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On monotone pointwise contractions in Banach spaces with a graph

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Abstract
In this work, we give a new definition of G-monotone pointwise contraction mappings in metric spaces endowed with a graph. Then we obtain sufficient conditions for the existence of a fixed point for such mappings. The proofs are based on the crucial inequality (GK).

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1 Introduction

The notion of asymptotic pointwise mappings was introduced in [1–4]. The use of ultrapower technique was useful in proving some related fixed point results. In the paper [3], the authors gave simple and elementary proofs for the existence of fixed point theorems for asymptotic pointwise mappings without the use of ultrapowers. In [5], most of these results were extended to metric spaces. In this paper, we introduce the new concept of G-monotone mappings in Banach spaces. Indeed, recently a new direction has been discovered dealing with the extension of the Banach contraction principle to metric spaces endowed with a partial order. The first attempt was successfully carried by Ran and Reurings [6]. In particular, they show how this extension is useful when dealing with some special matrix equations. Another similar approach was carried by Nieto and Rodríguez-López [7] who used such arguments in solving some differential equations. In [8], Jachymski gave a more general unified version of these extensions by considering graphs instead of a partial order. Recently, the author [9] showed the existence of fixed points for monotone multivalued mappings on a metric space with a graph.

In this work, we investigate the fixed point theory of pointwise G-monotone contraction mappings. In particular, we will extend the main result of [3] to the case of G-monotone mappings. Our approach is new and different from the ideas found in [6, 7]. This work was inspired by [10].

For more on metric fixed point theory, the reader may consult the book [11].

2 Graph basic definitions

The terminology of graph theory instead of partial ordering gives a wider picture and yields interesting generalization of the Banach contraction principle. In this section, we give the basic graph theory definitions and notations which will be used throughout.

A graph is an ordered pair (V,E), where V is a set and E is a binary relation on V ($E \subseteq V \times V$). Elements of E are called edges. We are concerned here with directed graphs (digraphs) that have a loop at every vertex (*i.e.*, $(a,a) \in E$ for each $a \in V$). Such digraphs are called reflexive. In this case $E \subseteq V \times V$ corresponds to a reflexive (and symmetric) binary relation on V. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices. By G^{-1} we denote the conversion of a graph G, *i.e.*, the graph obtained from G by reversing the direction of edges. Thus we have

$$E\bigl(G^{-1}\bigr) = \bigl\{(y,x) \mid (x,y) \in E(G)\bigr\}.$$

A digraph G is called an oriented graph if whenever $(u,v) \in E(G)$, then $(v,u) \notin E(G)$. The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\widetilde{G}) = E(G) \cup E\bigl(G^{-1}\bigr).$$

Given a digraph $G = (V,E)$, a (di)path of G is a sequence $a_0, a_1, \dots, a_n, \dots$ with $(a_i, a_{i+1}) \in E(G)$ for each $i = 0,1,2,\dots$. A finite path $(a_0, a_1, \dots, a_n)$ is said to have length $n+1$ for $n \in \mathbb{N}$. A closed directed path of length $n > 1$ from x to y, *i.e.*, $x = y$, is called a directed cycle. An acyclic digraph is a digraph that has no directed cycle. A digraph is connected if there is a finite (di)path joining any two of its vertices and it is weakly connected if $\widetilde{G}$ is connected.

Definition 2.1 A digraph G is transitive if

$$(x,y) \in E(G) \quad \text{and} \quad (y,z) \in E(G) \quad \Rightarrow \quad (x,z) \in E(G) \quad \text{for all } x,y,z \in V(G).$$

Definition 2.2 Let $(X, \|\cdot\|)$ be a Banach space. ω is called a weak-cluster point of a sequence $(x_n)_{n\in\mathbb{N}}$ in X if there exists a subsequence $(x_{\phi(n)})_{n\in\mathbb{N}}$ such that $(x_{\phi(n)})_{n\in\mathbb{N}}$ converges weakly to ω.

As Jachymski [8] did, we introduce the following property.

