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Modified Picard-Mann hybrid iteration process for total asymptotically nonexpansive mappings

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Abstract

In this paper, using the modified hybrid Picard-Mann iteration process, we establish Δ-convergence and strong convergence theorems for total asymptotically nonexpansive mappings on a *CAT*(0) space. Results established in the paper extend and improve a number of results in the literature. A numerical example is also given to examine the fastness of the proposed iteration process under different control conditions and initial points.

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Keywords: total asymptotically nonexpansive mappings; Δ-convergence and strong convergence theorem; *CAT*(0) space

1 Introduction

Let K be a nonempty, closed and convex subset of a normed linear space E. A mapping $T: K \to K$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for every x, y in K. In the last four decades, many papers have appeared in the literature on the iteration methods to approximate fixed points of a nonexpansive mapping, *cf.* [1–6] and the references therein. In the meantime, some generalizations of nonexpansive mappings have appeared, namely asymptotically nonexpansive mapping [7], asymptotically nonexpansive type mapping [8], asymptotically nonexpansive mappings in the intermediate sense [9].

Recently Alber *et al.* [10] made an effort to unify some generalization of nonexpansive mappings and introduced the notion of total asymptotically nonexpansive mappings. A mapping $T: K \to K$ is said to be *total asymptotically nonexpansive* if there exist nonnegative real sequences $\{k_n^{(1)}\}$ and $\{k_n^{(2)}\}$, $n \ge 1$, $k_n^{(1)}, k_n^{(2)} \to 0$ as $n \to \infty$, and a strictly increasing and continuous function $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that

$$\|T^n x - T^n y\| \le \|x - y\| + k_n^{(1)} \phi(\|x - y\|) + k_n^{(2)}$$

for every x, y in K. They further studied the iterative approximation of fixed point of total asymptotically nonexpansive mappings using a modified Mann iteration process. In the modified Mann iteration process [11] the sequence $\{x_n\}$ is generated by

$$\left.\begin{array}{l} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1, \end{array}\right\} \tag{1.1}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying certain conditions.

Iterative approximation of fixed points of total nonexpansive mappings has also been studied by [12–15].

In 2013, Khan [16] introduced a new iteration process for nonexpansive mappings, which he called 'Picard-Mann hybrid iteration process' and claimed that this process is independent of Picard and Mann iterative process and the convergence process is faster than Picard and Mann iteration process. The sequence $\{x_n\}$ in this process is given by

$$\left.\begin{array}{l} x_1 \in K, \\ x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 1, \end{array}\right\} \tag{1.2}$$

where $\{\alpha_n\}$ is in $(0,1)$.

On the other hand, in 2003, Kirk [17, 18] initiated the study of fixed point theory in metric spaces with nonpositive curvature. He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete $CAT(0)$ space always has a fixed point. After his work, fixed point theory in $CAT(0)$ spaces has been rapidly developed and many papers have appeared [19–24]. Nanjaras and Panyanak [25] proved the demiclosed principle for asymptotically nonexpansive mappings in $CAT(0)$ space and obtained a Δ-convergence theorem for the Mann iteration. Abbas *et al.* [26] proved the demiclosed principle for asymptotically nonexpansive mappings in the intermediate sense and established convergence theorems, Tang *et al.* [27] proved the demiclosed principle for total asymptotically nonexpansive mappings in a $CAT(0)$ space and obtained Δ-convergence theorem.

Motivated by the above recorded studies, in this paper, we propose a 'modified hybrid Picard-Mann' iteration process for iterative approximation of fixed points of total asymptotically nonexpansive mappings in $CAT(0)$ spaces. The sequence $\{x_n\}$ in this iteration is given by

$$\left.\begin{array}{l} x_1 \in C, \\ x_{n+1} = T^n y_n, \\ y_n = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n, \quad n \in \mathbb{N}, \end{array}\right\} \tag{1.3}$$

where C is a nonempty bounded closed and convex subset of a complete $CAT(0)$ space X and $\{\alpha_n\}$ is in $(0,1)$.

We establish some strong and Δ-convergence results of the iterative process (1.3) for total asymptotically nonexpansive mappings on a $CAT(0)$ space. Our results extend and improve the corresponding results of Chang *et al.* [28], Nanjaras and Panyanak [25] and others.

2 Preliminaries

Throughout the paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers.

