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# Menger-Hausdorff metric and common fixed point theorems in Menger probabilistic *G*-metric spaces

Qiang Tu, Chuanxi Zhu* and Zhaoqi Wu

*Correspondence:
chuanxizhu@126.com
Department of Mathematics,
Nanchang University, Nanchang,
330031, P.R. China

**Abstract**
In this paper, we introduce the concepts of generalized probabilistically bounded set $\Omega^*$ and Menger-Hausdorff metric $\widetilde{G}^*$ in Menger probabilistic *G*-metric spaces, and prove that $(\Omega^*, \widetilde{G}^*, \Delta)$ is also a Menger probabilistic *G*-metric space. Utilizing these concepts, we establish some common fixed point theorems for three hybrid pairs of mappings satisfying the common property (*E.A*) in Menger probabilistic *G*-metric spaces. Finally, an example is given to exemplify the theorems.

**MSC:** Primary 47H10; secondary 46S10

**Keywords:** Menger-Hausdorff metric; Menger probabilistic *G*-metric space; common fixed point; common property (*E.A*)

## 1 Introduction and preliminaries

As a generalization of a metric space, the concept of a probabilistic metric space has been introduced by Menger [1, 2]. Fixed point theory in a probabilistic metric space is an important branch of probabilistic analysis, and many results on the existence of fixed points or solutions of nonlinear equations in Menger *PM*-spaces have been studied by many scholars (see *e.g.* [3, 4]). Egbert [5] defined the notion of the distance between two sets in a Menger *PM*-space, *i.e.*, the so-called Menger-Hausdorff metric. In 2006, Mustafa and Sims [6] introduced the concept of a generalized metric space, and many fixed point results have been obtained by many authors (see *e.g.* [7–12]). On the other hand, Kaewcharoen and Kaewkhao [13] introduced the concept of a Hausdorff *G*-distance in a *G*-metric space. Moreover, Zhou *et al.* [14] defined the notion of a generalized probabilistic metric space or a *PGM*-space as a generalization of a *PM*-space and a *G*-metric space. After that, Zhu *et al.* [15] obtained some fixed point theorems in generalized probabilistic metric spaces. However, the concept of a Menger-Hausdorff $G^*$-metric in a *PGM*-space has not been introduced and studied yet.

To fill this gap, we introduce the concept of a generalized probabilistically bounded set and a Menger-Hausdorff $G^*$-metric in Menger probabilistic *G*-metric spaces, and we prove that $(\Omega^*, \widetilde{G}^*, \Delta)$ is also a Menger probabilistic *G*-metric space. Based on these, we obtain some useful results. As an application, we establish some common fixed point theorems for three hybrid pairs of mappings satisfying the common property (*E.A*) in Menger probabilistic *G*-metric spaces. Finally, an example is given to illustrate the theorems.

Throughout this paper, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, and $\mathbb{Z}^+$ be the set of all positive integers.

A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is nondecreasing left-continuous with $\sup_{t\in\mathbb{R}} F(t) = 1$ and $\inf_{t\in\mathbb{R}} F(t) = 0$.

We shall denote by $\mathscr{D}$ the set of all distribution functions while $H$ will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \le 0, \\ 1, & t > 0. \end{cases}$$

A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (for short, a $t$-norm) if the following conditions are satisfied:

(1) $\Delta(a,1) = a$;

(2) $\Delta(a,b) = \Delta(b,a)$;

(3) $a \ge b, c \ge d \Rightarrow \Delta(a,c) \ge \Delta(b,d)$;

(4) $\Delta(a,\Delta(b,c)) = \Delta(\Delta(a,b),c)$.

A typical example of $t$-norm is $\Delta_m$, where $\Delta_m(a,b) = \min\{a,b\}$, for each $a,b \in [0,1]$.

**Remark 1.1** From (4), it is not difficult to find that

$$\begin{aligned} \Delta\bigl(\Delta(a,b),\Delta(c,d)\bigr) &= \Delta\bigl(\Delta\bigl(\Delta(a,b),c\bigr),d\bigr) = \Delta\bigl(\Delta\bigl(\Delta(a,c),b\bigr),d\bigr) \\ &= \Delta\bigl(\Delta(a,c),\Delta(b,d)\bigr) = \cdots. \end{aligned}$$

**Definition 1.1** [16] A triplet $(X, \mathscr{F}, \Delta)$ is called a Menger probabilistic metric space (for short, a Menger $PM$-space) if $X$ is a nonempty set, $\Delta$ is a $t$-norm and $\mathscr{F}$ is a mapping from $X \times X$ into $\mathscr{D}$ satisfying the following conditions (we denote $\mathscr{F}(x,y)$ by $F_{x,y}$):

(MS-1) $F_{x,y}(t) = H(t)$ for all $t \in R$ if and only if $x = y$;

(MS-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in R$;

(MS-3) $F_{x,y}(t+s) \ge \Delta(F_{x,z}(t), F_{z,y}(s))$ for all $x,y,z \in X$ and $t,s \ge 0$.

Let $(X, \mathscr{F}, \Delta)$ be a $PM$-space and $A$ be a nonempty subset of $X$. Then the function

$$D_A(t) = \sup_{s<t} \inf_{x,y\in A} F_{x,y}(s), \quad t \in \mathbb{R}$$

is called the probabilistic diameter of $A$. If $\sup_{t>0} D_A(t) = 1$, then $A$ is said to be probabilistically bounded.

Let $(X, \mathscr{F}, \Delta)$ be a Menger $PM$-space and $\Omega$ be the family of all nonempty probabilistically bounded $\mathscr{T}$-closed subsets of $X$. For any $A, B \in \Omega$, define the distribution functions as follows:

$$\tilde{\mathscr{F}}(A,B)(t) = \tilde{F}_{A,B}(t) = \sup_{s<t} \Delta\Bigl(\inf_{x\in A}\sup_{y\in B} F_{x,y}(s), \inf_{y\in B}\sup_{x\in A} F_{x,y}(s)\Bigr), \quad s,t \in \mathbb{R},$$

$$\mathscr{F}(x,A)(t) = F_{x,A}(t) = \sup_{s<t}\sup_{y\in A} F_{x,y}(s), \quad s,t \in \mathbb{R},$$

where $\tilde{\mathscr{F}}$ is called the Menger-Hausdorff metric induced by $\mathscr{F}$.

**Lemma 1.1** [16] *Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space. Then for any $A, B, C \in \Omega$ and any $x, y \in X$, we have the following*:

(i) $\tilde{F}_{A,B}(t) = 1$ *if and only if* $A = B$;

(ii) $F_{x,A}(t) = 1$ *if and only if* $x \in A$;

(iii) *for any* $x \in A$, $F_{x,B}(t) \geq \tilde{F}_{A,B}(t)$, *for all* $t \geq 0$;

(iv) $F_{x,A}(t_1 + t_2) \geq \Delta(F_{x,y}(t_1), F_{y,A}(t_2))$, *for all* $t_1, t_2 \geq 0$;

(v) $F_{x,A}(t_1 + t_2) \geq \Delta(F_{x,B}(t_1), F_{A,B}(t_2))$, *for all* $t_1, t_2 \geq 0$;

(vi) $\tilde{F}_{A,C}(t_1 + t_2) \geq \Delta(\tilde{F}_{A,B}(t_1), \tilde{F}_{B,C}(t_2))$, *for all* $t_1, t_2 \geq 0$.

**Definition 1.2** [14] A Menger probabilistic $G$-metric space (for brevity, a *PGM*-space) is a triple $(X, G^*, \Delta)$, where $X$ is a nonempty set, $\Delta$ is a continuous $t$-norm and $G^*$ is a mapping from $X \times X \times X$ into $\mathscr{D}$ ($G^*_{x,y,z}$ denote the value of $G^*$ at the point $(x, y, z)$) satisfying the following conditions:

(PGM-1) $G^*_{x,y,z}(t) = 1$ for all $x, y, z \in X$ and $t > 0$ if and only if $x = y = z$;

(PGM-2) $G^*_{x,x,y}(t) \geq G^*_{x,y,z}(t)$ for all $x, y, z \in X$ with $z \neq y$ and $t > 0$;

(PGM-3) $G^*_{x,y,z}(t) = G^*_{x,z,y}(t) = G^*_{y,x,z}(t) = \cdots$ (symmetry in all three variables);

(PGM-4) $G^*_{x,y,z}(t + s) \geq \Delta(G^*_{x,a,a}(s), G^*_{a,y,z}(t))$ for all $x, y, z, a \in X$ and $s, t \geq 0$.

**Definition 1.3** [14] Let $(X, G^*, \Delta)$ be a Menger *PGM*-space and $x_0$ be any point in $X$. For any $\epsilon > 0$ and $\delta$ with $0 < \delta < 1$, and $(\epsilon, \delta)$-neighborhood of $x_0$ is the set of all points $y$ in $X$ for which $G^*_{x_0,y,y}(\epsilon) > 1 - \delta$ and $G^*_{y,x_0,x_0}(\epsilon) > 1 - \delta$. We write

$$N_{x_0}(\epsilon, \delta) = \{y \in X : G^*_{x_0,y,y}(\epsilon) > 1 - \delta, G^*_{y,x_0,x_0}(\epsilon) > 1 - \delta\},$$

which means that $N_{x_0}(\epsilon, \delta)$ is the set of all points $y$ in $X$ for which the probability of the distance from $x_0$ to $y$ being less than $\epsilon$ is greater than $1 - \delta$.