Let $(X, \|\cdot\|)$ be a Banach space and G be a reflexive digraph defined on X. We say that $E(G)$ has property $(*)$ if

$(*)$ for any sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ and ω is a weak-cluster point of $(x_n)_{n\in\mathbb{N}}$, then there exists a subsequence $(x_{\phi(n)})_{n\in\mathbb{N}}$ which converges weakly to ω and $(x_{\phi(n)}, \omega) \in E(G)$ for every $n \geq 1$.

Note that if G is a reflexive transitive digraph defined on X, then property $(*)$ implies the following property:

for any sequence $(x_n)_{n\geq 1}$ in X such that $(x_n, x_{n+1}) \in E(G)$ for $n \geq 1$ and ω is a weak-cluster point of $(x_n)_{n\geq 1}$, we have $(x_n, \omega) \in E(G)$ for every $n \geq 1$.

Let us finish this section with the following example of a transitive cyclic digraph which can not be generated by a partial order. Therefore our approach is different from the one used in [10] which is based on the use of a partial order in Banach and metric spaces.

Example 2.1 Let $(l_2, \|\cdot\|)$ be the classical Hilbert space. Define the digraph G on l_2 by: $(x,y) \in E(G)$ if and only if $x_i \leq y_i$, for $i = 2, \dots$, where $x = (x_n)$ and $y = (y_n)$ are in l_2.

Then G is reflexive, transitive for which G-intervals are convex and closed. Note that G contains cycles. Indeed, we have $(x,y) \in E(G)$ and $(y,x) \in E(G)$, where

$$x = (1,0,0,\dots) \quad \text{and} \quad y = (2,0,0,\dots).$$

3 Monotone pointwise contraction mappings

Let us start this section by defining G-monotone pointwise Lipschitzian mappings.

Definition 3.1 Let (X,d) be a metric space and G be a reflexive digraph defined on X. Let C be a nonempty subset of X. A mapping $T : C \to C$ is said to be

(1) G-monotone if T is edge preserving, *i.e.*, $(T(x), T(y)) \in E(G)$ whenever $(x,y) \in E(G)$ for any $x, y \in C$.

(2) G-monotone pointwise Lipschitzian if T is G-monotone and for any $x \in X$, there exists $k(x) \in [0,+\infty)$ such that

$$d\bigl(T(x), T(y)\bigr) \le k(x) d(x,y) \quad \text{for any } y \in C \text{ such that } (x,y) \in E(\widetilde{G}).$$

If $k(x) \in [0,1)$ for any $x \in X$, then T is said to be a G-monotone pointwise contraction mapping. If $k(x) \le 1$ for any $x \in X$, then T is said to be a G-monotone nonexpansive mapping. A fixed point of T is any element $x \in C$ such that $T(x) = x$. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$.

It is clear that the pointwise contractive concept was introduced to extend the contractive behavior in the Banach contraction principle.

Example 3.1 As Kirk did in [12], we consider K a bounded closed convex subset of the Hilbert space l^2. Let $F : K \to K$ be such that F is continuously Fréchet differentiable on a convex open set containing K. Then F is a pointwise contraction on K if and only if $\|F'_x\| < 1$ for each $x \in K$, where F'_x denotes the Fréchet derivative of F at x. Next consider the metric space

$$M = \{0,1\} \times K = \bigl\{(0,x), (1,x); x \in K\bigr\}.$$

The distance d on M is defined by

$$d\bigl((\varepsilon_1, x_1), (\varepsilon_2, x_2)\bigr) = |\varepsilon_1 - \varepsilon_2| + \|x_1 - x_2\|.$$

Let G be the graph with M as its vertex set and its edge set $E(G)$ defined by the following two conditions:

(1) $(0,x)$ and $(1,y)$ are not connected for any $x, y \in K$;

(2) (ε, x) and (ε, y) are connected if and only if $x \le y$ (using the natural pointwise order in l^2) for any $\varepsilon \in \{0,1\}$ and $x, y \in K$.