The following lemma plays an important role in our paper.

Lemma 2.1 ([23], Lemma 2.4) *Let X be a $CAT(0)$ space, then*

$$d\big((1-t)x \oplus ty, z\big) \leq (1-t)d(x,z) + td(y,z)$$

for all $t \in [0,1]$ and x, y, z are points in X.

Let C be a nonempty subset of a $CAT(0)$ space X.

Definition 2.1 A mapping $T\colon C \to C$ is said to be a *nonexpansive* mapping if

$$d(Tx, Ty) \leq d(x,y), \quad \forall x,y \in C.$$

Definition 2.2 A mapping $T\colon C \to C$ is said to be *asymptotically nonexpansive* if there is a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that

$$d\big(T^n x, T^n y\big) \leq k_n d(x,y), \quad \forall n \geq 1, x, y \in C.$$

Definition 2.3 A mapping $T\colon C \to C$ is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$d\big(T^n x, T^n y\big) \leq L d(x,y), \quad \forall n \geq 1, x, y \in C.$$

Definition 2.4 A mapping $T\colon C \to C$ is said to be $(\{\nu_n\},\{\mu_n\},\zeta)$-*total asymptotically nonexpansive* if there exist nonnegative sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\{\nu_n\} \to 0$, $\{\mu_n\} \to 0$ and a strictly increasing continuous function $\zeta : [0,\infty) \to [0,\infty)$ with $\zeta(0) = 0$ such that

$$d\big(T^n x, T^n y\big) \leq d(x,y) + \nu_n \zeta\big(d(x,y)\big) + \mu_n, \quad \forall n \geq 1, x, y \in C.$$

From the definitions, we see that each nonexpansive mapping is an asymptotically nonexpansive mapping with a sequence $\{k_n = 1\}$, and each asymptotically nonexpansive mapping is a $(\{\nu_n\},\{\mu_n\},\zeta)$- total asymptotically nonexpansive mapping with $\mu_n = 0$, $\nu_n = k_n - 1$, $n \geq 1$ and $\zeta(t) = t$, $t \geq 0$, and each asymptotically nonexpansive mapping is a uniformly L-Lipschitzian mapping with $L = \sup\{k_n\}$, $n \geq 1$.

Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space X. For $x \in X$, we set

$$r\big(x,\{x_n\}\big) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\big(\{x_n\}\big) = \inf\big\{r\big(x,\{x_n\}\big) : x \in X\big\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A\big(\{x_n\}\big) = \big\{x \in X : r\big(x,\{x_n\}\big) = r\big(\{x_n\}\big)\big\}.$$

In a $CAT(0)$ space, $A(\{x_n\})$ consists of exactly one point [22], Proposition 7.

A sequence $\{x_n\}$ in X is said to Δ-converge to $p \in X$ if p is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim x_n = p$ and call p the Δ-limit of $\{x_n\}$.

Lemma 2.2 *In a complete* $CAT(0)$ *space* X, *the following hold*:

(i) *Every bounded sequence in a complete* $CAT(0)$ *space always has a* Δ*-convergent subsequence* [24], *p.*3690.

(ii) *If* $\{x_n\}$ *is a bounded sequence in a closed convex subset* C *of* X, *then the asymptotic center of* $\{x_n\}$ *is in* C [29], *Proposition* 2.1.

(iii) *If* $\{x_n\}$ *is a bounded sequence in* X *with* $A(\{x_n\}) = \{p\}$, $\{u_n\}$ *is a subsequence of* $\{x_n\}$ *with* $A(\{u_n\}) = \{u\}$ *and the sequence* $\{d(x_n, u)\}$ *converges, then* $p = u$ [23], *Lemma* 2.8.

The following results are useful to prove our main result.

