Ax_n-Ax^*\|^2). \end{aligned} \tag{3.15}$$

From (3.10), (3.14), and (3.15), we get

$$\begin{aligned}&\langle x_n-x^*,A^*[(1-\zeta_n)I+\zeta_n S((1-\eta_n)I+\eta_n S)-I]Ax_n\rangle\\ &\quad=\frac{1}{2}(\|[(1-\zeta_n)I+\zeta_n S((1-\eta_n)I+\eta_n S)]Ax_n-Ax^*\|^2 \end{aligned}$$

$$
\begin{aligned}
&+ \big\|\big[(1-\zeta_n)I + \zeta_n S\big((1-\eta_n)I + \eta_n S\big) - I\big]Ax_n\big\|^2 - \big\|Ax_n - Ax^*\big\|^2\big) \\
&- \big\|\big[(1-\zeta_n)I + \zeta_n S\big((1-\eta_n)I + \eta_n S\big) - I\big]Ax_n\big\|^2 \\
\le{}& \frac{1}{2}\big(\big\|Ax_n - Ax^*\big\|^2 + \big\|\big[(1-\zeta_n)I + \zeta_n S\big((1-\eta_n)I + \eta_n S\big) - I\big]Ax_n\big\|^2 \\
&- \big\|Ax_n - Ax^*\big\|^2\big) - \big\|\big[(1-\zeta_n)I + \zeta_n S\big((1-\eta_n)I + \eta_n S\big) - I\big]Ax_n\big\|^2 \\
={}& -\frac{1}{2}\big\|\big[(1-\zeta_n)I + \zeta_n S\big((1-\eta_n)I + \eta_n S\big) - I\big]Ax_n\big\|^2.
\end{aligned}
\tag{3.16}
$$

So,

$$
\begin{aligned}
\big\|v_n - x^*\big\|^2 &= \big\|x_n - x^* + \delta A^*\big[(1-\zeta_n)I + \zeta_n S\big((1-\eta_n)I + \eta_n S\big) - I\big]Ax_n\big\|^2 \\
&\le \delta^2\|A\|^2\big\|\big[(1-\zeta_n)I + \zeta_n S\big((1-\eta_n)I + \eta_n S\big) - I\big]Ax_n\big\|^2 \\
&\quad + \big\|x_n - x^*\big\|^2 - \delta\big\|\big[(1-\zeta_n)I + \zeta_n S\big((1-\eta_n)I + \eta_n S\big) - I\big]Ax_n\big\|^2\big) \\
&= \big\|x_n - x^*\big\|^2 + \big(\delta^2\|A\|^2 - \delta\big)\big\|\big[(1-\zeta_n)I \\
&\quad + \zeta_n S\big((1-\eta_n)I + \eta_n S\big) - I\big]Ax_n\big\|^2 \\
&\le \big\|x_n - x^*\big\|^2.
\end{aligned}
\tag{3.17}
$$

It follows that

$$
\big\|x_n - x^* + \delta A^*\big[(1-\zeta_n)I + \zeta_n S\big((1-\eta_n)I + \eta_n S\big) - I\big]Ax_n\big\| \le \big\|x_n - x^*\big\|. \tag{3.18}
$$

Substituting (3.18) into (3.12), we deduce that

$$
\begin{aligned}
\big\|u_n - x^*\big\| &\le \alpha_n\rho\big\|x_n - x^*\big\| + \alpha_n\big\|f\big(x^*\big) - Bx^*\big\| + (1-\alpha_n\xi)\big\|x_n - x^*\big\| \\
&= \alpha_n\big\|f\big(x^*\big) - Bx^*\big\| + \big[1 - (\xi-\rho)\alpha_n\big]\big\|x_n - x^*\big\|.
\end{aligned}
\tag{3.19}
$$

From (3.11) and (3.19), we get

$$
\begin{aligned}
\big\|x_{n+1} - x^*\big\| &\le \big\|u_n - x^*\big\| \\
&\le \alpha_n\big\|f\big(x^*\big) - Bx^*\big\| + \big[1 - (\xi-\rho)\alpha_n\big]\big\|x_n - x^*\big\| \\
&\le \max\left\{\big\|x_n - x^*\big\|, \frac{\|f(x^*) - Bx^*\|}{\xi - \rho}\right\}.
\end{aligned}
$$

The boundedness of the sequence $\{x_n\}$ yields the result.

Next, we focus our analysis on the fact that the real sequence $\{\|x_n - x^*\|\}$ is either monotone decreasing at infinity (Case 1) or not (Case 2):

Case 1. There exists n_0 such that the sequence $\{\|x_n - x^*\|\}_{n\ge n_0}$ is decreasing.