Define the mapping $T : M \to M$ by

$$T\bigl((\varepsilon, x)\bigr) = \bigl(1 - \varepsilon, F(x)\bigr).$$

If we choose F to be G-monotone on K, *i.e.*, $(x,y) \in E(G)$ implies that $(F(x), F(y)) \in E(G)$ for any $x, y \in K$. Then T is a G-monotone pointwise contraction on M. Indeed, any two vertices of G are connected if and only if they have the same first component. Next we notice

$$\begin{aligned}
d\big(T\big((\varepsilon,x)\big),T\big((\varepsilon,y)\big)\big) &= d\big(\big(1-\varepsilon,F(x)\big),\big(1-\varepsilon,F(y)\big)\big) \\
&= \big\|F(x)-F(y)\big\| \\
&\leq \alpha(x)\|x-y\| \\
&= \alpha(x)d\big((\varepsilon,x),(\varepsilon,y)\big),
\end{aligned}$$

where $\alpha(x)\in[0,1)$, for any $\varepsilon\in\{0,1\}$ and $x,y\in K$ with $(x,y)\in E(G)$. Clearly we used the fact that F is a pointwise contraction on K. But T is not a pointwise contraction on M since

$$d\big(T\big((0,x)\big),T\big((1,x)\big)\big) = d\big((0,x),(1,x)\big) = 1$$

for any $x\in K$.

For more examples on fixed points of multivalued mappings on metric spaces endowed with a graph, see [13].

The fundamental fixed point result for pointwise contraction mappings is the following theorem.

Theorem 3.1 [1, 3] *Let C be a weakly compact convex subset of a Banach space and suppose that $T:C\to C$ is a pointwise contraction. Then T has a unique fixed point z. Moreover, the orbit $(T^n(x))_{n\geq1}$ converges to z for each $x\in C$.*

Note that if T is a G-monotone pointwise Lipschitzian mapping, then it is not necessarily continuous by contrast to the case of pointwise Lipschitzian mappings. Since the main focus of this paper is about the existence of the fixed points, we have the following result.

Theorem 3.2 *Let (X,d) be a metric space and G be a reflexive digraph defined on X. Let C be a nonempty subset of X. Let $T:C\to C$ be a G-monotone pointwise contraction. If $a\in\operatorname{Fix}(T)$, then for any $x\in X$ such that $(a,x)\in E(G)$, we have $(T^n(x))_{n\geq1}$ converges to a. In particular, if a and b are two fixed points of T and $(a,b)\in E(G)$, then we must have $a=b$.*

Proof Let $a\in\operatorname{Fix}(T)$ and $(a,x)\in E(G)$. Since T is G-monotone, we have $(a,T^n(x))\in E(G)$ for any $n\geq1$. Using the definition of pointwise contraction, we get

$$d\big(T^n(x),a\big)\leq k(a)d\big(T^{n-1}(x),a\big)\leq k(a)^n d(x,a)$$

for any $n\geq1$. Since $k(a)<1$, we conclude that $(T^n(x))_{n\in\mathbb{N}}$ converges to a. Obviously if a and b are two fixed points of T and $(a,b)\in E(G)$, then we have $(T^n(b))_{n\in\mathbb{N}}=(b)_{n\in\mathbb{N}}$ converges to a, which implies $a=b$. □

Remark 3.1 In both Banach and metric spaces [1, 5], the pointwise contraction mappings have at most one fixed point. But in the case of G-monotone pointwise contraction mappings, we may have more than one fixed point. Indeed Jachymski [8] proved that

G-contractions have a fixed point in each component of elements that are compatible. Since we do not assume the weak connectivity of the digraph G, we may have more than one component which implies the possibility to have more than one fixed point.

The crucial part in dealing with pointwise contractions is the existence of the fixed point. Usually it takes more assumptions than the classical Banach contraction principle.

4 Existence of fixed point of monotone pointwise contractions

In this section, we investigate the existence of a fixed point of G-monotone pointwise contraction mappings. Since Theorem 3.1 is done in the linear case, we will assume that $(X, \|\cdot\|)$ is a Banach space and G is a reflexive digraph defined on X. Moreover, the linear convexity of X is assumed to be compatible with the graph structure in the following sense:

(CG) If $(x,y) \in E(G)$ and $(w,z) \in E(G)$, then

$$\big(\alpha x + (1-\alpha)w, \alpha y + (1-\alpha)z\big) \in E(G)$$

for all $x, y, w, z \in X$ and $\alpha \in \mathbb{R}^+$.

