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New descent LQP alternating direction methods for solving a class of structured variational inequalities

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Abstract

In this paper, we propose two descent alternating direction methods based on a logarithmic-quadratic proximal method for structured variational inequalities. The first method can be viewed as an extension of the method proposed by Bnouhachem and Xu (Comput. Math. Appl. 67:671-680, 2014) by performing an additional step at each iteration. The second method generates the new iterate by searching the optimal step size along a new descent direction, which can be viewed as a refinement and improvement of the first one. Under certain conditions, the global convergence of the both methods is proved.

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1 Introduction

We consider the constrained convex programming problem with the following separate structure:

$$\min\{\theta_1(x) + \theta_2(y) | Ax + By = b, x \in \mathcal{R}^n_+, y \in \mathcal{R}^m_+\}, \tag{1.1}$$

where $\theta_1 : \mathcal{R}^n_+ \to \mathcal{R}$ and $\theta_2 : \mathcal{R}^m_+ \to \mathcal{R}$ are closed proper convex functions, $A \in \mathcal{R}^{l \times n}$, $B \in \mathcal{R}^{l \times m}$ are given matrices, and $b \in \mathcal{R}^l$ is a given vector.

A large number of problems can be modeled as problem (1.1). In practice, these classes of problems have very large size and, due to their practical importance, they have received a great deal of attention from many researchers. Various methods have been suggested to find the solution of problem (1.1). A popular approach is the alternating direction method (ADM) which was proposed by Gabay and Mercier [2] and Gabay [3]. The ADM can reduce the scale of variational inequalities by decomposing the original problem into a series of subproblems with a lower scale. To make the ADM more efficient and practical, some strategies have been studied; For further details, we refer to [4–13] and the references therein.

Let $\partial(\cdot)$ denote the sub-gradient operator of a convex function, and $f(x) \in \partial\theta_1(x)$ and $g(y) \in \partial\theta_2(y)$ are the sub-gradient of $\theta_1(x)$ and $\theta_2(y)$, respectively. By attaching a Lagrange

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multiplier vector $\lambda \in \mathcal{R}^l$ to the linear constraint $Ax + By = b$, problem (1.1) can be written in terms of finding $w \in \mathcal{W}$ such that

$$(w' - w)^\top Q(w) \geq 0, \quad \forall w' \in \mathcal{W}, \tag{1.2}$$

where

$$w = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix}, \qquad Q(w) = \begin{pmatrix} f(x) - A^\top \lambda \\ g(y) - B^\top \lambda \\ Ax + By - b \end{pmatrix}, \qquad \mathcal{W} = \mathcal{R}_+^n \times \mathcal{R}_+^m \times \mathcal{R}^l. \tag{1.3}$$

Problem (1.2)-(1.3) is referred to as *structured variational inequalities* (in short, SVI).

Very recently, Yuan and Li [14] developed the following logarithmic-quadratic proximal (LQP)-based decomposition method by applying the LQP terms to regularize the ADM subproblems: For a given $w^k = (x^k, y^k, \lambda^k) \in \mathcal{R}_{++}^n \times \mathcal{R}_{++}^m \times \mathcal{R}^l$, and $\mu \in (0,1)$, the new iterative $(x^{k+1}, y^{k+1}, \lambda^{k+1})$ is obtained via solving the following system:

$$f(x) - A^\top[\lambda^k - H(Ax + By^k - b)] + R[(x - x^k) + \mu(x^k - X_k^2 x^{-1})] = 0, \tag{1.4}$$

$$g(y) - B^\top[\lambda^k - H(Ax + By - b)] + S[(y - y^k) + \mu(y^k - Y_k^2 y^{-1})] = 0, \tag{1.5}$$

$$\lambda^{k+1} = \lambda^k - H(Ax^k + By^k - b), \tag{1.6}$$

where $H \in \mathcal{R}^{l \times l}$, $R \in \mathcal{R}^{n \times n}$, and $S \in \mathcal{R}^{m \times m}$ are symmetric positive definite.