**Lemma 1.2** [14] *Let $(X, G^*, \Delta)$ be a Menger PGM-space. Then $(X, G^*, \Delta)$ is a Hausdorff space in the topology introduced by the family $\{N_{x_0}(\epsilon, \delta)\}$ of $(\epsilon, \delta)$-neighborhoods.*

**Definition 1.4** [14] Let $(X, G^*, \Delta)$ be a *PGM*-space, and $\{x_n\}$ is a sequence in $X$.

(1) $\{x_n\}$ is said to be convergent to a point $x \in X$ (write $x_n \to x$), if for any $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\epsilon,\delta}$ such that $x_n \in N_{x_0}(\epsilon, \delta)$ whenever $n > M_{\epsilon,\delta}$;

(2) $\{x_n\}$ is called a *Cauchy* sequence, if for any $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\epsilon,\delta}$ such that $G^*_{x_n,x_m,x_l}(\epsilon) > 1 - \delta$ whenever $n, m, l > M_{\epsilon,\delta}$;

(3) $(X, G^*, \Delta)$ is said to be complete, if every *Cauchy* sequence in $X$ converges to a point in $X$.

We can analogously prove the following lemma as in Menger *PM*-spaces.

**Lemma 1.3** *Let $(X, G^*, \Delta)$ be a Menger PGM-space with $\Delta$ a continuous t-norm, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be sequences in $X$ and $x, y, z \in X$, if $\{x_n\} \to x$, $\{y_n\} \to y$ and $\{z_n\} \to z$ as $n \to \infty$. Then*

(1) $\liminf_{n\to\infty} G^*_{x_n,y_n,z_n}(t) \geq G^*_{x,y,z}(t)$ *for all* $t > 0$;

(2) $G^*_{x,y,z}(t + 0) \geq \limsup_{n\to\infty} G^*_{x_n,y_n,z_n}(t)$ *for all* $t > 0$.

Particularly, if $t_0$ is a continuous point of $G_{x,y,z}(\cdot)$, then $\lim_{n\to\infty} G_{x_n,y_n,z_n}(t_0) = G_{x,y,z}(t_0)$.

**Definition 1.5** [17] A pair of self-mappings $S$ and $T$ on $X$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence point, *i.e.*, if $Tu = Su$ for some $u \in X$ implies that $TSu = STu$.

**Definition 1.6** [18] Let $F_1, F_2 \in \mathscr{D}$. The algebraic sum $F_1 \oplus F_2$ of $F_1$ and $F_2$ is defined by

$$(F_1 \oplus F_2)(t) = \sup_{t_1+t_2=t} \min\{F_1(t_1), F_2(t_2)\}$$

for all $t \in \mathbb{R}$.

As a generalization, we give the following definition.

**Definition 1.7** Let $F_1, F_2, F_3 \in \mathscr{D}$. The algebraic sum $F_1 \oplus F_2 \oplus F_3$ of $F_1$, $F_2$, and $F_3$ is defined by

$$(F_1 \oplus F_2 \oplus F_3)(t) = \sup_{t_1+t_2+t_3=t} \min\{F_1(t_1), F_2(t_2), F_3(t_3)\}$$

for all $t \in \mathbb{R}$.

**Remark 1.2** Let $F_3(t) = H(t)$. Then Definition 1.6 and Definition 1.7 are equivalent.

For two functions $f$ and $g$, $f > g$ means that $f(t) \geq g(t)$ and there exists some $t_0$ such that $f(t_0) > g(t_0)$.

**Definition 1.8** [19] Let $f$ and $g$ be self-mappings of a set $X$. If $w = fx = gx$ for some $x$ in $X$, then $x$ is called a coincidence point of $f$ and $g$, and $w$ is called point of coincidence of $f$ and $g$.

In the sequel, we will denote by $C(f, F)$ the set of all coincidence points of $f$ and $F$.

We recall the definitions of property $(E.A)$ for a hybrid pair of mappings and common property $(E.A)$ for two hybrid pairs of mappings in Menger $PM$-spaces.

**Definition 1.9** [20] Let $(X, \mathscr{F}, \Delta)$ be a Menger $PM$-space, $(\Omega, \tilde{\mathscr{F}}, \Delta)$ be the induced Menger $PM$-space, $f : X \to X$ be a self-mapping and $F : X \to \Omega$ be a multivalued mapping. A pair of mappings $(f, F)$ is said to satisfy the property $(E.A)$, if there exist a sequence $\{x_n\}$ in $X$, some $a \in X$, and $A \in \Omega$, such that $\lim_{n\to\infty} fx_n = a \in A = \lim_{n\to\infty} Fx_n$.

**Definition 1.10** [20] Let $(X, \mathscr{F}, \Delta)$ be a Menger $PM$-space and $(\Omega, \tilde{\mathscr{F}}, \Delta)$ be the induced Menger $PM$-space, $f, g : X \to X$, and $F, G : X \to \Omega$. Two pairs of mappings $(f, F)$ and $(g, G)$ are said to satisfy the common property $(E.A)$ if there exist two sequences $\{x_n\}$, $\{y_n\}$ in $X$, some $u \in X$ and $A, B \in \Omega$, such that

$$\lim_{n\to\infty} Fx_n = A, \qquad \lim_{n\to\infty} Gy_n = B, \qquad \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = u \in A \cap B.$$

## 2 Menger-Hausdorff metric in Menger *PGM*-spaces

In this section, we first introduce some new concepts in Menger $PGM$-spaces, and then establish some useful results in Menger $PGM$-spaces.

**Definition 2.1** Let $A$ be a nonempty subset of $X$. The function $D_A^*$ defined by

$$D_A^*(t) = \sup_{s<t} \inf_{p,q,r\in A} G_{p,q,r}^*(s)$$

is called the generalized probabilistic diameter of $A$.

**Definition 2.2** A nonempty subset $A$ of $X$ is said to be

(1) generalized probabilistically bounded, if $\sup_{t>0} D_A^*(t) = 1$;
(2) generalized probabilistically semi-bounded, if $0 < \sup_{t>0} D_A^*(t) < 1$;
(3) generalized probabilistically unbounded, if $\sup_{t>0} D_A^*(t) = 0$.

**Lemma 2.1** *If $A$ and $B$ are two nonempty subsets of $X$, then*

$$D_{A\cup B}^*(x+y) \geq \Delta\bigl(D_A^*(x), D_B^*(y)\bigr). \tag{2.1}$$

*Proof* Let $x$, $y$ be given, for (2.1), we first prove that

$$\inf_{p,q,r\in A\cup B} G_{p,q,r}^*(x+y) \geq \Delta\Bigl(\inf_{p,q,r\in A} G_{p,q,r}^*(x), \inf_{p,q,r\in B} G_{p,q,r}^*(y)\Bigr). \tag{2.2}$$

Case (1):

$$\inf_{p,q,r\in A\cup B} G_{p,q,r}^*(x+y) = \inf_{\substack{p\in A\\ q,r\in B}} G_{p,q,r}^*(x+y). \tag{2.3}$$

For any $p, q, r \in X$, we have

$$G_{p,q,r}^*(x+y) \geq \Delta\bigl(G_{p,a,a}^*(x), G_{a,q,r}^*(y)\bigr).$$

Taking the infimum on both sides of this inequality as $p$ ranges over $A$, $a$ ranges over $A\cap B$, and $r$, $q$ range over $B$, and using (2.3), we have

$$\begin{aligned}
\inf_{p,q,r\in A\cup B} G_{p,q,r}^*(x+y) &= \inf_{\substack{p\in A\\ q,r\in B}} G_{p,q,r}^*(x+y) \geq \inf_{\substack{p\in A,q,r\in B\\ a\in A\cap B}} \Delta\bigl(G_{p,a,a}^*(x), G_{a,q,r}^*(y)\bigr) \\
&\geq \Delta\Bigl(\inf_{\substack{p\in A\\ a\in A\cap B}} G_{p,a,a}^*(x), \inf_{\substack{q,r\in B\\ a\in A\cap B}} G_{a,q,r}^*(y)\Bigr) \\
&\geq \Delta\Bigl(\inf_{\substack{p\in A\\ a\in A}} G_{p,a,a}^*(x), \inf_{\substack{q,r\in B\\ a\in B}} G_{a,q,r}^*(y)\Bigr) \\
&\geq \Delta\Bigl(\inf_{p,q,r\in A} G_{p,q,r}^*(x), \inf_{p,q,r\in B} G_{p,q,r}^*(y)\Bigr).
\end{aligned}$$

So, (2.2) is proved.

Case (2): $\inf_{p,q,r\in A\cup B} G_{p,q,r}^*(x+y) < \inf_{\substack{p\in A,\\ q,r\in B}} G_{p,q,r}^*(x+y)$. Then one of the following equalities:

(a) $\inf_{p,q,r\in A\cup B} G^*_{p,q,r}(x+y) = \inf_{p,q,r\in A} G^*_{p,q,r}(x+y)$,
(b) $\inf_{p,q,r\in A\cup B} G^*_{p,q,r}(x+y) = \inf_{p,q,r\in B} G^*_{p,q,r}(x+y)$
and
(c) $\inf_{p,q,r\in A\cup B} G^*_{p,q,r}(x+y) = \inf_{\substack{p\in B \\ q,r\in A}} G^*_{p,q,r}(x+y)$
holds.