Note that the classical proof of Theorem 3.1 will not work in the setting of G-monotone mappings. The main difficulty encountered in this setting has to do with the fact that the mappings do not have a good behavior on the entire sets. They do have a good behavior only on connected points. For this reason, our investigation is based on a constructive approach initiated by Krasnoselskii [14].

Lemma 4.1 *Let $(X, \|\cdot\|)$ be a Banach space and G be a reflexive digraph defined on X. Assume that $E(G)$ has properties $(*)$ and* (CG). *Let C be a nonempty convex subset of X. Let $T : C \to C$ be a G-monotone mapping. Fix $\lambda \in (0,1)$ and $x_1 \in C$. Consider the Krasnoselskii iteration sequence $(x_n)_{n\geq 1} \subset C$ defined by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T(x_n), \quad n \geq 1. \tag{KIS}$$

(i) *If $(x_1, T(x_1)) \in E(G)$, then we have $(x_n, x_{n+1}) \in E(G)$ for any $n \geq 1$.*
(ii) *If $(T(x_1), x_1) \in E(G)$, then we have $(x_{n+1}, x_n) \in E(G)$ for any $n \geq 1$.*

Proof We will prove (i). The proof of (ii) is similar and will be omitted. As $(x_1, T(x_1)) \in E(G)$ and $(x_1, x_1) \in E(G)$, we have by property (CG)

$$\big((1-\lambda)x_1 + \lambda x_1, (1-\lambda)x_1 + \lambda T(x_1)\big) \in E(G),$$

i.e., $(x_1, x_2) \in E(G)$. Now assume that $(x_{n-1}, x_n) \in E(G)$ for $n > 1$. Since T is G-monotone, we have $(T(x_{n-1}), T(x_n)) \in E(G)$. Using property (CG) again, we get

$$\big((1-\lambda)x_{n-1} + \lambda T(x_{n-1}), (1-\lambda)x_n + \lambda T(x_n)\big) \in E(G),$$

i.e., $(x_n, x_{n+1}) \in E(G)$. Hence, by induction, we have $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 1$. □

In order to show that the main property satisfied by the sequence is defined by (KIS), we need the following result which may be found in [14, 15]. We will give its proof here.

Lemma 4.2 *Let $(X, \|\cdot\|)$ be a Banach space and G be a reflexive digraph defined on X. Assume that $E(G)$ has properties $(*)$ and* (CG). *Let C be a nonempty convex subset of X. Let $T : C \to C$ be a G-monotone nonexpansive mapping. Assume there exists $x_1 \in C$ such that $(x_1, T(x_1)) \in E(\widetilde{G})$. Consider the sequence $(x_n)_{n\geq 1}$ defined by* (KIS). *Then we have*

$$(1 + n\lambda)\big\|T(x_i) - x_i\big\| \leq \big\|T(x_{i+n}) - x_i\big\| + (1-\lambda)^{-n}\big(\big\|T(x_i) - x_i\big\| - \big\|T(x_{i+n}) - x_{i+n}\big\|\big) \qquad (1)$$

for any $i, n \in \mathbb{N}$.

Proof Without loss of any generality, we may assume $(x_1, T(x_1)) \in E(G)$. We will prove this inequality by induction on $n \in \mathbb{N}$. The inequality is obvious when $n = 0$. Fix $n \geq 1$ and assume the inequality holds for any $i \in \mathbb{N}$. In particular, we have

$$\begin{aligned}(1 + n\lambda)\big\|T(x_{i+1}) - x_{i+1}\big\| \leq{} & \big\|T(x_{i+1+n}) - x_{i+1}\big\| + (1-\lambda)^{-n}\big\|T(x_{i+1}) - x_{i+1}\big\| \\ & - (1-\lambda)^{-n}\big\|T(x_{i+1+n}) - x_{i+1+n}\big\|.\end{aligned}$$