Note that the LQP method was presented originally in [15]. It seems that it is easier to solve a series of systems of nonlinear equations than to solve a series of sub-variational inequalities in many cases. Later, Bnouhachem *et al.* [16], Bnouhachem and Ansari [17], and Li [18] proposed some LQP alternating direction methods and made the LQP alternating direction method more practical. Each iteration of the above methods contains a prediction and a correction. The predictor is obtained via solving (1.4)-(1.6) and the new iterate is obtained by a convex combination of the previous point and the one generated by a projection-type method along a descent direction for [17, 18]. In 2014, Bnouhachem and Xu [1] proposed the following LQP alternating direction: For a given $w^k = (x^k, y^k, \lambda^k) \in \mathcal{R}_{++}^n \times \mathcal{R}_{++}^m \times \mathcal{R}^l$, and $\mu \in (0,1)$, the predictor $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k) \in \mathcal{R}_{++}^n \times \mathcal{R}_{++}^m \times \mathcal{R}^l$ is obtained via solving the following system:

$$f(x) - A^\top[\lambda^k - H(Ax + By - b)] + R[(x - x^k) + \mu(x^k - X_k^2 x^{-1})] =: \xi_x^k \approx 0, \tag{1.7a}$$

$$g(y) - B^\top[\lambda^k - H(Ax + By - b)] + S[(y - y^k) + \mu(y^k - Y_k^2 y^{-1})] =: \xi_y^k \approx 0, \tag{1.7b}$$

$$\tilde{\lambda}^k = \lambda^k - H(A\tilde{x}^k + B\tilde{y}^k - b), \tag{1.7c}$$

where

$$\|G^{-1}\xi^k\|_G^2 \leq \frac{1 - \mu}{1 + \mu} \eta^2 \|w^k - \tilde{w}^k\|_G^2, \quad \eta \in (0,1), \tag{1.8}$$

$$\xi^k = \begin{pmatrix} \xi_x^k \\ \xi_y^k \\ 0 \end{pmatrix}, \tag{1.9}$$

and

$$G = \begin{pmatrix} (1+\mu)R & & \\ & (1+\mu)S & \\ & & H^{-1} \end{pmatrix} \qquad (1.10)$$

is a positive definite (block diagonal) matrix.

Take the new iterate $w^{k+1} = (x^{k+1}, y^{k+1}, \lambda^{k+1})$ as the solution of the following system:

$$\tau\left[f\left(\tilde{x}^k\right) - A^\top \tilde{\lambda}^k\right] + R\left[\left(x - x^k\right) + \mu\left(x^k - X_k^2 x^{-1}\right)\right] = 0, \qquad (1.11a)$$

$$\tau\left[g\left(\tilde{y}^k\right) - B^\top \tilde{\lambda}^k\right] + S\left[\left(y - y^k\right) + \mu\left(y^k - Y_k^2 y^{-1}\right)\right] = 0, \qquad (1.11b)$$

$$\lambda^{k+1} = \lambda^k - \tau H\left(A\tilde{x}^k + B\tilde{y}^k - b\right). \qquad (1.11c)$$

Each iteration of the above method contains a prediction and a correction. The predictor is obtained via solving (1.4)-(1.6) and the new iterate is computed directly by an explicit formula derived from the original LQP method.

Inspired and motivated by the above research, we present two methods to solve SVI. The first one can be viewed as an extension of the method proposed in [1] by performing an additional step at each iteration. The second method can be viewed as an extension of the first one by using a new descent direction which provides a significant refinement and improvement of the first one. We also study the global convergence of the both methods method under certain conditions.

2 Iterative methods and convergence results

In this section, we suggest and analyze two new modified logarithmic-quadratic proximal alternating direction methods for solving structured variational inequalities. The following lemma provides some basic properties of the projection onto Ω.