If (a) holds, we have

$$\begin{aligned}\inf_{p,q,r\in A\cup B} G^*_{p,q,r}(x+y) &= \inf_{p,q,r\in A} G^*_{p,q,r}(x+y) \geq \inf_{p,q,r\in A} G^*_{p,q,r}(x) \\ &\geq \Delta\Bigl(\inf_{p,q,r\in A} G^*_{p,q,r}(x), 1\Bigr) \geq \Delta\Bigl(\inf_{p,q,r\in A} G^*_{p,q,r}(x), \inf_{p,q,r\in B} G^*_{p,q,r}(y)\Bigr).\end{aligned}$$

Then (2.2) is proved.

Similarly, we can prove that (2.2) is satisfied if (b) or (c) holds.

Finally, by (2.2) and the continuity of $\Delta$, we have

$$\begin{aligned}D^*_{A\cup B}(x+y) &= \sup_{s+t<x+y} \inf_{p,q,r\in A\cup B} G^*_{p,q,r}(s+t) \geq \sup_{\substack{s<x \\ t<y}} \inf_{p,q,r\in A\cup B} G^*_{p,q,r}(s+t) \\ &\geq \Delta\Bigl(\sup_{s<x} \inf_{p,q,r\in A} G^*_{p,q,r}(s), \sup_{t<y} \inf_{p,q,r\in B} G^*_{p,q,r}(y)\Bigr) = \Delta\bigl(D^*_A(x), D^*_B(y)\bigr).\end{aligned}$$

This completes the proof. □

**Lemma 2.2** *Let* $(X, G^*, \Delta)$ *be a Menger PGM-space with a continuous t-norm.*

(1) *If* $A$ *is a generalized probabilistically bounded set, then* $D^*_A$ *is a distribution function.*
(2) *If* $A, B \subseteq X$ *are two generalized probabilistically bounded sets, then* $A \cup B$ *is also a generalized probabilistically bounded set.*

*Proof* (1) Since $A$ is a generalized probabilistically bounded set, by Definition 2.1, it is easy to see that $D^*_A(t)$ is nondecreasing in $t$, $D^*_A(0) = 0$, $\sup_{t>0} D^*_A(t) = 1$ and $D^*_A(t)$ is left-continuous in $t$. This shows that $D^*_A(t)$ is a distribution function.

(2) Since $A$ and $B$ are generalized probabilistically bounded sets, from Lemma 2.1 and the continuity of $\Delta$, we have $\sup_{t>0} D^*_{A\cup B}(t) \geq \Delta(\sup_{t>0} D^*_A(\frac{t}{2}), \sup_{t>0} D^*_B(\frac{t}{2})) = \Delta(1,1) = 1$. This completes the proof. □

**Remark 2.1** By Lemma 2.2(2), we claim that if $A$, $B$, $C$ are generalized probabilistically bounded sets, then $A \cup B \cup C$ is also a generalized probabilistically bounded set.

In the remainder of this paper, we always assume that $(X, G^*, \Delta)$ is a Menger *PGM*-space with a continuous $t$-norm $\Delta$ and $\Omega^*$ be the family of all nonempty $\mathscr{T}$-closed generalized probabilistically bounded sets.

**Definition 2.3** For $A, B, C \in \Omega^*$, define the mapping $\widetilde{G}^* : \Omega^* \times \Omega^* \times \Omega^* \to \mathscr{D}$ by

$$\widetilde{G}^*_{A,B,C}(t) = \min\bigl\{g_{A,B}(t), g_{B,C}(t), g_{C,A}(t)\bigr\},$$

where $g_{A,B}(t) = \sup_{s<t} \Delta(\inf_{x\in A} \sup_{y\in B} g_{x,y}(s), \inf_{y\in B} \sup_{x\in A} g_{x,y}(s))$, $g_{x,y}(s) = \Delta(G^*_{x,x,y}(s), G^*_{x,y,y}(s))$.

Then $\widetilde{G}^*$ is called the Menger-Hausdorff metric induced by $G^*$.

**Definition 2.4** Let $A, B, C \in \Omega^*$ and $x, y, z \in X$.

(1) The generalized probabilistic distance between two points $x$, $y$ and a set $C$ is the function $\widetilde{G}^*_{x,y,C}(t)$ defined by

$$\widetilde{G}^*_{x,y,C}(t) = \sup_{s<t} \sup_{z\in C} G^*_{x,y,z}(s), \quad t \geq 0.$$

(2) The generalized probabilistic distance between a point $x$ and two sets $B$, $C$ is the function $\widetilde{G}^*_{x,B,C}(t)$ defined by

$$\widetilde{G}^*_{x,B,C}(t) = \min\{g_{x,B}(t), g_{B,C}(t), g_{x,C}(t)\}, \quad t \geq 0,$$

where $g_{x,B}(t) = \sup_{s<t} \sup_{y\in B} g_{x,y}(s)$.

**Lemma 2.3** *For any $A, B, D \in \Omega^*$ and $a, b > 0$, we have*

$$g_{A,B}(a+b) \geq \Delta\bigl(g_{A,D}(a), g_{D,B}(b)\bigr).$$

*Proof* For any $x, y, z \in X$ and $s, t > 0$, we have

$$G^*_{x,y,y}(t+s) \geq \Delta\bigl(G^*_{x,z,z}(t), G^*_{z,y,y}(s)\bigr)$$

and

$$G^*_{x,x,y}(t+s) \geq \Delta\bigl(G^*_{x,x,z}(t), G^*_{z,z,y}(s)\bigr)$$

for all $z \in D$. Using the continuity and monotonicity of $\Delta$, we have the following inequalities:

$$\sup_{y\in B} G^*_{x,y,y}(t+s) \geq \Delta\Bigl(\sup_{z\in D} G^*_{x,z,z}(t), \inf_{z\in D} \sup_{y\in B} G^*_{z,y,y}(s)\Bigr)$$

and

$$\sup_{y\in B} G^*_{x,x,y}(t+s) \geq \Delta\Bigl(\sup_{z\in D} G^*_{x,x,z}(t), \inf_{z\in D} \sup_{y\in B} G^*_{z,z,y}(s)\Bigr).$$

Thus, we have

$$\inf_{x\in A} \sup_{y\in B} G^*_{x,y,y}(t+s) \geq \Delta\Bigl(\inf_{x\in A} \sup_{z\in D} G^*_{x,z,z}(t), \inf_{z\in D} \sup_{y\in B} G^*_{z,y,y}(s)\Bigr), \tag{2.4}$$

$$\inf_{x\in A} \sup_{y\in B} G^*_{x,x,y}(t+s) \geq \Delta\Bigl(\inf_{x\in A} \sup_{z\in D} G^*_{x,x,z}(t), \inf_{z\in D} \sup_{y\in B} G^*_{z,z,y}(s)\Bigr). \tag{2.5}$$

Similarly, we can get

$$\inf_{y\in B} \sup_{x\in A} G^*_{x,x,y}(t+s) \geq \Delta\Bigl(\inf_{z\in D} \sup_{x\in A} G^*_{x,x,z}(t), \inf_{y\in B} \sup_{z\in D} G^*_{z,z,y}(s)\Bigr), \tag{2.6}$$

$$\inf_{y\in B} \sup_{x\in A} G^*_{x,y,y}(t+s) \geq \Delta\Bigl(\inf_{z\in D} \sup_{x\in A} G^*_{x,z,z}(t), \inf_{y\in B} \sup_{z\in D} G^*_{z,y,y}(s)\Bigr). \tag{2.7}$$

Since $\Delta$ is associative, by combining (2.4), (2.5), (2.6), and (2.7), we obtain