Lemma 4.1 implies that $(x_m, x_{m+1}) \in E(G)$ for any $m \geq 1$. Since T is a G-monotone nonexpansive mapping, we get

$$\big\|T(x_{m+1}) - T(x_m)\big\| \leq \|x_{m+1} - x_m\|,$$

and $(T(x_m), T(x_{m+1})) \in E(G)$ for any $m \geq 1$. Since

$$\begin{aligned}\big\|T(x_{i+1+n}) - x_{i+1}\big\| &\leq (1-\lambda)\big\|T(x_{i+n+1}) - x_i\big\| + \lambda\big\|T(x_{i+n+1}) - T(x_i)\big\| \\ &\leq (1-\lambda)\big\|T(x_{i+n+1}) - x_i\big\| + \lambda\sum_{k=0}^{n}\big\|T(x_{i+k+1}) - T(x_{i+k})\big\| \\ &\leq (1-\lambda)\big\|T(x_{i+n+1}) - x_i\big\| + \lambda\sum_{k=0}^{n}\|x_{i+k+1} - x_{i+k}\|,\end{aligned}$$

we get

$$\begin{aligned}(1 + n\lambda)\big\|T(x_{i+1}) - x_{i+1}\big\| \leq{} & (1-\lambda)\big\|T(x_{i+n+1}) - x_i\big\| + \lambda\sum_{k=0}^{n}\|x_{i+k+1} - x_{i+k}\| \\ & + (1-\lambda)^{-n}\big(\big\|T(x_{i+1}) - x_{i+1}\big\| - \big\|T(x_{i+1+n}) - x_{i+1+n}\big\|\big),\end{aligned}$$

which implies

$$\begin{aligned}\big\|T(x_{i+n+1}) - x_i\big\| \geq{} & \frac{(1+n\lambda)}{(1-\lambda)}\big\|T(x_{i+1}) - x_{i+1}\big\| - \frac{\lambda}{(1-\lambda)}\sum_{k=0}^{n}\|x_{i+k+1} - x_{i+k}\| \\ & - (1-\lambda)^{-n-1}\big(\big\|T(x_{i+1}) - x_{i+1}\big\| - \big\|T(x_{i+1+n}) - x_{i+1+n}\big\|\big).\end{aligned}$$

Note that $(\|T(x_m) - x_m\|)_{m\geq 1}$ is a decreasing sequence. Indeed, we have

$$\|x_{m+1} - x_m\| = \lambda\big\|T(x_m) - x_m\big\|, \quad m \geq 1.$$

So to show that $(\|T(x_m) - x_m\|)_{m\geq 1}$ is decreasing, we only need to prove that $(\|x_{m+1} - x_m\|)_{m\geq 1}$ is decreasing, which is true since

$$\begin{aligned}\|x_{m+2} - x_{m+1}\| &\leq (1-\lambda)\|x_{m+1} - x_m\| + \lambda\big\|T(x_{m+1}) - T(x_m)\big\| \\ &\leq (1-\lambda)\|x_{m+1} - x_m\| + \lambda\|x_{m+1} - x_m\| = \|x_{m+1} - x_m\|\end{aligned}$$

for any $m \geq 1$. Using this fact and $1 + n\lambda \leq (1-\lambda)^{-n}$, we get