Lemma 2.1 *Let G be a symmetry positive definite matrix and Ω be a nonempty closed convex subset of R^l. We denote by $P_{\Omega,G}(\cdot)$ the projection under the G-norm, that is,*

$$P_{\Omega,G}(v) = \mathrm{argmin}\left\{\|v - u\|_G : u \in \Omega\right\}.$$

Then

$$\left(z - P_{\Omega,G}[z]\right)^\top G\left(P_{\Omega,G}[z] - v\right) \geq 0, \quad \forall z \in R^l, v \in \Omega; \qquad (2.1)$$

$$\left\|P_{\Omega,G}[u] - P_{\Omega,G}[v]\right\|_G \leq \|u - v\|_G, \quad \forall u, v \in R^l; \qquad (2.2)$$

$$\left\|u - P_{\Omega,G}[z]\right\|_G^2 \leq \|z - u\|_G^2 - \left\|z - P_{\Omega,G}[z]\right\|_G^2, \quad \forall z \in R^l, u \in \Omega. \qquad (2.3)$$

For convenience, we make the following standard assumptions to guarantee that the problem under consideration is solvable and the proposed methods are well defined.

Assumption A $f(x)$ is monotone with respect to $\mathcal{R}_{++}^n$ and $g(y)$ is monotone with respect to $\mathcal{R}_{++}^m$.

Assumption B The solution set of SVI, denoted by $\mathcal{W}^*$, is nonempty.

Then the iterative scheme of the first method is given as follows.

Algorithm 2.1

Step 0. *The initial step*:

Given $\varepsilon > 0$, $\mu \in (0,1)$ and $w^1 = (x^1, y^1, \lambda^1) \in \mathcal{R}_{++}^n \times \mathcal{R}_{++}^m \times \mathcal{R}^l$. Set $k = 1$.

Step 1. *Prediction step*:

Compute $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k) \in \mathcal{R}_{++}^n \times \mathcal{R}_{++}^m \times \mathcal{R}^l$ by solving the system (1.7a)-(1.7c).

Step 2. *Convergence verification*:

If $\{\max \|x^k - \tilde{x}^k\|_\infty, \|y^k - \tilde{y}^k\|_\infty, \|\lambda^k - \tilde{\lambda}^k\|_\infty\} < \epsilon$, then stop.

Step 3. *Correction step*:

Compute $\bar{w}^k = (\bar{x}^k, \bar{y}^k, \bar{\lambda}^k)$ by solving the following system:

$$\frac{1-\mu}{1+\mu}\alpha_k\big[f(\tilde{x}^k) - A^\top\tilde{\lambda}^k\big] + R\big[(x - x^k) + \mu(x^k - X_k^2 x^{-1})\big] = 0, \tag{2.4a}$$

$$\frac{1-\mu}{1+\mu}\alpha_k\big[g(\tilde{y}^k) - B^\top\tilde{\lambda}^k\big] + S\big[(y - y^k) + \mu(y^k - Y_k^2 y^{-1})\big] = 0, \tag{2.4b}$$

$$\lambda^{k+1} = \lambda^k - \frac{1-\mu}{1+\mu}\alpha_k H\big(A\tilde{x}^k + B\tilde{y}^k - b\big), \tag{2.4c}$$

where

$$\alpha_k = \frac{\varphi_k}{\|d_k\|_G^2}, \tag{2.5}$$

$$\varphi_k := \|x^k - \tilde{x}^k\|_R^2 + \|y^k - \tilde{y}^k\|_S^2 + \|\lambda^k - \lambda^{k+1}\|_{H^{-1}}^2 + (w^k - \tilde{w}^k)^\top \xi^k, \tag{2.6}$$

and

$$d_k := w^k - \tilde{w}^k + G^{-1}\xi^k. \tag{2.7}$$

The new iterate $w^{k+1}(\tau_k) = (x^{k+1}, y^{k+1}, \lambda^{k+1})$ is given by

$$w^{k+1}(\tau_k) = \rho w^k + (1 - \rho)P_\mathcal{W}\big[w^k - \tau_k d_k'\big], \quad \rho \in (0,1), \tag{2.8}$$

where

$$d_k' := w^k - \bar{w}^k. \tag{2.9}$$

Set $k := k + 1$ and go to Step 1.

τ_k is a positive scalar.

How to choose a suitable step length τ_k to force convergence will be discussed later.