Case 2. For any n_0, there exists an integer $m \ge n_0$ such that $\|x_m - x^*\| \le \|x_{m+1} - x^*\|$.

More precisely, regarding the situation when $\{\|x_n - x^*\|\}$ is monotonous at infinity (Case 1) and bounded (hence convergent), we prove that its only possible limit is zero.

In Case 1, we assume there exists some integer $n_0 > 0$ such that $\{\|x_n - x^*\|\}$ is decreasing for all $n \ge n_0$. In this case, we know that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Returning to (3.12), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \|u_n-x^*\|^2\\ &\le \left[\alpha_n\rho\|x_n-x^*\|+\alpha_n\|f(x^*)-Bx^*\|+(1-\alpha_n\xi)\|v_n-x^*\|\right]^2\\ &=\alpha_n^2\left(\rho\|x_n-x^*\|+\|f(x^*)-Bx^*\|\right)^2+2\alpha_n(1-\alpha_n\xi)\left(\rho\|x_n-x^*\|\right.\\ &\quad\left.+\|f(x^*)-Bx^*\|\right)\|v_n-x^*\|+(1-\alpha_n\xi)^2\|v_n-x^*\|^2\\ &\le \alpha_n\left(\rho\|x_n-x^*\|+\|f(x^*)-Bx^*\|\right)\left(3\|x_n-x^*\|+\|f(x^*)-Bx^*\|\right)\\ &\quad+(1-\alpha_n\xi)\|v_n-x^*\|^2\\ &\le (1-\alpha_n\xi)\left(\delta^2\|A\|^2-\delta\right)\left\|\left[(1-\zeta_n)I+\zeta_nS\left((1-\eta_n)I+\eta_nS\right)-I\right]Ax_n\right\|^2\\ &\quad+M\alpha_n+(1-\alpha_n\xi)\|x_n-x^*\|^2\\ &\le M\alpha_n+\|x_n-x^*\|^2, \end{aligned}\tag{3.20}$$

where $M > 0$ is a constant such that

$$\sup_n\left\{\left(\rho\|x_n-x^*\|+\|f(x^*)-Bx^*\|\right)\left(3\|x_n-x^*\|+\|f(x^*)-Bx^*\|\right)\right\}\le M.$$

Hence,

$$\begin{aligned}&(1-\alpha_n\xi)\left(\delta-\delta^2\|A\|^2\right)\left\|\left[(1-\zeta_n)I+\zeta_nS\left((1-\eta_n)I+\eta_nS\right)-I\right]Ax_n\right\|^2\\ &\quad\le (1-\alpha_n\xi)\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+M\alpha_n.\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n-x^*\|$ exists and $\alpha_n\to 0$, we obtain

$$\lim_{n\to\infty}\left\|\left[(1-\zeta_n)I+\zeta_nS\left((1-\eta_n)I+\eta_nS\right)-I\right]Ax_n\right\|=0.\tag{3.21}$$

Therefore,

$$\lim_{n\to\infty}\left\|Ax_n-S\left((1-\eta_n)I+\eta_nS\right)Ax_n\right\|=0.$$

We have

$$\begin{aligned}\|Ax_n-SAx_n\| &\le \left\|Ax_n-S\left((1-\eta_n)I+\eta_nS\right)Ax_n\right\|\\ &\quad+\left\|S\left((1-\eta_n)I+\eta_nS\right)Ax_n-SAx_n\right\|\\ &\le \left\|Ax_n-S\left((1-\eta_n)I+\eta_nS\right)Ax_n\right\|+L_1\eta_n\|Ax_n-SAx_n\|.\end{aligned}$$

It follows that

$$\|Ax_n-SAx_n\|\le\frac{1}{1-L_1\eta_n}\left\|Ax_n-S\left((1-\eta_n)I+\eta_nS\right)Ax_n\right\|.$$

Hence,

$$\lim_{n\to\infty}\|Ax_n-SAx_n\|=0.\tag{3.22}$$

Note that

$$\begin{aligned}\|u_n - x_n\| &= \big\|\delta A^*\big[(1-\zeta_n)I + \zeta_n S\big((1-\eta_n)I + \eta_n S\big) - I\big]Ax_n \\ &\quad + \alpha_n\big(Bx_n + \delta BA^*\big((1-\zeta_n)I + \zeta_n S\big((1-\eta_n)I + \eta_n S\big) - I\big)Ax_n - f(x_n)\big)\big\| \\ &\le \delta\|A\|\big\|\big[(1-\zeta_n)I + \zeta_n S\big((1-\eta_n)I + \eta_n S\big) - I\big]Ax_n\big\| \\ &\quad + \alpha_n\big\|Bx_n + \delta BA^*\big[(1-\zeta_n)I + \zeta_n S\big((1-\eta_n)I + \eta_n S\big) - I\big]Ax_n - f(x_n)\big\|.\end{aligned}$$