Lemma 2.3 ([25], Lemma 4.5) *Let* X *be a* $CAT(0)$ *space*, $x \in X$ *be a given point and* $\{t_n\}$ *be a sequence in* $[b, c]$ *with* $b, c \in (0,1)$ *and* $0 < b(1-c) \leq \frac{1}{2}$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be any sequences in* X *such that*

$$\begin{cases} \limsup_{n\to\infty} d(x_n, x) \leq r, \\ \limsup_{n\to\infty} d(y_n, x) \leq r \text{ and} \\ \limsup_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r, \end{cases}$$

for some $r \geq 0$. *Then* $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 2.4 ([30], Lemma 2) *Let* $\{a_n\}$, $\{\lambda_n\}$ *and* $\{c_n\}$ *be the sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1 + \lambda_n)a_n + c_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $\sum_{n=1}^{\infty} c_n < \infty$, *then* $\lim_{n\to\infty} a_n$ *exists. If there exists a subsequence* $\{a_{n_i}\} \subset \{a_n\}$ *such that* $a_{n_i} \to 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 ([28], Theorem 2.8) *Let* C *be a closed convex subset of a complete* $CAT(0)$ *space* X *and let* $T: C \to C$ *be a total asymptotically nonexpansive and uniformly L-Lipschitzian mapping. Let* $\{x_n\}$ *be a bounded sequence in* C *such that* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ *and* $\Delta\text{-}\lim_{n\to\infty} x_n = p$. *Then* $Tp = p$.

The following existence result is also needed.

Lemma 2.6 ([31], Corollary 3.2) *Let* C *be a nonempty bounded closed convex subset of a complete* $CAT(0)$ *space* X. *If* $T: C \to C$ *is a continuous total asymptotically nonexpansive mapping, then* T *has a fixed point*.

3 Main results

We now establish a $\triangle$-convergence result for the modified Picard-Mann hybrid iterative process.

Theorem 3.1 *Let C be a bounded closed and convex subset of a complete $CAT(0)$ space X. Let $T: C \to C$ be a uniformly L-Lipschitzian and $(\{\nu_n\}, \{\mu_n\}, \zeta)$-total asymptotically nonexpansive mapping. If the following conditions are satisfied*:

(1) $\sum_{n=1}^{\infty} \nu_n < \infty$; $\sum_{n=1}^{\infty} \mu_n < \infty$;

(2) *there exist constants* $a, b \in (0,1)$ *with* $0 < b(1-a) \leq \frac{1}{2}$ *such that* $\{\alpha_n\} \subset [a,b]$;

(3) *there exists a constant* $M^* > r$ *such that* $\zeta(r) \leq M^* r$, $r \geq 0$;

then the sequence $\{x_n\}$ *defined by* (1.3) Δ*-converges to some point* $p \in F(T)$, *where* $F(T)$ *is the set of fixed points of* T.

Proof Since T is Lipschitz continuous, $F(T) \neq \emptyset$ by Lemma 2.6. We divide the proof of Theorem 3.1 into three steps.

Step-I. First we prove that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$.

Step-II. Next we prove that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Step-III. Finally to show that the sequence $\{x_n\}$ Δ-converges to a fixed point of T, we prove that

$$W_\Delta(x_n) = \bigcup_{\{u_n\}\subset\{x_n\}} A\bigl(\{u_n\}\bigr) \subseteq F(T)$$

and $W_\Delta(x_n)$ consists of exactly one point.

Proof of Step-I: For each $p \in F(T)$, we have

$$\begin{aligned} d(y_n, p) &= d\bigl((1-\alpha_n)x_n \oplus \alpha_n T^n x_n, p\bigr) \\ &\leq (1-\alpha_n) d(x_n, p) + \alpha_n d\bigl(T^n x_n, p\bigr) \\ &\leq (1-\alpha_n) d(x_n, p) + \alpha_n \bigl\{d(x_n, p) + \nu_n \zeta\, d(x_n, p) + \mu_n\bigr\} \\ &\leq d(x_n, p) + \beta_n \nu_n \zeta\, d(x_n, p) + \mu_n \\ &\leq \bigl(1 + \nu_n M^*\bigr) d(x_n, p) + \mu_n. \end{aligned} \tag{3.1}$$

Also,

$$\begin{aligned} d(x_{n+1}, p) &= d\bigl(T^n y_n, p\bigr) \\ &\leq d(y_n, p) + \nu_n \zeta\, d(y_n, p) + \mu_n \\ &\leq \bigl(1 + \nu_n M^*\bigr) d(y_n, p) + \mu_n \\ &\leq \bigl(1 + \nu_n M^*\bigr)\bigl(1 + \nu_n M^*\bigr) d(x_n, p) + \bigl(1 + \nu_n M^*\bigr)\mu_n + \mu_n \\ &\leq \bigl(1 + \nu_n M^*\bigl(2 + \nu_n M^*\bigr)\bigr) d(x_n, p) + \bigl(2 + \nu_n M^*\bigr)\mu_n \\ &= (1 + \sigma_n) d(x_n, p) + \rho_n, \end{aligned}$$

where $\sigma_n = \nu_n M^*(2 + \nu_n M^*)$ and $\rho_n = (2 + \nu_n M^*)\mu_n$.