Theorem 2.1 [1] *For given $w^k = (x^k, y^k, \lambda^k) \in \mathcal{R}_{++}^n \times \mathcal{R}_{++}^m \times \mathcal{R}^l$, let $\bar{w}^k = (\bar{x}^k, \bar{y}^k, \bar{\lambda}^k)$ be generated by* (2.4a)-(2.4c). *Then for any $w^* = (x^*, y^*, \lambda^*) \in \mathcal{W}^*$, we have*

$$\|w^k - w^*\|_G^2 - \|\bar{w}^k - w^*\|_G^2 \geq \frac{1-\mu}{1+\mu}\Phi(\alpha_k), \tag{2.10}$$

where

$$\Phi(\alpha_k) := 2\alpha_k \varphi_k - \alpha_k^2 \|d_k\|_G^2. \tag{2.11}$$

Theorem 2.2 [1] *For given $w^k \in \mathcal{R}_{++}^n \times \mathcal{R}_{++}^m \times \mathcal{R}^l$, let $\tilde{w}^k$ be generated by (1.7b)-(1.7c), then we have the following*:

$$\alpha_k \geq \frac{1}{2} \tag{2.12}$$

and

$$\Phi(\alpha_k) \geq \frac{(1 - \eta^2)(1 - \mu)}{4(1 + \mu)} \|w^k - \tilde{w}^k\|_G^2. \tag{2.13}$$

How should one choose values of τ_k to ensure that $w^{k+1}(\tau_k)$ is closer to the solution set than w^k? For this purpose, we define

$$\Theta_1(\tau_k) = \|w^k - w^*\|_G^2 - \|w^{k+1}(\tau_k) - w^*\|_G^2. \tag{2.14}$$

Theorem 2.3 *Let $w^* = (x^*, y^*, \lambda^*) \in \mathcal{W}^*$, then we have*

$$\Theta_1(\tau_k) \geq (1 - \rho)\left(\tau_k\{\|d_k'\|_G^2 + \|w^k - w^*\|_G^2 - \|\bar{w}^k - w^*\|_G^2\} - \tau_k^2\|d_k'\|_G^2\right). \tag{2.15}$$

Proof Since $w^* = (x^*, y^*, \lambda^*) \in \mathcal{W}^*$ and $w_*^k(\tau_k) = P_{\mathcal{W}}[w^k - \tau_k d_k']$, it follows from (2.3) that

$$\|w_*^k(\tau_k) - w^*\|_G^2 \leq \|w^k - \tau_k d_k' - w^*\|_G^2 - \|w^k - \tau_k d_k' - w_*^k(\tau_k)\|_G^2. \tag{2.16}$$

On the other hand, we have

$$
\begin{aligned}
\|w^{k+1}(\tau_k) - w^*\|_G^2 &= \|\rho(w^k - w^*) + (1 - \rho)(w_*^k(\tau_k) - w^*)\|_G^2 \\
&= \rho^2\|w^k - w^*\|_G^2 + (1 - \rho)^2\|w_*^k(\tau_k) - w^*\|_G^2 \\
&\quad + 2\rho(1 - \rho)(w^k - w^*)^\top G(w_*^k(\tau_k) - w^*).
\end{aligned}
$$

Using the identity

$$2(a + b)^\top Gb = \|a + b\|_G^2 - \|a\|_G^2 + \|b\|_G^2,$$

for $a = w^k - w_*^k(\tau_k)$, $b = w_*^k(\tau_k) - w^*$, and (2.16), we obtain

$$
\begin{aligned}
\|w^{k+1}(\tau_k) - w^*\|_G^2 &= \rho^2\|w^k - w^*\|_G^2 + (1 - \rho)^2\|w_*^k(\tau_k) - w^*\|_G^2 \\
&\quad + \rho(1 - \rho)\{\|w^k - w^*\|_G^2 - \|w^k - w_*^k(\tau_k)\| + \|w_*^k(\tau_k) - w^*\|_G^2\} \\
&= \rho\|w^k - w^*\|_G^2 + (1 - \rho)\|w_*^k(\tau_k) - w^*\|_G^2 - \rho(1 - \rho)\|w^k - w_*^k(\tau_k)\|_G^2 \\
&\leq \rho\|w^k - w^*\|_G^2 + (1 - \rho)\|w^k - \tau_k d_k' - w^*\|_G^2 \\
&\quad - (1 - \rho)\|w^k - \tau_k d_k' - w_*^k(\tau_k)\|_G^2 - \rho(1 - \rho)\|w^k - w_*^k(\tau_k)\|_G^2
\end{aligned}
$$