This together with (3.21) implies that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.23}$$

From (3.10) and (3.20), we deduce

$$\begin{aligned}\big\|x_{n+1} - x^*\big\|^2 &\le \big\|u_n - x^*\big\|^2 - \beta_n(\gamma_n - \beta_n)\big\|u_n - T\big((1-\gamma_n)u_n + \gamma_n Tu_n\big)\big\|^2 \\ &\le \big\|x_n - x^*\big\|^2 + \alpha_n M - \beta_n(\gamma_n - \beta_n)\big\|u_n - T\big((1-\gamma_n)u_n + \gamma_n Tu_n\big)\big\|^2.\end{aligned}$$

It follows that

$$\beta_n(\gamma_n - \beta_n)\big\|u_n - T\big((1-\gamma_n)u_n + \gamma_n Tu_n\big)\big\|^2 \le \big\|x_n - x^*\big\|^2 - \big\|x_{n+1} - x^*\big\|^2 + \alpha_n M.$$

Therefore,

$$\lim_{n\to\infty}\big\|u_n - T\big((1-\gamma_n)u_n + \gamma_n Tu_n\big)\big\| = 0. \tag{3.24}$$

Observe that

$$\begin{aligned}\|u_n - Tu_n\| &\le \big\|u_n - T\big((1-\gamma_n)u_n + \gamma_n Tu_n\big)\big\| + \big\|T\big((1-\gamma_n)u_n + \gamma_n Tu_n\big) - Tu_n\big\| \\ &\le \big\|u_n - T\big((1-\gamma_n)u_n + \gamma_n Tu_n\big)\big\| + L\gamma_n\|u_n - Tu_n\|.\end{aligned}$$

Thus,

$$\|u_n - Tu_n\| \le \frac{1}{1 - L\gamma_n}\big\|u_n - T\big((1-\gamma_n)u_n + \gamma_n Tu_n\big)\big\|.$$

This together with (3.24) implies that

$$\lim_{n\to\infty}\|u_n - Tu_n\| = 0. \tag{3.25}$$

Now, we show that

$$\limsup_{n\to\infty}\big\langle f\big(x^*\big) - Bx^*, u_n - x^*\big\rangle \le 0.$$

Choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\big\langle f\big(x^*\big) - Bx^*, u_n - x^*\big\rangle = \lim_{i\to\infty}\big\langle f\big(x^*\big) - Bx^*, u_{n_i} - x^*\big\rangle. \tag{3.26}$$

Since the sequence $\{u_{n_i}\}$ is bounded, we can choose a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ such that $u_{n_{i_j}} \rightharpoonup z$. For the sake of convenience, we assume (without loss of generality) that $u_{n_i} \rightharpoonup z$. Consequently, we derive from the above conclusions that

$$x_{n_i} \rightharpoonup z \quad \text{and} \quad Ax_{n_i} \rightharpoonup Az. \tag{3.27}$$

By the demi-closedness of $T - I$ and $S - I$, we deduce $Az \in \operatorname{Fix}(S)$ and $z \in \operatorname{Fix}(T)$. That is to say, $z \in \Gamma$.

Therefore,

$$\begin{aligned} \limsup_{n\to\infty} \langle f(x^*) - Bx^*, u_n - x^*\rangle &= \lim_{i\to\infty} \langle f(x^*) - Bx^*, u_{n_i} - x^*\rangle \\ &= \lim_{i\to\infty} \langle f(x^*) - Bx^*, z - x^*\rangle \\ &\leq 0. \end{aligned} \tag{3.28}$$