$$\begin{aligned}
&\inf_{x\in A}\sup_{y\in B} g_{x,y}(t+s)\\
&\quad = \Delta\Big(\inf_{x\in A}\sup_{y\in B} G^*_{x,y,y}(t+s), \inf_{x\in A}\sup_{y\in B} G^*_{x,x,y}(t+s)\Big)\\
&\quad \geq \Delta\Big(\Delta\Big(\inf_{x\in A}\sup_{z\in D} G^*_{x,z,z}(t), \inf_{z\in D}\sup_{y\in B} G^*_{z,y,y}(s)\Big), \Delta\Big(\inf_{x\in A}\sup_{z\in D} G^*_{x,x,z}(t), \inf_{z\in D}\sup_{y\in B} G^*_{z,z,y}(s)\Big)\Big)\\
&\quad = \Delta\Big(\Delta\Big(\inf_{x\in A}\sup_{z\in D} G^*_{x,z,z}(t), \inf_{x\in A}\sup_{z\in D} G^*_{x,x,z}(t)\Big), \Delta\Big(\inf_{z\in D}\sup_{y\in B} G^*_{z,y,y}(s), \inf_{z\in D}\sup_{y\in B} G^*_{z,z,y}(s)\Big)\Big)\\
&\quad = \Delta\Big(\inf_{x\in A}\sup_{z\in D} g_{x,z}(t), \inf_{z\in D}\sup_{y\in B} g_{y,z}(s)\Big), \qquad (2.8)\\
&\inf_{y\in B}\sup_{x\in A} g_{x,y}(t+s)\\
&\quad = \Delta\Big(\inf_{y\in B}\sup_{x\in A} G^*_{x,y,y}(t+s), \inf_{y\in B}\sup_{x\in A} G^*_{x,x,y}(t+s)\Big)\\
&\quad \geq \Delta\Big(\Delta\Big(\inf_{z\in D}\sup_{x\in A} G^*_{x,z,z}(t), \inf_{y\in B}\sup_{z\in D} G^*_{z,y,y}(s)\Big), \Delta\Big(\inf_{z\in D}\sup_{x\in A} G^*_{x,x,z}(t), \inf_{y\in B}\sup_{z\in D} G^*_{z,z,y}(s)\Big)\Big)\\
&\quad = \Delta\Big(\Delta\Big(\inf_{z\in D}\sup_{x\in A} G^*_{x,z,z}(t), \inf_{z\in D}\sup_{x\in A} G^*_{x,x,z}(t)\Big), \Delta\Big(\inf_{y\in B}\sup_{z\in D} G^*_{z,y,y}(s), \inf_{y\in B}\sup_{z\in D} G^*_{z,z,y}(s)\Big)\Big)\\
&\quad = \Delta\Big(\inf_{z\in D}\sup_{x\in A} g_{x,z}(t), \inf_{y\in B}\sup_{z\in D} g_{y,z}(s)\Big). \qquad (2.9)
\end{aligned}$$

By (2.8) and (2.9), we have

$$\begin{aligned}
&g_{A,B}(a+b)\\
&\quad = \sup_{t+s<a+b} \Delta\Big(\inf_{x\in A}\sup_{y\in B} g_{x,y}(t+s), \inf_{y\in B}\sup_{x\in A} g_{x,y}(t+s)\Big)\\
&\quad \geq \sup_{t+s<a+b} \Delta\Big(\Delta\Big(\inf_{x\in A}\sup_{z\in D} g_{x,z}(t), \inf_{z\in D}\sup_{y\in B} g_{y,z}(s)\Big), \Delta\Big(\inf_{z\in D}\sup_{x\in A} g_{x,z}(t), \inf_{y\in B}\sup_{z\in D} g_{y,z}(s)\Big)\Big)\\
&\quad = \sup_{t+s<a+b} \Delta\Big(\Delta\Big(\inf_{x\in A}\sup_{z\in D} g_{x,z}(t), \inf_{z\in D}\sup_{x\in A} g_{x,z}(t)\Big), \Delta\Big(\inf_{z\in D}\sup_{y\in B} g_{y,z}(s), \inf_{y\in B}\sup_{z\in D} g_{y,z}(s)\Big)\Big)\\
&\quad \geq \Delta\Big(\sup_{t<a} \Delta\Big(\inf_{x\in A}\sup_{z\in D} g_{x,z}(t), \inf_{z\in D}\sup_{x\in A} g_{x,z}(t)\Big), \sup_{s<b} \Delta\Big(\inf_{z\in D}\sup_{y\in B} g_{y,z}(s), \inf_{y\in B}\sup_{z\in D} g_{y,z}(s)\Big)\Big)\\
&\quad = \Delta\big(g_{A,D}(a), g_{D,B}(b)\big).
\end{aligned}$$

This completes the proof. □

**Theorem 2.1** $(\Omega^*, \widetilde{G}^*, \Delta)$ *is a Menger PGM-space*.

*Proof* First, we prove that $\widetilde{G}^*$ is a distribution function. By the definition of $\widetilde{G}^*(t)$, it is easy to see that $\widetilde{G}^*(t)$ is nondecreasing and left-continuous in $t$ and $\widetilde{G}^*(0) = 0$. Now, we prove

$$\sup_{t>0} \widetilde{G}^*(t) = 1.$$

In fact, since $A, B, C \in \Omega^*$, we know $A \cup B \cup C \in \Omega^*$. By the continuity of $\Delta$, we have

$$\sup_{t>0} \widetilde{G}^*(t) = \sup_{t>0} \min\big\{g_{A,B}(t), g_{B,C}(t), g_{C,A}(t)\big\} = \min\Big\{\sup_{t>0} g_{A,B}(t), \sup_{t>0} g_{B,C}(t), \sup_{t>0} g_{C,A}(t)\Big\}$$

and

$$
\begin{aligned}
\sup_{t>0} g_{A,B}(t) &= \sup_{\substack{t>0\\ s<t}} \Delta\Big\{\Delta\Big[\inf_{x\in A}\sup_{y\in B} G^*_{x,y,y}(s), \inf_{x\in A}\sup_{y\in B} G^*_{x,x,y}(s)\Big],\\
&\quad \Delta\Big[\inf_{y\in B}\sup_{x\in A} G^*_{x,y,y}(s), \inf_{y\in B}\sup_{x\in A} G^*_{x,x,y}(s)\Big]\Big\}\\
&\geq \Delta\Big\{\sup_{\substack{t>0\\ s<t}} \Delta\Big[\inf_{x\in A}\inf_{y\in B} G^*_{x,y,y}(s), \inf_{x\in A}\inf_{y\in B} G^*_{x,x,y}(s)\Big],\\
&\quad \sup_{\substack{t>0\\ s<t}} \Delta\Big[\inf_{y\in B}\inf_{x\in A} G^*_{x,y,y}(s), \inf_{y\in B}\inf_{x\in A} G^*_{x,x,y}(s)\Big]\Big\}\\
&\geq \Delta\Big\{\Delta\Big[\sup_{\substack{t>0\\ s<t}} \inf_{x,y\in A\cup B} G^*_{x,y,y}(s), \sup_{\substack{t>0\\ s<t}} \inf_{x,y\in A\cup B} G^*_{x,x,y}(s)\Big],\\
&\quad \Delta\Big[\sup_{\substack{t>0\\ s<t}} \inf_{x,y\in A\cup B} G^*_{x,y,y}(s), \sup_{\substack{t>0\\ s<t}} \inf_{x,y\in A\cup B} G^*_{x,x,y}(s)\Big]\Big\}\\
&\geq \Delta\Big\{\Delta\Big[\sup_{\substack{t>0\\ s<t}} \inf_{x,y,z\in A\cup B} G^*_{x,y,z}(s), \sup_{\substack{t>0\\ s<t}} \inf_{x,y,z\in A\cup B} G^*_{x,y,z}(s)\Big],\\
&\quad \Delta\Big[\sup_{\substack{t>0\\ s<t}} \inf_{x,y,z\in A\cup B} G^*_{x,y,z}(s), \sup_{\substack{t>0\\ s<t}} \inf_{x,y,z\in A\cup B} G^*_{x,y,z}(s)\Big]\Big\}\\
&= \Delta\big\{\Delta\big[D^*_{A\cup B,}, D^*_{A\cup B}\big], \Delta\big[D^*_{A\cup B,}, D^*_{A\cup B}\big]\big\}\\
&= \Delta\big(\Delta(1,1), \Delta(1,1)\big)\\
&= 1.
\end{aligned}
$$

Similarly, we have $\sup_{t>0} g_{B,C}(t) = 1$ and $\sup_{t>0} g_{C,A}(t) = 1$. This shows that $\widetilde{G}^*$ is a mapping from $\Omega^* \times \Omega^* \times \Omega^*$ into $\mathscr{D}$.

Next, we will show that $\widetilde{G}^*(t)$ satisfies the following:

(1) $\widetilde{G}^*_{A,B,C}(t) = 1$ for all $t > 0$ if and only if $A = B$;

(2) $\widetilde{G}^*_{A,A,B}(t) \geq \widetilde{G}^*_{A,B,C}(t)$ for all $A, B, C \in \Omega^*$ with $B \neq C$ and $t > 0$;

(3) $\widetilde{G}^*_{A,B,C}(t) = \widetilde{G}^*_{A,C,B}(t) = \widetilde{G}^*_{B,A,C}(t) = \cdots$ (symmetry in all three variables);

(4) $\widetilde{G}^*_{A,B,C}(t + s) \geq \Delta(\widetilde{G}^*_{A,D,D}(t), \widetilde{G}^*_{D,B,C}(t))$ for all $A, B, C \in \Omega^*$ with $t > 0$.

• (i) If $\widetilde{G}^*_{A,B,C}(t) = 1$ for all $t > 0$, then for any $\epsilon > 0$, we have

$$g_{A,B}(\epsilon) = g_{B,C}(\epsilon) = g_{C,A}(\epsilon) = 1.$$

By $g_{A,B}(\epsilon) = 1$, we have

$$\sup_{s<\epsilon}\inf_{x\in A}\sup_{y\in B} \Delta\big(G^*_{x,x,y}(s), G^*_{x,y,y}(s)\big) = 1, \tag{2.10}$$

$$\sup_{s<\epsilon}\inf_{y\in B}\sup_{x\in A} \Delta\big(G^*_{x,x,y}(s), G^*_{x,y,y}(s)\big) = 1. \tag{2.11}$$

From (2.10), it follows that $\sup_{s<\epsilon}\sup_{y\in B} \Delta(G^*_{x,x,y}(s), G^*_{x,y,y}(s)) = 1$ for all $x \in A$. Therefore, for any $a \in A$ and $\lambda > 0$, there exists $b^* \in B$, such that

$$\Delta\big(G^*_{a,a,b^*}(\epsilon), G^*_{a,b^*,b^*}(\epsilon)\big) > 1 - \lambda.$$

So, we have

$$G^*_{a,a,b^*}(\epsilon) > 1 - \lambda \quad \text{and} \quad G^*_{a,b^*,b^*}(\epsilon) > 1 - \lambda.$$

This shows that the point $a$ is an accumulation point of $B$ and hence $a \in B$, *i.e.*, $A \subseteq B$.