$$
\begin{aligned}
&= \left\| w^k - w^* \right\|_G^2 - (1 - \rho)\big\{ \left\| w^k - w_*^k(\tau_k) - \tau_k d_k' \right\|_G^2 + \rho \left\| w^k - w_*^k(\tau_k) \right\|_G^2 \\
&\quad - \tau_k^2 \left\| d_k' \right\|_G^2 + 2\tau_k \big(w^k - w^* \big)^\top G d_k' \big\} \\
&\le \left\| w^k - w^* \right\|_G^2 - (1 - \rho)\big\{ 2\tau_k \big(w^k - w^* \big)^\top G d_k' - \tau_k^2 \left\| d_k' \right\|_G^2 \big\}.
\end{aligned}
$$

Using the definition of $\Theta_1(\tau_k)$, we get

$$
\begin{aligned}
\Theta_1(\tau_k) &\ge (1 - \rho)\big\{ 2\tau_k \big(w^k - w^* \big)^\top G d_k' - \tau_k^2 \left\| d_k' \right\|_G^2 \big\} \\
&= (1 - \rho)\big(2\tau_k \big\{ \left\| d_k' \right\|_G^2 - \big(w^* - \bar{w}^k \big)^\top G d_k' \big\} - \tau_k^2 \left\| d_k' \right\|_G^2 \big).
\end{aligned}
\tag{2.17}
$$

The identity

$$
\big(w^* - \bar{w}^k \big)^\top G d_k' = \frac{1}{2}\big(\left\| \bar{w}^k - w^* \right\|_G^2 - \left\| w^k - w^* \right\|_G^2 \big) + \frac{1}{2}\left\| d_k' \right\|_G^2
\tag{2.18}
$$

implies

$$
\left\| d_k' \right\|_G^2 - 2\big(w^* - \bar{w}^k \big)^\top G d_k' = \left\| w^k - w^* \right\|_G^2 - \left\| \bar{w}^k - w^* \right\|_G^2.
\tag{2.19}
$$

Substituting (2.19) in (2.17), we get the assertion of this lemma. $\qquad\square$

Using Theorem 2.1 and Theorem 2.3, we get

$$
\Theta_1(\tau_k) \ge (1 - \rho)\Lambda_1(\tau_k),
\tag{2.20}
$$

where

$$
\Lambda_1(\tau_k) = \tau_k \left\{ \left\| d_k' \right\|_G^2 + \left(\frac{1 - \mu}{1 + \mu} \right)\Phi(\alpha_k) \right\} - \tau_k^2 \left\| d_k' \right\|_G^2.
\tag{2.21}
$$

$\Lambda_1(\tau_k)$ measures the progress obtained in the kth iteration. It is natural to choose a step length τ_k which maximizes the progress. Note that $\Lambda_1(\tau_k)$ is a quadratic function of τ_k and it reaches its maximum at

$$
\tau_k^* = \frac{\left\| d_k' \right\|_G^2 + \big(\frac{1 - \mu}{1 + \mu} \big)\Phi(\alpha_k)}{2\left\| d_k' \right\|_G^2}
$$

and

$$
\Lambda_1\big(\tau_k^* \big) = \frac{\tau_k^* \big\{ \left\| d_k' \right\|_G^2 + \big(\frac{1 - \mu}{1 + \mu} \big)\Phi(\alpha_k) \big\}}{2}.
\tag{2.22}
$$