Using (2.4), we have

$$\begin{aligned} \|u_n - x^*\|^2 &= \|(I - \alpha_n B)(v_n - x^*) + \alpha_n (f(x_n) - Bx^*)\|^2 \\ &\leq (1 - \alpha_n \xi)\|v_n - x^*\|^2 + 2\alpha_n \langle f(x_n) - Bx^*, u_n - x^*\rangle \\ &\leq (1 - \alpha_n \xi)\|x_n - x^*\|^2 + 2\alpha_n \langle f(x_n) - Bx^*, u_n - x^*\rangle \\ &= (1 - \alpha_n \xi)\|x_n - x^*\|^2 + 2\alpha_n \langle f(x_n) - f(x^*), u_n - x^*\rangle \\ &\quad + 2\alpha_n \langle f(x^*) - Bx^*, u_n - x^*\rangle \\ &= (1 - \alpha_n \xi)\|x_n - x^*\|^2 + 2\alpha_n \rho \|x_n - x^*\| \|u_n - x^*\| \\ &\quad + 2\alpha_n \langle f(x^*) - Bx^*, u_n - x^*\rangle \\ &\leq (1 - \alpha_n \xi)\|x_n - x^*\|^2 + \alpha_n \rho \|x_n - x^*\|^2 + \alpha_n \rho \|u_n - x^*\|^2 \\ &\quad + 2\alpha_n \langle f(x^*) - Bx^*, u_n - x^*\rangle. \end{aligned} \tag{3.29}$$

Therefore,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \|u_n - x^*\|^2 \\ &\leq \left[1 - \frac{(\xi - 2\rho)\alpha_n}{1 - \alpha_n \rho}\right] \|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - \alpha_n \rho} \langle f(x^*) - Bx^*, u_n - x^*\rangle. \end{aligned} \tag{3.30}$$

Applying Lemma 2.8 and (3.28) to (3.30), we deduce $x_n \to x^*$.

In Case 2 above, we know that, for any integer n_0, there exists another integer $p \geq n_0$ such that $\|x_p - x^*\| \leq \|x_{p+1} - x^*\|$. Let n_0 be such that $\|x_{n_0} - x^*\| \leq \|x_{n_0+1} - x^*\|$. Set $\omega_n = \{\|x_n - x^*\|\}$. Then we have

$$\omega_{n_0} \leq \omega_{n_0+1}.$$

Define an integer sequence $\{\tau_n\}$ for all $n \geq n_0$ as follows:

$$\tau(n) = \max\{l \in \mathbb{N} \mid n_0 \leq l \leq n, \omega_l \leq \omega_{l+1}\}.$$

It is clear that $\tau(n)$ is a non-decreasing sequence satisfying

$$\lim_{n\to\infty} \tau(n) = \infty$$

and

$$\omega_{\tau(n)} \leq \omega_{\tau(n)+1},$$

for all $n \geq n_0$.

By a similar argument to that of Case 1, we can obtain

$$\lim_{n\to\infty} \|SAx_{\tau(n)} - Ax_{\tau(n)}\| = 0$$

and

$$\lim_{n\to\infty} \|u_{\tau(n)} - Tu_{\tau(n)}\| = 0.$$

This implies that

$$\omega_w(u_{\tau(n)}) \subset \Gamma.$$

Thus, we obtain

$$\limsup_{n\to\infty} \langle f(x^*) - Bx^*, u_{\tau(n)} - x^* \rangle \leq 0. \tag{3.31}$$

Since $\omega_{\tau(n)} \leq \omega_{\tau(n)+1}$, we have from (3.30)

$$\begin{aligned} \omega^2_{\tau(n)} &\leq \omega^2_{\tau(n)+1} \\ &\leq \left[1 - \frac{(\xi - 2\rho)\alpha_{\tau(n)}}{1 - \alpha_{\tau(n)}\rho}\right]\omega^2_{\tau(n)} + \frac{2\alpha_{\tau(n)}}{1 - \alpha_{\tau(n)}\rho}\langle f(x^*) - Bx^*, u_{\tau(n)} - x^* \rangle. \end{aligned} \tag{3.32}$$

It follows that

$$\omega^2_{\tau(n)} \leq \frac{2}{\xi - 2\rho}\langle f(x^*) - Bx^*, u_{\tau(n)} - x^* \rangle. \tag{3.33}$$

Combining (3.31) and (3.33), we have

$$\limsup_{n\to\infty} \omega_{\tau(n)} \leq 0,$$

and hence

$$\lim_{n\to\infty} \omega_{\tau(n)} = 0. \tag{3.34}$$

From (3.32), we deduce

$$\limsup_{n\to\infty} \omega^2_{\tau(n)+1} \leq \limsup_{n\to\infty} \omega^2_{\tau(n)}.$$

This together with (3.34) implies that

$$\lim_{n\to\infty} \omega_{\tau(n)+1} = 0.$$

Applying Lemma 2.9 to get

$$0 \le \omega_n \le \max\{\omega_{\tau(n)}, \omega_{\tau(n)+1}\}.$$

Therefore, $\omega_n \to 0$. That is, $x_n \to x^*$. This completes the proof. □

From Algorithm 3.4 and Theorem 3.5, we can deduce easily the following algorithms and corollaries.