It follows from condition (i) and Lemma 2.4 that $\lim_{n\to\infty} d(x_n, p)$ exists.

Proof of Step-II. It follows from Step-I that $\lim_{n\to\infty} d(x_n, p)$ exists for each given $p \in F$. Without loss of generality, we can assume that

$$\lim_{n\to\infty} d(x_n, p) = r \geq 0.$$

From (3.1) we have

$$\liminf_{n\to\infty} d(y_n, p) \leq \limsup_{n\to\infty} d(y_n, p) \leq \lim_{n\to\infty} \{(1 + \nu_n M^*) d(x_n, p) + \mu_n\} = r \geq 0. \tag{3.2}$$

Also,

$$\begin{aligned} d(T^n y_n, p) &= d(T^n y_n, T^n p) \\ &\leq d(y_n, p) + \nu_n \zeta\, d(y_n, p) + \mu_n \\ &\leq (1 + \nu_n M^*) d(y_n, p) + \mu_n, \quad \forall n \geq 1, \end{aligned}$$

so we have

$$\limsup_{n\to\infty} d(T^n y_n, p) \leq r.$$

Similarly,

$$\limsup_{n\to\infty} d(T^n x_n, p) \leq r.$$

Now,

$$d(x_{n+1}, p) \leq (1 + \nu_n M^*) d(y_n, p) + \mu_n,$$

taking limit infimum, we have

$$\liminf_{n\to\infty} d(x_{n+1}, p) \leq \liminf_{n\to\infty} d(y_n, p),$$

i.e.,

$$r \leq \liminf_{n\to\infty} d(y_n, p). \tag{3.3}$$

From (3.2) and (3.3), we have

$$\lim_{n\to\infty} d(y_n, p) = r,$$

i.e.,

$$r = \lim_{n\to\infty} d(y_n, p) = \lim_{n\to\infty} d((1 - \alpha_n) x_n \oplus \alpha_n T^n x_n, p).$$

Therefore, by Lemma 2.3, we obtain

$$\lim_{n\to\infty} d(x_n, T^n x_n) = 0. \tag{3.4}$$

Also,

$$\begin{aligned} d(T^n y_n, T^n x_n) &\le d(y_n, x_n) + \nu_n \zeta d(y_n, x_n) + \mu_n \\ &\le (1 + \nu_n M^*) d(y_n, x_n) + \mu_n \\ &\le (1 + \nu_n M^*) d((1-\alpha_n) T^n x_n \oplus \alpha_n x_n, x_n) + \mu_n \\ &\le (1 + \nu_n M^*) \{(1-\alpha_n) d(T^n x_n, x_n) + \alpha_n d(x_n, x_n)\} + \mu_n, \end{aligned}$$

from (3.4), we have

$$\lim_{n\to\infty} d(T^n y_n, T^n x_n) = 0, \tag{3.5}$$

and

$$\begin{aligned} d(x_{n+1}, x_n) &\le d(x_{n+1}, T^n y_n) + d(T^n y_n, T^n x_n) + d(T^n x_n, x_n) \\ &= d(T^n y_n, T^n y_n) + d(T^n y_n, T^n x_n) + d(T^n x_n, x_n), \end{aligned}$$

using (3.4) and (3.5) we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Finally,

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) + d(T^{n+1} x_{n+1}, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n) \\ &\le d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) + Ld(x_n, x_{n+1}) + Ld(T^n x_n, x_n) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence, Step-II is proved.

Proof of Step-III. Let $u \in W_\Delta(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.2(i), there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{n\to\infty} v_n = v \in C$. By Lemma 2.5, $v \in F(T)$. Since $\{d(u_n, v)\}$ converges, by Lemma 2.2(iii), $u = v$. This shows that $W_\Delta(x_n) \subseteq F(T)$.

Now we prove that $W_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F(T)$. Finally, since $\{d(x_n, v)\}$ converges, by Lemma 2.2(iii), we have $x = v \in F(T)$. This shows that $W_\Delta(x_n) = \{x\}$. □

We now establish some strong convergence results.