$$\begin{aligned}\big\|T(x_{i+n+1}) - x_i\big\| &\geq (1-\lambda)^{-n-1}\big[\big\|T(x_{i+n+1}) - x_{i+n+1}\big\| - \big\|T(x_{i+1}) - x_{i+1}\big\|\big] \\ &\quad + \frac{(1+n\lambda)}{(1-\lambda)}\big\|T(x_{i+1}) - x_{i+1}\big\| - \frac{\lambda^2(n+1)}{(1-\lambda)}\big\|T(x_i) - x_i\big\| \\ &= (1-\lambda)^{-n-1}\big[\big\|T(x_{i+n+1}) - x_{i+n+1}\big\| - \big\|T(x_i) - x_i\big\|\big] \\ &\quad + \left(\frac{(1+n\lambda)}{(1-\lambda)} - (1-\lambda)^{-n-1}\right)\big\|T(x_{i+1}) - x_{i+1}\big\| \\ &\quad + \left((1-\lambda)^{-n-1} - \frac{\lambda^2(n+1)}{(1-\lambda)}\right)\big\|T(x_i) - x_i\big\| \\ &\geq (1-\lambda)^{-n-1}\big[\big\|T(x_{i+n+1}) - x_{i+n+1}\big\| - \big\|T(x_i) - x_i\big\|\big] \\ &\quad + \left(\frac{(1+n\lambda)}{(1-\lambda)} - (1-\lambda)^{-n-1}\right)\big\|T(x_i) - x_i\big\| \\ &\quad + \left((1-\lambda)^{-n-1} - \frac{\lambda^2(n+1)}{(1-\lambda)}\right)\big\|T(x_i) - x_i\big\| \\ &= (1-\lambda)^{-n-1}\big[\big\|T(x_{i+n+1}) - x_{i+n+1}\big\| - \big\|T(x_i) - x_i\big\|\big] \\ &\quad + \big(1 + (n+1)\lambda\big)\big\|T(x_i) - x_i\big\|.\end{aligned}$$

This is our inequality when we take $n+1$ instead of n. Therefore by induction inequality (1) is true for any $i, n \in \mathbb{N}$. □

As a direct consequence of Lemma 4.2, we get the following result.

Theorem 4.1 *Let* $(X, \|\cdot\|)$ *be a Banach space and* G *be a reflexive digraph defined on* X. *Assume that* $E(G)$ *has properties* $(*)$ *and* (CG). *Let* C *be a bounded nonempty convex subset of* X. *Let* $T : C \to C$ *be a* G-*monotone nonexpansive mapping. Assume that there exists* $x_1 \in C$ *such that* $(x_1, T(x_1)) \in E(\widetilde{G})$. *Consider the sequence* $(x_n)_{n\geq 1}$ *defined by* (KIS). *Then we have* $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$.

Proof Using Lemma 4.2, we know that the inequality

$$(1 + n\lambda)\big\|T(x_i) - x_i\big\| \leq \big\|T(x_{i+n}) - x_i\big\| + (1-\lambda)^{-n}\big(\big\|T(x_i) - x_i\big\| - \big\|T(x_{i+n}) - x_{i+n}\big\|\big)$$

holds for any $i, n \in \mathbb{N}$. Since $(\|x_n - T(x_n)\|)_{n\geq 1}$ is decreasing, we set $\lim_{n\to+\infty} \|x_n - T(x_n)\| = R$. Then, if we let $i \to +\infty$ in the above inequality, we obtain

$$(1 + n\lambda)R \leq \delta(C)$$

for any $n \in \mathbb{N}$, where $\delta(C) = \sup\{\|x - y\|; x, y \in C\} < +\infty$. Obviously this will imply a contradiction if we assume $R \neq 0$. Therefore we must have

$$\lim_{n\to+\infty} \|x_n - T(x_n)\| = 0. \qquad \square$$

Such a sequence $(x_n)_{n\geq 1}$ is known as an approximate fixed point sequence of T. Assume that there exists $x_1 \in C$ such that $(x_1, T(x_1)) \in E(G)$. Let ω be a weak-cluster point of (x_n). Since $E(G)$ has property $(*)$ and $(x_n, x_{n+1}) \in E(G)$, for any $n \geq 1$, there exists a subsequence $(x_{\phi(n)})$ of (x_n) such that $(x_{\phi(n)})$ weakly converges to ω and $(x_{\phi(n)}, \omega) \in E(G)$ for any $n \geq 1$. If we assume that T is a G-monotone pointwise contraction, then T is a G-monotone nonexpansive mapping. Moreover, we have

$$\|T(x_{\phi(n)}) - T(\omega)\| \leq k(\omega)\|x_{\phi(n)} - \omega\|, \quad n \geq 1. \tag{2}$$

Assume that X satisfies the large Opial property [16] which says that for any sequence $(y_n) \subset X$ which weakly converges to y, we have

$$\liminf_{n\to\infty} \|y_n - y\| \leq \liminf_{n\to\infty} \|y_n - z\|$$

for any $z \in X$. Using inequality (2) above, we get

$$\liminf_{n\to\infty} \|T(x_{\phi(n)}) - T(\omega)\| \leq k(\omega) \liminf_{n\to\infty} \|x_{\phi(n)} - \omega\|.$$