From (2.11), we can prove that $B \subseteq A$. Therefore, we have $A = B$.

Similarly, we can also prove that $B = C$, $C = A$. So, we have $A = B = C$.

Conversely, if $A = B = C$, then for any $t > 0$, we have

$$\widetilde{G}^*_{A,B,C}(t) = \min\{g_{A,A}(t), g_{A,A}(t), g_{A,A}(t)\} = g_{A,A}(t).$$

For any $s \in (0,1)$,

$$\begin{aligned} g_{A,A}(t) &\geq \Delta\Big(\inf_{a\in A}\sup_{b\in A} g_{a,b}(s), \inf_{a\in A}\sup_{b\in A} g_{a,b}(s)\Big) \\ &= \Delta\Big(\inf_{a\in A}\sup_{b\in A}\Delta\big(G^*_{a,a,b}(s), G^*_{a,b,b}(s)\big), \inf_{a\in A}\sup_{b\in A}\Delta\big(G^*_{a,a,b}(s), G^*_{a,b,b}(s)\big)\Big) \\ &= \Delta(1,1) = 1. \end{aligned}$$

Therefore (1) is satisfied.

- (ii) $\widetilde{G}^*_{A,A,B}(t) = \min\{g_{A,A}(t), g_{A,B}(t), g_{A,B}(t)\} = g_{A,B}(t) \geq \min\{g_{A,B}(t), g_{B,C}(t), g_{C,A}(t)\} = \widetilde{G}^*_{A,B,C}(t)$. So, (2) is satisfied.
- (iii) It is obvious that (3) holds.
- (iv) From Definition 2.3, we have

$$\begin{aligned} &\widetilde{G}^*_{A,B,C}(t+s) = \min\{g_{A,B}(t+s), g_{B,C}(t+s), g_{C,A}(t+s)\}, \\ &\widetilde{G}^*_{A,D,D}(t) = \min\{g_{A,D}(t), g_{D,D}(t), g_{A,D}(t)\} = g_{A,D}(t), \\ &\widetilde{G}^*_{D,B,C}(s) = \min\{g_{D,B}(s), g_{B,C}(s), g_{C,D}(s)\}. \end{aligned}$$

We just need to show

$$\min\{g_{A,B}(t+s), g_{B,C}(t+s), g_{C,A}(t+s)\} \geq \Delta\big(g_{A,D}(t), \min\{g_{D,B}(s), g_{B,C}(s), g_{C,D}(s)\}\big).$$

In fact,

$$\begin{aligned} g_{B,C}(t+s) &\geq g_{B,C}(s) \geq \min\{g_{D,B}(s), g_{B,C}(s), g_{C,D}(s)\} \\ &\geq \Delta\big(g_{A,D}(t), \min\{g_{D,B}(s), g_{B,C}(s), g_{C,D}(s)\}\big), \\ g_{A,B}(t+s) &\geq \Delta\big(g_{A,D}(t), g_{D,B}(s)\big) \geq \Delta\big(g_{A,D}(t), \min\{g_{D,B}(s), g_{B,C}(s), g_{C,D}(s)\}\big), \\ g_{C,A}(t+s) &\geq \Delta\big(g_{A,D}(t), g_{D,C}(s)\big) \geq \Delta\big(g_{A,D}(t), \min\{g_{D,B}(s), g_{B,C}(s), g_{C,D}(s)\}\big). \end{aligned}$$

So, (4) is also satisfied. This completes the proof. □

**Remark 2.2** By the proof process of Lemma 2.3 and Theorem 2.1, we can also prove that $(X, g, \Delta)$ and $(\Omega^*, g, \Delta)$ are Menger *PM*-spaces. We call $(X, g, \Delta)$ the *PM*-space induced by $(X, G^*, \Delta)$, and $(\Omega^*, g, \Delta)$ is the *PM*-space induced by $(X, g, \Delta)$. So, the properties in Lemma 1.1 can be applied to $(X, g, \Delta)$ and $(\Omega^*, g, \Delta)$.

**Example 2.1** Let $(X,d)$ be a metric space and $x,y,z \in X$, $G^*_{x,y,z}(t) = \frac{t}{t+\max\{d(x,y),d(y,z),d(x,z)\}}$ for all $t \geq 0$, then $(X, G^*, \Delta_m)$ is a Menger $PGM$-space. In fact, $G^*_{x,y,z}(0) = 0$, $\sup_{t>0} G^*_{x,y,z}(t) = 1$, and $G^*_{x,y,z}(t)$ is nondecreasing and continuous in $t$, so $G^*_{x,y,z}(t)$ is a distribution function. Obviously, $G^*_{x,y,z}(t)$ satisfy (PGM-1), (PGM-2), and (PGM-3). Next, we will show that (PGM-4) is also satisfied. Since $d(x,y) \leq d(x,a) + d(a,y)$ and $d(x,z) \leq d(x,a) + d(a,z)$, we have

$$\begin{aligned} G^*_{x,y,z}(t+s) &= \frac{t+s}{t+s+\max\{\max\{d(x,y),d(y,z),d(x,z)\}\}} \\ &\geq \frac{t+s}{t+s+\max\{d(x,a)+d(a,y),d(y,z),d(x,a)+d(a,z)\}} \\ &\geq \frac{t+s}{t+s+d(x,a)+\max\{d(a,y),d(y,z),d(a,z)\}} \\ &\geq \min\left\{\frac{t}{t+d(x,a)}, \frac{s}{s+\max\{d(a,y),d(y,z),d(a,z)\}}\right\} \\ &= \min\left\{G^*_{x,a,a}(t), G^*_{a,y,z}(s)\right\}, \end{aligned}$$

which implies that (PGM-4) is satisfied. So $(X, G^*, \Delta_m)$ is a Menger $PGM$-space. Then

$$g_{x,y}(t) = \min\left\{G^*_{x,x,y}(t), G^*_{x,y,y}(t)\right\} = \frac{t}{t+d(x,y)}$$

and

$$\begin{aligned} g_{A,B}(t) &= \min\left\{\inf_{x\in A}\sup_{y\in B}\frac{t}{t+d(x,y)}, \inf_{y\in B}\sup_{x\in A}\frac{t}{t+d(x,y)}\right\} \\ &= \min\left\{\frac{t}{t+\inf_{x\in A}\sup_{y\in B} d(x,y)}, \frac{t}{t+\inf_{y\in B}\sup_{x\in A} d(x,y)}\right\} \\ &= \frac{t}{t+\max\{\inf_{x\in A}\sup_{y\in B} d(x,y), \inf_{y\in B}\sup_{x\in A} d(x,y)\}} = \frac{t}{t+\delta(A,B)}. \end{aligned}$$

Thus,

$$\begin{aligned} \widetilde{G}^*_{A,B,C}(t) &= \min\left\{g_{A,B}(t), g_{B,C}(t), g_{A,C}(t)\right\} \\ &= \min\left\{\frac{t}{t+\delta(A,B)}, \frac{t}{t+\delta(B,C)}, \frac{t}{t+\delta(A,C)}\right\} \\ &= \frac{t}{t+\max\{\delta(A,B),\delta(B,C),\delta(A,C)\}}, \end{aligned}$$

where $\delta(A,B) = \max\{\inf_{x\in A}\sup_{y\in B} d(x,y), \inf_{y\in B}\sup_{x\in A} d(x,y)\}$. Then $(\Omega^*, \widetilde{G}^*_{A,B,C}(t), \Delta_m)$ is a Menger $PGM$-space induced by $(X, G^*, \Delta_m)$.

**Theorem 2.2** *Let $(\Omega^*, \widetilde{G}^*, \Delta)$ be a Menger PGM-space. Then for any $A,B,C,D \in \Omega^*$ and $x,y,z \in X$, we have the following*:

(1) $\widetilde{G}^*_{x,B,C}(t) = 1$ *if and only if* $x \in B = C$;
(2) $\widetilde{G}^*_{x,x,B}(t) \geq \widetilde{G}^*_{x,B,B}(t) \geq \widetilde{G}^*_{x,B,C}(t) \geq \widetilde{G}^*_{A,B,C}(t)$ *for all* $x \in A$ *and* $t \geq 0$;
(3) $\widetilde{G}^*_{x,B,C}(t+s) \geq \Delta(\widetilde{G}^*_{x,D,D}(t), \widetilde{G}^*_{D,B,C}(s))$ *for all* $s,t \geq 0$;
(4) $\widetilde{G}^*_{x,y,C}(t+s) \geq \Delta(\widetilde{G}^*_{x,a,a}(t), \widetilde{G}^*_{a,y,C}(s))$ *for all* $s,t \geq 0$ *and* $a \in X$.