Using (2.13), we have

$$
\begin{aligned}
\tau_k^* &\ge \frac{(1 - \mu)^2(1 - \eta^2)}{4(1 + \mu)^2}\left(\frac{\left\| d_k' \right\|_G^2 + \left\| w^k - \tilde{w}^k \right\|_G^2}{2\left\| d_k' \right\|_G^2} \right) \\
&\ge \frac{(1 - \mu)^2(1 - \eta^2)}{8(1 + \mu)^2},
\end{aligned}
\tag{2.23}
$$

which implies

$$
\begin{aligned}
\Lambda_1\big(\tau_k^*\big) &\geq \frac{\tau_k^*(\frac{1-\mu}{1+\mu})\Phi(\alpha_k)}{2} \\
&\geq \frac{(1-\mu)^4(1-\eta^2)^2}{64(1+\mu)^4}\big\|w^k - \tilde{w}^k\big\|_G^2.
\end{aligned}
\tag{2.24}
$$

Remark 2.1 If $\rho = 0$ and $\tau_k^* = 1$, Algorithm 2.1 reduces to the method proposed in [1]. Since τ_k^* is to maximize the profit function $\Lambda_1(\tau)$, we have

$$
\Lambda_1\big(\tau_k^*\big) \geq \Lambda_1(1).
\tag{2.25}
$$

Inequalities (2.20) and (2.25) show theoretically that Algorithm 2.1 is expected to make more progress than the method proposed in [1] at each iteration, and so it explains theoretically that Algorithm 2.1 outperforms the method proposed in [1].

Inspired and motivated by the first method, we next propose the following new LQP alternating direction method for solving SVI.

Algorithm 2.2

Step 0. *The initial step*:
 Given $\varepsilon > 0$, $\mu \in (0,1)$, and $w^1 = (x^1, y^1, \lambda^1) \in \mathcal{R}_{++}^n \times \mathcal{R}_{++}^m \times \mathcal{R}^l$ and $d_0'' = (0,0,0)$. Set $k = 1$.

Step 1. *Prediction step*:
 Compute $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k) \in \mathcal{R}_{++}^n \times \mathcal{R}_{++}^m \times \mathcal{R}^l$ by solving the system (1.7a)-(1.7c).

Step 2. *Convergence verification*:
 If $\{\max \|x^k - \tilde{x}^k\|_\infty, \|y^k - \tilde{y}^k\|_\infty, \|\lambda^k - \tilde{\lambda}^k\|_\infty\} < \epsilon$, then stop.

Step 3. *Correction step*:
 Compute $\bar{w}^k = (\bar{x}^k, \bar{y}^k, \bar{\lambda}^k)$ by solving the system (2.4a)-(2.4c). The new iterate $w^{k+1}(\beta_k)$ is given by

$$
w^{k+1}(\beta_k) = \rho w^k + (1-\rho)P_{\mathcal{W}}\big[w^k - \beta_k d_k''\big], \quad \rho \in (0,1),
\tag{2.26}
$$

where

$$
d_k'' := d_k' + \sigma_k d_{k-1}'',
\tag{2.27}
$$

$$
\sigma_k = \max\left(0, \frac{-d_k'^\top G d_{k-1}''}{\|d_{k-1}''\|_G^2}\right),
$$

and

$$
\beta_k = \frac{\|d_k'\|_G^2 + (\frac{1-\mu}{1+\mu})\Phi(\alpha_k)}{2\|d_k''\|_G^2}.
\tag{2.28}
$$

Set $k := k + 1$ and go to Step 1.

The following lemma has motivated us to choose the new descent direction as d_k''.