Using Theorem 4.1, we conclude that

$$\liminf_{n\to\infty} \|x_{\phi(n)} - T(\omega)\| \leq k(\omega) \liminf_{n\to\infty} \|x_{\phi(n)} - \omega\|.$$

The large Opial property will imply

$$\liminf_{n\to\infty} \|x_{\phi(n)} - \omega\| \leq \liminf_{n\to\infty} \|x_{\phi(n)} - T(\omega)\| \leq k(\omega) \liminf_{n\to\infty} \|x_{\phi(n)} - \omega\|.$$

Since $k(\omega) < 1$, we obtain $\liminf_{n\to\infty} \|x_{\phi(n)} - \omega\| = 0$. Combined with the conclusion of Theorem 4.1, we get $T(\omega) = \omega$, *i.e.*, ω is a fixed point of T. In other words, we proved the following result.

Theorem 4.2 *Let $(X, \|\cdot\|)$ be a Banach space and G be a reflexive digraph defined on X. Assume that $E(G)$ has properties $(*)$ and* (CG). *Assume X satisfies the large Opial property. Let C be a weakly compact nonempty convex subset of X. Let $T : C \to C$ be a G-monotone pointwise contraction. Assume that there exists $x_1 \in C$ such that $(x_1, T(x_1)) \in E(G)$. Then T has a fixed point.*

We do not know whether the conclusion of Theorem 4.2 holds if X does not satisfy the large Opial condition. However, if the digraph G is transitive, we may get the conclusion of Theorem 4.2 without such condition. Indeed instead of assuming property $(*)$, we will assume that G-intervals are closed. Recall that a G-interval is any subset of the form

(i) $[a, \to) = \{x \in X; (a, x) \in E(G)\}$,
(ii) $(\leftarrow, a] = \{x \in X; (x, a) \in E(G)\}$

for any $a \in X$. Using the convexity properties of G, we conclude that G-intervals are convex and closed. Therefore they are also weakly-closed. Under these assumptions, we have the following result which is the analogue to Theorem 3.1.

Theorem 4.3 *Let* $(X, \|\cdot\|)$ *be a Banach space and* G *be a reflexive transitive digraph defined on* X. *Assume that* $E(G)$ *has property* (CG) *and* G*-intervals are closed. Let* C *be a weakly compact nonempty convex subset of* X. *Let* $T : C \to C$ *be a* G*-monotone pointwise contraction. Assume that there exists* $x_1 \in C$ *such that* $(x_1, T(x_1)) \in E(\widetilde{G})$. *Then* T *has a fixed point.*

Proof Without loss of any generality, we may assume that $(x_1, T(x_1)) \in E(G)$. Let (x_n) be the sequence generated by x_1 and defined by (KIS). Since C is weakly compact, then (x_n) has a weak-cluster point $\omega \in C$. Since G is transitive, then we have $(x_n, \omega) \in E(G)$ for any $n \geq 1$. Since T is G-monotone, we get $(T(x_n), T(\omega)) \in E(G)$ for any $n \geq 1$. Using the conclusion of Theorem 4.1, we know that ω is also a weak-cluster point of $(T(x_n))$. Since the G-intervals are weakly closed, we conclude that $(\omega, T(\omega)) \in E(G)$. Consider the set

$$C_\omega = [\omega, \to) \cap C = \bigl\{x \in C; (\omega, x) \in E(G)\bigr\}.$$