*Proof* (1) If $\widetilde{G}^*_{x,B,C}(t) = \min\{g_{x,B}(t), g_{B,C}(t), g_{x,C}(t)\} = 1$, then we have $g_{x,B}(t) = g_{B,C}(t) = g_{x,C}(t) = 1$, which implies that $x \in B$, $x \in C$, $B = C$, that is, $x \in B = C$.

Conversely, it is obvious that $\widetilde{G}^*_{x,B,C}(t) = 1$ holds.

(2) From Definition 2.4 and Lemma 1.1, we have

$$\begin{aligned}\widetilde{G}^*_{x,x,B}(t) &= \sup_{y\in B} G^*_{x,x,y}(t) \geq g_{x,B}(t) = \widetilde{G}^*_{x,B,B}(t) \geq \min\bigl\{g_{x,B}(t), g_{B,C}(t), g_{x,C}(t)\bigr\}\\ &= \widetilde{G}^*_{x,B,C}(t) \geq \min\bigl\{g_{A,B}(t), g_{B,C}(t), g_{A,C}(t)\bigr\} = \widetilde{G}^*_{A,B,C}(t).\end{aligned}$$

So, (2) is proved.

(3) By Definition 2.4 and Lemma 1.1, we have $\widetilde{G}^*_{x,B,C}(t+s) = \min\{g_{x,B}(t+s), g_{B,C}(t+s), g_{x,C}(t+s)\}$,

$$\begin{aligned}&g_{x,B}(t+s) \geq \Delta\bigl(g_{x,D}(t), g_{D,B}(s)\bigr) = \Delta\bigl(\widetilde{G}^*_{x,D,D}(t), g_{D,B}(s)\bigr) \geq \Delta\bigl(\widetilde{G}^*_{x,D,D}(t), \widetilde{G}^*_{D,B,C}(s)\bigr),\\ &g_{B,C}(t+s) \geq g_{B,D}(s) \geq \widetilde{G}^*_{D,B,C}(s) \geq \Delta\bigl(\widetilde{G}^*_{x,D,D}(t), \widetilde{G}^*_{D,B,C}(s)\bigr),\\ &g_{x,C}(t+s) \geq \Delta\bigl(g_{x,D}(t), g_{D,C}(s)\bigr) = \Delta\bigl(\widetilde{G}^*_{x,D,D}(t), g_{D,C}(s)\bigr) \geq \Delta\bigl(\widetilde{G}^*_{x,D,D}(t), \widetilde{G}^*_{D,B,C}(s)\bigr).\end{aligned}$$

So, (3) is proved.

(4) By Lemma 1.1, we have

$$\widetilde{G}^*_{x,y,C}(t+s) = \sup_{z\in C} G^*_{x,y,z}(t+s) \geq \sup_{z\in C} \Delta\bigl(\widetilde{G}^*_{x,a,a}(t), \widetilde{G}^*_{a,y,z}(s)\bigr) = \Delta\bigl(\widetilde{G}^*_{x,a,a}(t), \widetilde{G}^*_{a,y,C}(s)\bigr). \qquad \square$$

**Remark 2.3** By (1), (2), and the proof of Lemma 2.2, it is easy to prove that $\widetilde{G}^*_{x,x,B}(t) = 1$ if and only if $x \in B$, and $\widetilde{G}^*_{x,B,B}(t) = 1$ if and only if $x \in B$.

## 3 Common fixed point theorems in Menger *PGM*-spaces

In this section, we will give some common fixed point theorems in Menger probabilistic $G$-metric spaces. To this end, we first introduce the concept of common property $(E.A)$ for three hybrid pairs of mappings in Menger probabilistic $G$-metric spaces.

**Definition 3.1** Let $(X, G^*, \Delta)$ be a Menger *PM*-space and $(\Omega^*, \widetilde{G}^*, \Delta)$ be the induced Menger *PM*-space, $f, h, r : X \to X$ and $F, H, R : X \to \Omega^*$. Three pairs of mappings $(f, F)$, $(h, H)$, and $(r, R)$ are said to satisfy the common property $(E.A)$ if there exist three sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ in $X$, some $u \in X$ and $A, B, C \in \Omega^*$, such that

$$\begin{aligned}&\lim_{n\to\infty} Fx_n = A, \qquad \lim_{n\to\infty} Hy_n = B, \qquad \lim_{n\to\infty} Ry_n = C,\\ &\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hy_n = \lim_{n\to\infty} rz_n = u \in A \cap B \cap C.\end{aligned}$$

We are now ready to give the common fixed point theorems in Menger probabilistic $G$-metric spaces.

**Theorem 3.1** *Let* $(X, G^*, \Delta)$ *be a Menger PGM-space with a continuous t-norm on* $[0,1] \times [0,1]$ *and* $(\Omega^*, \widetilde{G}^*, \Delta)$ *be the induced Menger PGM-space. Suppose that* $f, h, r : X \to X$ *and* $F, H, R : X \to \Omega^*$ *are mappings satisfying the following conditions*:

(1) $(f, F)$, $(h, H)$, *and* $(r, R)$ *satisfy the common property* $(E.A)$;

(2) *$f(X)$, $h(X)$, and $r(X)$ are $\mathscr{T}$-closed subsets of $X$;*
(3) *for any $x, y, z \in X$ with $Fx$, $Hy$, and $Rz$ not all equal and some $1 \le k \le 3$,*

$$\widetilde{G}^*_{Fx,Hy,Rz} > \min\left\{G^*_{fx,hy,rz}, \tfrac{3}{k}\left[\widetilde{G}^*_{Fx,hy,rz} \oplus \widetilde{G}^*_{fx,Hy,rz} \oplus \widetilde{G}^*_{fx,hy,Rz}\right]\right\}, \tag{3.1}$$

*where $\frac{3}{k}[\widetilde{G}^*_{Fx,hy,rz} \oplus \widetilde{G}^*_{fx,Hy,rz} \oplus \widetilde{G}^*_{fx,hy,Rz}](t)$ means $[\widetilde{G}^*_{Fx,hy,rz} \oplus \widetilde{G}^*_{fx,Hy,rz} \oplus \widetilde{G}^*_{fx,hy,Rz}](\frac{3}{k}t)$. Then $(f,F)$, $(h,H)$, and $(r,R)$ each has a coincidence point. Moreover, if $ffv = fv$ for $v \in C(f,F)$, $hhv = hv$ for $v \in C(h,H)$, and $rrv = rv$ for $v \in C(r,R)$, then $f$, $h$, $r$, $F$, $H$, and $R$ have a common fixed point in $X$.*

*Proof* Since $(f,F)$, $(h,H)$, and $(r,R)$ satisfy the common property $(E.A)$, there exist $\{x_n\}, \{y_n\}, \{z_n\} \subset X$, some $u \in X$ and $A, B, C \in \Omega^*$, such that

$$\begin{aligned} &\lim_{n\to\infty} Fx_n = A, \qquad \lim_{n\to\infty} Hy_n = B, \qquad \lim_{n\to\infty} Tz_n = C, \\ &\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hy_n = \lim_{n\to\infty} rz_n = u \in A \cap B \cap C. \end{aligned} \tag{3.2}$$

Since $f(X)$ is $\mathscr{T}$-closed, there exists some $v \in X$, such that $u = fv$. We claim that $fv \in Fv$. Suppose this is not true, then $fv \notin Fv$. By $u = fv \in B$, we have $B \neq Fv$. Thus, there exists some $t_0 > 0$, such that

$$\widetilde{G}^*_{Fv,B,C}\left(\frac{3t_0}{k}\right) > \widetilde{G}^*_{Fv,B,C}(t_0). \tag{3.3}$$

(Otherwise, for all $t > 0$, $\widetilde{G}^*_{Fv,B,C}(t) = \widetilde{G}^*_{Fv,B,C}(\frac{3t}{k}) = \cdots = \widetilde{G}^*_{Fv,B,C}((\frac{3}{k})^n t) \to 1$ as $n \to \infty$, that is, $\widetilde{G}^*_{Fv,B,C}(t) = 1$, for all $t > 0$, which is a contradiction.)

Without loss of generality, we can assume that $t_0$ is a continuous point of $\widetilde{G}^*_{Fv,B,C}(\cdot)$. In fact, by the left continuity of the distribution function, we know that there exists some $\delta > 0$, such that

$$\widetilde{G}^*_{Fv,B,C}\left(\frac{3t}{k}\right) > \widetilde{G}^*_{Fv,B,C}(t), \quad \forall t \in (t_0 - \delta, t_0].$$

Since the distribution function is nondecreasing, the discontinuous points are at most a countable set. Thus, when $t_0$ is not a continuous point of $\widetilde{G}^*_{Fv,B,C}(\cdot)$, we can always choose a point $t_1$ in $(t_0 - \delta, t_0]$ to replace $t_0$.

Noting that $\lim_{n\to\infty} fx_n = u \notin Fv$ and $u \in B = \lim_{n\to\infty} Hy_n$, we have $Fv \neq \lim_{n\to\infty} Hy_n$, so there exists some $n_0 \in \mathbb{Z}^+$, such that for all $n \ge n_0$, $Hy_n \neq Fv$.