Algorithm 3.6 **Initialization:** Let $x_0 \in H_1$ be arbitrary.
Iterative step: For $n \ge 0$ let

$$\begin{cases} v_n = x_n + \delta A^*[(1-\zeta_n)I + \zeta_n S((1-\eta_n)I + \eta_n S) - I]Ax_n, \\ u_n = \alpha_n f(x_n) + (1-\alpha_n)v_n, \\ x_{n+1} = (1-\beta_n)u_n + \beta_n T((1-\gamma_n)u_n + \gamma_n Tu_n), \quad n \in \mathbb{N}, \end{cases} \tag{3.35}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$, $\{\gamma_n\}_{n\in\mathbb{N}}$, $\{\zeta_n\}_{n\in\mathbb{N}}$, and $\{\eta_n\}_{n\in\mathbb{N}}$ are five real number sequences in $(0,1)$ and δ is a constant in $(0, \frac{1}{\|A\|^2})$.

Corollary 3.7 *Suppose $T - I$ and $S - I$ are demiclosed at* 0. *Assume that the following conditions are satisfied*:
(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < a_1 < \beta_n < c_1 < \gamma_n < b_1 < \frac{1}{\sqrt{1+L_2^2}+1}$;
(C4) $0 < a_2 < \zeta_n < c_2 < \eta_n < b_2 < \frac{1}{\sqrt{1+L_1^2}+1}$.
Then the sequence $\{x_n\}$ generated by algorithm (3.35) *converges strongly to $x^* = P_\Gamma(f)x^*$.*

Algorithm 3.8 **Initialization:** Let $x_0 \in H_1$ be arbitrary.
Iterative step: For $n \ge 0$ let

$$\begin{cases} v_n = x_n + \delta A^*[(1-\zeta_n)I + \zeta_n S((1-\eta_n)I + \eta_n S) - I]Ax_n, \\ x_{n+1} = (1-\beta_n)(1-\alpha_n)v_n + \beta_n T((1-\gamma_n)(1-\alpha_n)v_n + \gamma_n T(1-\alpha_n)v_n), \quad n \in \mathbb{N}, \end{cases} \tag{3.36}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$, $\{\gamma_n\}_{n\in\mathbb{N}}$, $\{\zeta_n\}_{n\in\mathbb{N}}$, and $\{\eta_n\}_{n\in\mathbb{N}}$ are five real number sequences in $(0,1)$ and δ is a constant in $(0, \frac{1}{\|A\|^2})$.

Corollary 3.9 *Suppose $T - I$ and $S - I$ are demiclosed at* 0. *Assume that the following conditions are satisfied*:
(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < a_1 < \beta_n < c_1 < \gamma_n < b_1 < \frac{1}{\sqrt{1+L_2^2}+1}$;

(C4) $0 < a_2 < \zeta_n < c_2 < \eta_n < b_2 < \frac{1}{\sqrt{1+L_1^2}+1}$.

Then the sequence $\{x_n\}$ *generated by algorithm* (3.36) *converges strongly to* $x^* = P_\Gamma(0)x^*$, *which is the minimum norm element in* Γ.

Remark 3.10 From Remark 2.3, we know that if S and T are quasi-nonexpansive operators or directed operators or demicontractive operators, the above corollaries are still valid.

Note that the pseudo-contractive operator satisfies the following demi-closedness principle.

Lemma 3.11 ([30]) *Let H be a real Hilbert space, C a closed convex subset of H. Let $U : C \to C$ be a continuous pseudo-contractive operator. Then*

(i) $\mathrm{Fix}(U)$ *is a closed convex subset of* C,

(ii) $(I - U)$ *is demiclosed at zero.*

Corollary 3.12 *Suppose that $S : H_2 \to H_2$ is an L_1-Lipschitzian pseudo-contractive operator with $L_1 > 1$ and $T : H_1 \to H_1$ is an L_2-Lipschitzian pseudo-contractive operator with $L_2 > 1$. Assume the following conditions are satisfied*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $0 < a_1 < \beta_n < c_1 < \gamma_n < b_1 < \frac{1}{\sqrt{1+L_2^2}+1}$;

(C4) $0 < a_2 < \zeta_n < c_2 < \eta_n < b_2 < \frac{1}{\sqrt{1+L_1^2}+1}$.

Then the sequence $\{x_n\}$ *generated by algorithm* (3.9) *converges strongly to* $x^* = P_\Gamma(f + I - B)x^*$.

Remark 3.13 Our algorithms and results provide a unified framework for the study of the two-set split common fixed point problem.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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