Then C_ω is a nonempty closed convex subset of C. Hence C_ω is weakly compact. Let $x \in C_\omega$, then we have $(T(\omega), T(x)) \in E(G)$ since T is G-monotone. Using the transitivity of G, we get $(\omega, T(x)) \in E(G)$, *i.e.*, $T(x) \in C_\omega$. Next we consider the type function $\tau : C_\omega \to [0, +\infty)$ defined by

$$\tau(x) = \limsup_{n\to+\infty} \|x_n - x\|.$$

It is obvious that τ is convex and continuous. Since C_ω is weakly compact and convex, we conclude that there exists $z \in C_\omega$ such that $\tau(z) = \inf\{\tau(x); x \in C_\omega\}$. Since $(x_n, \omega) \in E(G)$, by the transitivity of G, we get $(x_n, z) \in E(G)$ for any $n \geq 1$. Since T is a G-monotone pointwise contraction, we get

$$\bigl\|T(x_n) - T(z)\bigr\| \leq k(z)\|x_n - z\|, \quad n = 1, 2, \ldots.$$

Hence

$$\limsup_{n\to+\infty} \bigl\|T(x_n) - T(z)\bigr\| \leq k(z) \limsup_{n\to+\infty} \|x_n - z\|.$$

Using the conclusion of Theorem 4.1, we get

$$\limsup_{n\to+\infty} \bigl\|x_n - T(z)\bigr\| = \limsup_{n\to+\infty} \bigl\|T(x_n) - T(z)\bigr\| \leq k(z) \limsup_{n\to+\infty} \|x_n - z\|,$$

i.e., $\tau(T(z)) \leq k(z)\tau(z)$. Since $\tau(z) \leq \tau(T(z))$ and $k(z) < 1$, we get $\tau(z) = 0$, *i.e.*, $\limsup_{n\to+\infty} \|x_n - z\| = 0$. So (x_n) converges to z. Since $\tau(T(z)) \leq k(z)\tau(z)$, we get $\tau(T(z)) = 0$ which implies also that (x_n) converges to $T(z)$. Therefore we must have $T(z) = z$, *i.e.*, z is a fixed point of T. □

Competing interests
The author declares that he has no competing interests.

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References

1. Kirk, WA: Fixed points of asymptotic contractions. J. Math. Anal. Appl. **277**, 645-650 (2003)
2. Kirk, WA: Asymptotic pointwise contractions. In: Plenary Lecture: The 8th International Conference on Fixed Point Theory and Its Applications, Chiang Mai University, Thailand, 16-22 July (2007)
3. Kirk, WA, Xu, H-K: Asymptotic pointwise contractions. Nonlinear Anal. **69**, 4706-4712 (2008)
4. Reich, S, Zaslavski, AJ: A convergence theorem for asymptotic contractions. J. Fixed Point Theory Appl. **4**, 27-33 (2008)
5. Hussain, N, Khamsi, MA: On asymptotic pointwise contractions in metric spaces. Nonlinear Anal., Theory Methods Appl. **71**, 4423-4429 (2009)
6. Ran, ACM, Reurings, MCB: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**, 1435-1443 (2004)
7. Nieto, JJ, Rodríguez-López, R: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22**, 223-239 (2005)
8. Jachymski, J: The contraction principle for mappings on a metric space with a graph. Proc. Am. Math. Soc. **136**, 1359-1373 (2007)
9. Alfuraidan, MR: Remarks on monotone multivalued mappings on a metric space with a graph. J. Inequal. Appl. **2015**, 202 (2015). doi:10.1186/s13660-015-0712-6
10. Abdou, AAN, Khamsi, MA: On monotone pointwise contractions in Banach and metric spaces. Preprint
11. Khamsi, MA, Kirk, WA: An Introduction to Metric Spaces and Fixed Point Theory. Wiley, New York (2001)
12. Kirk, WA: Mappings of generalized contractive type. J. Math. Anal. Appl. **32**, 567-572 (1970)
13. Nicolae, A, O'Regan, D, Petrusel, A: Fixed point theorems for singlevalued and multivalued generalized contractions in metric spaces endowed with a graph. Georgian Math. J. **18**, 307-327 (2011)
14. Goebel, K, Kirk, WA: Topics in Metric Fixed Point Theory. Cambridge Stud. Adv. Math., vol. 28. Cambridge University Press, Cambridge (1990)
15. Goebel, K, Kirk, WA: Iteration processes for nonexpansive mappings. Contemp. Math. **21**, 115-123 (1983)
16. Opial, Z: Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. **73**, 591-597 (1967)