Lemma 2.2 *For any $k \geq 1$, we have*

$$d_{k-1}''^{\top} G(w^k - w^*) \geq 0. \tag{2.29}$$

Proof From (2.18) and (2.10) it is easy to show that

$$\begin{aligned}
(w^k - w^*)^{\top} G d_k' &\geq \frac{\|d_k'\|_G^2 + (\frac{1-\mu}{1+\mu})\Phi(\alpha_k)}{2} \\
&\geq \frac{(1-\mu)^2(1-\eta^2)}{8(1+\mu)^2}\|w^k - \tilde{w}^k\|_G^2. \tag{2.30}
\end{aligned}$$

Note that (2.29) is trivially true for $k = 1$ since $d_0'' = (0,0,0)$. By induction, consider any $k \geq 2$ and assume that

$$d_{k-2}''^{\top} G(w^{k-1} - w^*) \geq 0.$$

Using the definition of d_{k-1}'', we have

$$\begin{aligned}
d_{k-1}''^{\top} G(w^k - w^*) &= d_{k-1}''^{\top} G(w^{k-1} - w^*) + d_{k-1}''^{\top} G(w^k - w^{k-1}) \\
&= d_{k-1}'^{\top} G(w^{k-1} - w^*) + \sigma_{k-1} d_{k-2}''^{\top} G(w^{k-1} - w^*) \\
&\quad + d_{k-1}''^{\top} G(w^k - w^{k-1}) \\
&\geq d_{k-1}'^{\top} G(w^{k-1} - w^*) + d_{k-1}''^{\top} G(w^k - w^{k-1}) \\
&\geq \frac{\|d_{k-1}'\|_G^2 + (\frac{1-\mu}{1+\mu})\Phi(\alpha_{k-1})}{2} \\
&\quad - \|d_{k-1}''\|_G \|P_{\mathcal{W}}[w^{k-1} - \beta_{k-1} d_{k-1}''] - w^{k-1}\|_G \\
&\geq \frac{\|d_{k-1}'\|_G^2 + (\frac{1-\mu}{1+\mu})\Phi(\alpha_{k-1})}{2} - \beta_{k-1}\|d_{k-1}''\|_G^2 \\
&= 0,
\end{aligned}$$

where the second inequality follows from (2.30). Hence, the lemma is proved. □

Remark 2.2 From (2.30) and Lemma 2.2, we get

$$(w^k - w^*)^{\top} G d_k'' \geq \frac{(1-\mu)^2(1-\eta^2)}{8(1+\mu)^2}\|w^k - \tilde{w}^k\|_G^2.$$

Then $-d_k''$ is a descent direction of $\frac{1}{2}\|w^k - w^*\|^2$ at the point w^k. Since $-d_k''$ is a descent direction of the distance function at w^k, along $-d_k''$, one can find a new iterate which is closer to the solution set. Due to this fact, we construct (2.26).

Lemma 2.3 *Let $w^* = (x^*, y^*, \lambda^*) \in \mathcal{W}^*$, then we have*

$$\Theta_2(\beta_k) \geq (1-\rho)\big(\beta_k\{\|d_k'\|_G^2 + \|w^k - w^*\|_G^2 - \|\bar{w}^k - w^*\|_G^2\} - \beta_k^2\|d_k''\|_G^2\big), \tag{2.31}$$

where

$$\Theta_2(\beta_k) = \|w^k - w^*\|_G^2 - \|w^{k+1}(\beta_k) - w^*\|_G^2. \tag{2.32}$$

Proof Similar to (2.17), we have

$$\begin{aligned}
\|w^{k+1}(\beta_k) - w^*\|_G^2 &\le \|w^k - w^*\|_G^2 - (1-\rho)\{2\beta_k(w^k - w^*)^\top G d_k'' - \beta_k^2 \|d_k''\|_G^2\} \\
&= \|w^k - w^*\|_G^2 - (1-\rho)\{2\beta_k(w^k - w^*)^\top G(d_k' + \sigma_k d_{k-1}'') - \beta_k^2 \|d_k''\|_G^2\} \\
&\le \|w^k - w^*\|_G^2 - (1-\rho)\{2\beta_k(w^k - w^*)^\top G d_k' - \beta_k^2 \|d_k''\|_G^2\}.
\end{aligned}$$

Using (2.19) and the definition of $\Theta_2(\beta_k)$, we get the assertion of this lemma. $\qquad\square$