From (3.1) we know that

$$\widetilde{G}^*_{Fv,Hy_n,Rz_n} > \min\left\{G^*_{fv,hy_n,rz_n}, \tfrac{3}{k}\left[\widetilde{G}^*_{Fv,hy_n,rz_n} \oplus \widetilde{G}^*_{fv,Hy_n,rz_n} \oplus \widetilde{G}^*_{fv,hy_n,Rz_n}\right]\right\}. \tag{3.4}$$

It is easy to verify that

$$\liminf_{n\to\infty}\left[\widetilde{G}^*_{Fv,hy_n,rz_n} \oplus \widetilde{G}^*_{fv,Hy_n,rz_n} \oplus \widetilde{G}^*_{fv,hy_n,Rz_n}\right]\left(\frac{3}{k}t_0\right) \ge \widetilde{G}^*_{Fv,u,u}\left(\frac{3}{k}t_0\right). \tag{3.5}$$

In fact, for any $\delta_1, \delta_2 \in (0, \frac{3}{k}t_0)$, we have

$$\begin{aligned}
&[\widetilde{G}^*_{Fv,hy_n,rz_n} \oplus \widetilde{G}^*_{fv,Hy_n,rz_n} \oplus \widetilde{G}^*_{fv,hy_n,Rz_n}]\left(\frac{3}{k}t_0\right) \\
&\quad \geq \min\left\{\widetilde{G}^*_{Fv,hy_n,rz_n}\left(\frac{3}{k}t_0 - \delta_1 - \delta_2\right), \widetilde{G}^*_{fv,Hy_n,rz_n}(\delta_1), \widetilde{G}^*_{fv,hy_n,Rz_n}(\delta_2)\right\}.
\end{aligned}$$

Since $fv = u \in [(B = \lim_{n\to\infty} Hy_n) \cap (C = \lim_{n\to\infty} Ry_n)]$, by Lemma 1.3 and Theorem 2.2(1), we get

$$\liminf_{n\to\infty}[\widetilde{G}^*_{Fv,hy_n,rz_n} \oplus \widetilde{G}^*_{fv,Hy_n,rz_n} \oplus \widetilde{G}^*_{fv,hy_n,Rz_n}]\left(\frac{3}{k}t_0\right) \geq \widetilde{G}^*_{Fv,u,u}\left(\frac{3}{k}t_0 - \delta_1 - \delta_2\right).$$

Letting $\delta_1, \delta_2 \to 0$, by the left continuity of the distribution function, we obtain (3.5).

Noting that $t_0$ is the continuous point of $\widetilde{G}^*_{Fv,B,C}(\cdot)$, by Lemma 1.3, we have

$$\lim_{n\to\infty} \widetilde{G}^*_{Fv,Hy_n,Rz_n}(t_0) = \widetilde{G}^*_{Fv,B,C}(t_0).$$

Thus, letting $n \to \infty$ in (3.4) and using (3.5), we obtain

$$\widetilde{G}^*_{Fv,B,C}(t_0) \geq \min\left\{1, \widetilde{G}^*_{Fv,u,u}\left(\frac{3}{k}t_0\right)\right\} = \widetilde{G}^*_{Fv,u,u}\left(\frac{3}{k}t_0\right),$$

that is,

$$\widetilde{G}^*_{Fv,B,C}(t_0) \geq \widetilde{G}^*_{Fv,u,u}\left(\frac{3}{k}t_0\right).$$

But since $fv \in B$, by Theorem 2.2(3) and (3.3), we obtain

$$\widetilde{G}^*_{Fv,u,u}\left(\frac{3}{k}t_0\right) > \widetilde{G}^*_{Fv,B,C}(t_0),$$

which is a contradiction. So, we get $fv \in Fv$.

On the other hand, since $h(X)$ is $\mathscr{T}$-closed, there exists some $w \in X$, such that $u = hw$. We claim that $hw \in Hw$. Suppose this is not true, that is, $hw \notin Hw$. Noting that $u = hw \in C$, we have $C \neq Hw$. Similarly, we know that there exists some $t_1 > 0$, such that

$$\widetilde{G}^*_{Fv,Hw,C}\left(\frac{3}{k}t_1\right) > \widetilde{G}^*_{Fv,Hw,C}(t_1). \tag{3.6}$$

Similarly, without loss of generality, we can assume that $t_1$ is a continuous point of $\widetilde{G}^*_{Fv,Hw,C}(\cdot)$.

Noting that $\lim_{n\to\infty} rz_n = u \notin Hw$ and $u \in C = \lim_{n\to\infty} Rz_n$, there exists some $n_1 \in \mathbb{Z}^+$, such that for all $n \geq n_1$, $Rz_n \neq Hw$.

From (3.1) we know that

$$\widetilde{G}^*_{Fv,Hw,Rz_n}(t_1) > \min\left\{G^*_{fv,hw,rz_n}(t_1), [\widetilde{G}^*_{Fv,hw,rz_n} \oplus \widetilde{G}^*_{fv,Hw,rz_n} \oplus \widetilde{G}^*_{fv,hw,Rz_n}]\left(\frac{3}{k}t_1\right)\right\}. \tag{3.7}$$

Similarly, we can verify that

$$\liminf_{n\to\infty}\bigl[\widetilde{G}^*_{Fv,Hw,rz_n}\oplus\widetilde{G}^*_{fv,Hw,rz_n}\oplus\widetilde{G}^*_{fv,hw,Rz_n}\bigr]\left(\frac{3}{k}t_1\right)\geq\widetilde{G}^*_{u,Hw,u}\left(\frac{3}{k}t_1\right).\tag{3.8}$$

Noting that $t_1$ is a continuous point of $\widetilde{G}^*_{Fv,Hw,C}(\cdot)$, by Lemma 1.3, we have

$$\lim_{n\to\infty}\widetilde{G}^*_{Fv,Hw,Rz_n}(t_1)=\widetilde{G}^*_{Fv,Hw,C}(t_1).$$

Thus, letting $n\to\infty$ in (3.7) and using (3.8), we obtain

$$\widetilde{G}^*_{Fv,Hw,C}(t_1)\geq\min\left\{1,\widetilde{G}^*_{u,Hw,u}\left(\frac{3}{k}t_1\right)\right\}=\widetilde{G}^*_{u,Hw,u}\left(\frac{3}{k}t_1\right)\geq\widetilde{G}^*_{Fv,Hw,C}\left(\frac{3}{k}t_1\right),$$

which is a contradiction. So, we get $hw\in Hw$.

Since $r(X)$ is $\mathscr{T}$-closed, there exists some $a\in X$, such that $u=ra$. We claim that $ra\in Ra$. Suppose this is not true, that is, $ra\notin Ra$. Noting that $u=ra\in A$, we have $A\neq Ra$. Similarly, we know that there exists some $t_2>0$, such that

$$\widetilde{G}^*_{A,Hw,Ra}\left(\frac{3}{k}t_2\right)>\widetilde{G}^*_{A,Hw,Ra}(t_2).$$

Similarly, without loss of generality, we can assume that $t_2$ is a continuous point of $\widetilde{G}^*_{A,Hw,Ra}(\cdot)$.

Noting that $\lim_{n\to\infty}fx_n=u\notin Ra$ and $u\in A=\lim_{n\to\infty}Fx_n$, there exists some $n_2\in\mathbb{Z}^+$, such that for all $n\geq n_2$, $Fx_n\neq Ra$.

From (3.1), we know that

$$\widetilde{G}^*_{Fx_n,Hw,Ra}(t_2)>\min\left\{G^*_{fx_n,hw,ra}(t_2),\bigl[\widetilde{G}^*_{Fx_n,hw,ra}\oplus\widetilde{G}^*_{fx_n,Hw,ra}\oplus\widetilde{G}^*_{fx_n,hw,Ra}\bigr]\left(\frac{3}{k}t_2\right)\right\}.$$

Similarly, it is easy to prove that $u=ra\in Ra$. This implies that $v$ is a coincidence point of $(f,F)$, $w$ is a coincidence point of $(h,H)$, and $a$ is a coincidence point of $(r,R)$.

Since $v\in C(f,F)$, $w\in C(h,H)$, and $a\in C(r,R)$, we have $u=fv=ffv=fu\in Fv$, $u=hw=hhw=hu\in Hw$, and $u=ra=rra=ru\in Rw$. Next, we prove that $Fv=Fu$, $Hw=Hu$, and $Ra=Ru$.

(1) First, we assert that $Fv=Hw$. In fact, suppose that $Fv\neq Hw$. Then, by (3.1), there exists some $t_3>0$, such that

$$\widetilde{G}^*_{Fv,Hw,Ra}(t_3)>\min\left\{G^*_{fv,hw,ra}(t_3),\bigl[\widetilde{G}^*_{Fv,hw,ra}\oplus\widetilde{G}^*_{fv,Hw,ra}\oplus\widetilde{G}^*_{fv,hw,Ra}\bigr]\left(\frac{3}{k}t_3\right)\right\}.$$

This implies that

$$\widetilde{G}^*_{Fv,Hw,Ra}(t_3)>1,$$

which is a contradiction, and thus we have $Fv=Hw$.

(2) Next, we assert that $Fu=Hw$. In fact, suppose that $Fu\neq Hw$. Then, by (3.1), there exists some $t_4>0$, such that

$$\widetilde{G}^*_{Fu,Hw,Ra}(t_4)>\min\left\{G^*_{fu,hw,ra}(t_4),\bigl[\widetilde{G}^*_{Fu,hw,ra}\oplus\widetilde{G}^*_{fu,Hw,ra}\oplus\widetilde{G}^*_{fu,hw,Ra}\bigr]\left(\frac{3}{k}t_4\right)\right\}.$$

This implies that

$$\widetilde{G}^*_{Fu,Hw,Ra}(t_4) > 1,$$

which is a contradiction, and thus we have $Fu = Hw$. Combining these two facts yields $Fv = Fu$. Similarly, we can prove that $Hw = Ra = Hu$ and $Ra = Fv = Ru$. Thus, we have $u = fu \in Fu$, $u = hu \in Hu$, and $u = ru \in Ru$, that is, $u$ is the common fixed point of $f$, $h$, $r$, $F$, $H$, and $R$. This completes the proof. □

Setting $f = h = r$ and $F = H = R$, we obtain the following result.

**Theorem 3.2** *Let $(X, G^*, \Delta)$ be a Menger PGM-space with a continuous t-norm on $[0,1] \times [0,1]$ and $(\Omega^*, \widetilde{G}^*, \Delta)$ be the induced Menger PGM-space. Suppose that $f : X \to X$ and $F : X \to \Omega^*$ are mappings satisfying the following conditions*:

(1) *$(f, F)$ satisfies the property $(E.A)$*;
(2) *$f(X)$ is a $\mathscr{T}$-closed subset of $X$*;
(3) *for any $x, y, z \in X$ with $Fx$, $Fy$, and $Fz$ not all equal and some $1 \leq k \leq 3$*,

$$\widetilde{G}^*_{Fx,Fy,Fz}(t) > \min\bigl\{G^*_{fx,fy,fz}, {}_{\frac{3}{k}}\bigl[\widetilde{G}^*_{Fx,fy,fz} \oplus \widetilde{G}^*_{fx,Fy,fz} \oplus \widetilde{G}^*_{fx,fy,Fz}\bigr]\bigr\},$$

*where ${}_{\frac{3}{k}}[\widetilde{G}^*_{Fx,fy,fz} \oplus \widetilde{G}^*_{fx,Fy,fz} \oplus \widetilde{G}^*_{fx,fy,Fz}](t)$ means $[\widetilde{G}^*_{Fx,fy,fz} \oplus \widetilde{G}^*_{fx,Fy,fz} \oplus \widetilde{G}^*_{fx,fy,Fz}](\frac{3}{k}t)$.*
*Then $f$ and $F$ have a coincidence point. Moreover, if $ffv = fv$ for $v \in C(f, F)$, then $f$ and $F$ have a common fixed point in $X$.*

## 4 An example

In this section, we will provide an example to show the validity of Theorem 3.1.

**Example 4.1** Let $X = (-2, 2)$ and define

$$G^*_{x,y,z}(t) = \frac{t}{t + \max\{|x-y|, |y-z|, |z-x|\}},$$
$$\widetilde{G}^*_{A,B,C}(t) = \frac{t}{t + \max\{\delta(A,B), \delta(B,C), \delta(A,C)\}}$$

for all $x, y, z \in X$, $A, B, C \in \Omega^*$, and $t \geq 0$. Then, by Example 2.1, $(X, G^*, \Delta_m)$ and $(\Omega^*, \widetilde{G}^*, \Delta_m)$ are $PGM$-spaces. Define $f, h, r : X \to X$ and $F, H, R : X \to \Omega^*$ as follows:

$$fx = \begin{cases} \frac{5}{6}, & x \in (-2,-1) \cup (1,2); \\ \frac{1}{3}x, & x \in [-1,1], \end{cases} \qquad Fx = \begin{cases} [0, \frac{2}{3}], & x \in (-2,-1) \cup (1,2); \\ [\frac{1}{3}x, 0], & x \in [-1,0]; \\ [0, \frac{1}{3}x], & x \in [0,1], \end{cases}$$

$$hx = \begin{cases} \frac{4}{5}, & x \in (-2,-1) \cup (1,2); \\ \frac{1}{3}x, & x \in [-1,1], \end{cases} \qquad Hx = \begin{cases} [0, \frac{1}{2}], & x \in (-2,-1) \cup (1,2); \\ [0, -\frac{1}{4}x], & x \in [-1,0]; \\ [-\frac{1}{4}x, 0], & x \in [0,1], \end{cases}$$

$$rx = \begin{cases} \frac{7}{8}, & x \in (-2,-1) \cup (1,2); \\ \frac{1}{3}x, & x \in [-1,1], \end{cases} \qquad Rx = \begin{cases} [0, \frac{3}{4}], & x \in (-2,-1) \cup (1,2); \\ [\frac{2}{3}x, 0], & x \in [-1,0]; \\ [0, \frac{2}{3}x], & x \in [0,1]. \end{cases}$$

Consider the sequences $\{x_n = \frac{1}{n+1}\}$ and $\{y_n = -\frac{1}{n+1}\}$ in $X$. Then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} rx_n = 0 \in \lim_{n\to\infty} Fx_n \cap \lim_{n\to\infty} Hx_n \cap \lim_{n\to\infty} Rx_n,$$

which shows that $(f,F)$, $(h,H)$, and $(r,R)$ satisfy the common property $(E.A)$. Also $f(X)$, $h(X)$, and $r(X)$ are $\mathscr{T}$-closed subsets of $X$. By a routine calculation, one can verify that (3.1) holds for all $x,y,z \in X$, $t > 0$, and some $1 \le k < 3$.

In fact, if $x,y,z \in (-2,-1) \cup (1,2)$, for any $t > 0$,

$$\begin{aligned}
\widetilde{G}^*_{Fx,Hy,Rz}(t) &= \frac{t}{t + \max\{\delta([0,\frac{2}{3}],[0,\frac{1}{2}]), \delta([0,\frac{3}{4}],[0,\frac{1}{2}]), \delta([0,\frac{2}{3}],[0,\frac{3}{4}])\}} = \frac{t}{t+0} = 1, \\
\widetilde{G}^*_{fx,hy,rz}(t) &= \frac{t}{t + \max\{(\frac{5}{6} - \frac{4}{5}), (\frac{7}{8} - \frac{4}{5}), (\frac{7}{8} - \frac{5}{6})\}} = \frac{t}{t + \frac{3}{40}} < 1.
\end{aligned}$$

So, we have

$$\widetilde{G}^*_{Fx,Hy,Rz}(t) > \widetilde{G}^*_{fx,hy,rz}(t) \ge \min\bigl\{G^*_{fx,hy,rz}(t), \tfrac{3}{k}\bigl[\widetilde{G}^*_{Fx,hy,rz} \oplus \widetilde{G}^*_{fx,Hy,rz} \oplus \widetilde{G}^*_{fx,hy,Rz}\bigr](t)\bigr\}.$$

Similarly, if $x,y,z \in [-1,0]$, or $x,y,z \in [0,1]$, we also have

$$\widetilde{G}^*_{Fx,Hy,Rz}(t) = 1 > \widetilde{G}^*_{fx,hy,rz}(t) \ge \min\bigl\{G^*_{fx,hy,rz}(t), \tfrac{3}{k}\bigl[\widetilde{G}^*_{Fx,hy,rz} \oplus \widetilde{G}^*_{fx,Hy,rz} \oplus \widetilde{G}^*_{fx,hy,Rz}\bigr](t)\bigr\}.$$

If $x,y \in (-2,-1) \cup (1,2)$, $z \in [0,1]$, we have

$$\begin{aligned}
\widetilde{G}^*_{Fx,Hy,Rz}(t) &= \frac{t}{t + \max\{\delta([0,\frac{2}{3}],[0,\frac{1}{2}]), \delta([\frac{1}{4}x,0],[0,\frac{1}{2}]), \delta([0,\frac{2}{3}],[\frac{1}{4}x,0])\}} = \frac{t}{t+0} = 1, \\
\widetilde{G}^*_{fx,hy,rz}(t) &= \frac{t}{t + \max\{(\frac{5}{6} - \frac{4}{5}), |\frac{1}{3}z - \frac{4}{5}|, |\frac{1}{3}z - \frac{5}{6}|\}} \le \frac{t}{t + \frac{1}{30}} < 1.
\end{aligned}$$

So, we have

$$\widetilde{G}^*_{Fx,Hy,Rz}(t) > \widetilde{G}^*_{fx,hy,rz}(t) \ge \min\bigl\{G^*_{fx,hy,rz}(t), \tfrac{3}{k}\bigl[\widetilde{G}^*_{Fx,hy,rz} \oplus \widetilde{G}^*_{fx,Hy,rz} \oplus \widetilde{G}^*_{fx,hy,Rz}\bigr](t)\bigr\}.$$

Similarly, it is easy to verify (3.1) for the other cases. Thus, all the conditions of Theorem 3.1 are satisfied and 0 is the unique coincidence point of $(f,F)$, $(h,H)$, and $(r,R)$. Furthermore, noting that $ff0 = f0$, $hh0 = h0$, and $rr0 = r0$, 0 remains the common fixed point of $(f,F)$, $(h,H)$, and $(r,R)$.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally. All authors read and approved the final manuscript.

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