Using Theorem 2.1 and Lemma 2.3, we get

$$\Theta_2(\beta_k) \ge (1-\rho)\Lambda_2(\beta_k), \tag{2.33}$$

where

$$\begin{aligned}
\Lambda_2(\beta_k) &= \beta_k\left\{\|d_k'\|_G^2 + \left(\frac{1-\mu}{1+\mu}\right)\Phi(\alpha_k)\right\} - \beta_k^2 \|d_k'\|_G^2 \\
&= \frac{\beta_k\{\|d_k'\|_G^2 + (\frac{1-\mu}{1+\mu})\Phi(\alpha_k)\}}{2}.
\end{aligned} \tag{2.34}$$

Remark 2.3 From (2.27), it is easy to prove that

$$\|d_k''\|_G^2 \le \|d_k'\|_G^2,$$

which implies that

$$\tau_k^* \le \beta_k.$$

From (2.20), (2.22), (2.33), and (2.34), we have

$$\Theta_1(\tau_k^*) \ge (1-\rho)\Lambda_1(\tau_k^*) = \frac{\tau_k^*(1-\rho)\{\|d_k'\|_G^2 + (\frac{1-\mu}{1+\mu})\Phi(\alpha_k)\}}{2}$$

and

$$\Theta_2(\beta_k) \ge (1-\rho)\Lambda_2(\beta_k) = \frac{\beta_k(1-\rho)\{\|d_k'\|_G^2 + (\frac{1-\mu}{1+\mu})\Phi(\alpha_k)\}}{2}.$$

From the above inequalities, we obtain

$$\Lambda_2(\beta_k) \ge \Lambda_1(\tau_k^*). \tag{2.35}$$

Inequality (2.35) shows theoretically that Algorithm 2.2 can be expected to make more progress than Algorithm 2.1 at each iteration, and so it explains theoretically that Algorithm 2.2 outperforms Algorithm 2.1.

Now, we mainly focus on investigating the convergence of the proposed methods. The following theorem plays a crucial role in the convergence of the proposed methods.

Theorem 2.4 *Let w^{k+1} be the new iterate generated by*

$$w^{k+1} = w^{k+1}\big(\tau_k^*\big) \quad or \quad w^{k+1} = w^{k+1}(\beta_k).$$

Then, for any $w^* \in \mathcal{W}^*$, w^k and $\tilde{w}^k$ are bounded, and

$$\big\| w^{k+1} - w^* \big\|^2 \leq \big\| w^k - w^* \big\|^2 - c \big\| w^k - \tilde{w}^k \big\|_G^2, \tag{2.36}$$

where

$$c := \frac{(1 - \mu)^4 (1 - \eta^2)^2}{64(1 + \mu)^4} > 0.$$

Proof Inequality (2.36) follows from (2.20), (2.24), (2.33), and (2.35) immediately. It follows from (2.36) that

$$\big\| w^{k+1} - w^* \big\| \leq \big\| w^k - w^* \big\| \leq \cdots \leq \big\| w^0 - w^* \big\|,$$

and thus, $\{w^k\}$ is a bounded sequence.

It follows from (2.36) that

$$\sum_{k=0}^{\infty} c \big\| w^k - \tilde{w}^k \big\|_G^2 < +\infty,$$

which means that

$$\lim_{k \to \infty} \big\| w^k - \tilde{w}^k \big\|_G = 0.$$

Since $\{w^k\}$ is a bounded sequence, we conclude that $\{\tilde{w}^k\}$ is also bounded. □

The convergence of the proposed method can be proved by using similar arguments to [1]. Hence the proof is omitted.

Theorem 2.5 [1] *The sequence $\{w^k\}$ generated by the proposed methods converges to some w^∞ which is a solution of SVI.*

3 Conclusions

In this paper, we proposed two new modified logarithmic-quadratic proximal alternating direction methods for solving structured variational inequalities. The first one can be viewed as an extension of the method proposed in [1] by performing an additional step at each iteration and another optimal step length is employed to reach substantial progress in each iteration while the second method generates the new iterate by searching the optimal step size along a new descent direction. It is proved theoretically that the lower-bound of the progress obtained by the second method is greater than that by the first one. Global convergence of the proposed method is proved under mild assumptions.

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