Theorem 3.2 *Let X, C, T, $\{\alpha_n\}$, $\{\beta_n\}$, $\{x_n\}$ satisfy the hypothesis of Theorem* 3.1. *Then the sequence $\{x_n\}$ generated by* (1.3) *converges strongly to a fixed point of T if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0,$$

where $d(x, F(T)) = \inf\{d(x, p) : p \in F(T)\}$.

Proof Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. As proved in Step-I of Theorem 3.1, $\lim_{n\to\infty} d(x_n, F(T))$ exists for all $p \in F(T)$. Thus, by hypothesis, $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in C. Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists a positive integer n_0 such that for all $n \geq n_0$,

$$d\bigl(x_n, F(T)\bigr) < \frac{\varepsilon}{4}.$$

In particular, $\inf\{d(x_{n_0}, p) : p \in F(T)\} < \frac{\varepsilon}{4}$. Thus, there exists $p^* \in F(T)$ such that

$$d\bigl(x_{n_0}, p^*\bigr) < \frac{\varepsilon}{2}.$$

Now, for all $m, n \geq n_0$, we have

$$d(x_{n+m}, x_n) \leq d\bigl(x_{n+m}, p^*\bigr) + d\bigl(x_n, p^*\bigr) \leq 2d\bigl(x_{n_0}, p^*\bigr) < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon,$$

i.e., $\{x_n\}$ is a Cauchy sequence in the closed subset C of a complete $CAT(0)$ space and hence it converges to a point q in C. Now, $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(q, F(T)) = 0$ and closedness of $F(T)$ forces q to be in $F(T)$. This completes the proof. □

Senter and Dotson [32], p.375, introduced the concept of Condition (I) as follows.

Definition 3.1 A mapping $T: C \to C$ is said to *satisfy Condition (I)* if there exists a nondecreasing function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that

$$d(x, Tx) \geq f\bigl(d\bigl(x, F(T)\bigr)\bigr)$$

for all $x \in C$.

It is weaker than demicompactness for a nonexpansive mapping T defined on a bounded set. Since every completely continuous mapping $T: K \to K$ is continuous and demicompact, so it satisfies Condition (I). Recently, Kim [33] gave an interesting example of total asymptotically nonexpansive self-mapping satisfying Condition (I).

Example 1 ([33], Example 3.7) Let $X := \mathbb{R}$ and $C := [0, 2]$. Define $T: C \to C$ by the formula

$$Tx = \begin{cases} 1, & x \in [0, 1]; \\ \frac{1}{\sqrt{3}}\sqrt{4 - x^2}, & x \in [1, 2]. \end{cases}$$

Here T is a uniformly continuous and total asymptotically nonexpansive mapping with $F(T) = \{1\}$. Also T satisfies Condition (I), but T is not Lipschitzian and hence it is not an asymptotically nonexpansive mapping.

Using Condition (I), we now establish the following strong convergence result for total asymptotically nonexpansive mapping.

Theorem 3.3 *Let X, C, T, $\{\alpha_n\}$, $\{\beta_n\}$, $\{x_n\}$ satisfy the hypothesis of Theorem* 3.1 *and let T be a mapping satisfying Condition* (*I*). *Then the sequence $\{x_n\}$ generated by* (1.3) *converges strongly to a fixed point of T.*

Proof As proved in Theorem 3.2, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Also, by Step-II of Theorem 3.1, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It follows from Condition (I) that

$$\lim_{n\to\infty} f\bigl(d\bigl(x_n, F(T)\bigr)\bigr) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

That is, $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a non-decreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r > 0$, we obtain

$$\lim_{n\to\infty} d\bigl(x_n, F(T)\bigr) = 0.$$

Now all the conditions of Theorem 3.3 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a point of $F(T)$. □

4 Numerical example

In this section, using Example 1 of a total asymptotically nonexpansive mapping, we compare the convergence of modified Picard-Mann hybrid iteration process (1.3) with the modified Mann iteration process (1.1).

Table 1 Iterates of modified Mann and modified Picard-Mann hybrid iterations

Iterate	$x_1 = 1.1$		$x_1 = 1.5$		$x_1 = 1.9$	
	Iteration (1.1)	Iteration (1.3)	Iteration (1.1)	Iteration (1.3)	Iteration (1.1)	Iteration (1.3)
x_2	1.03218253804965	0.98903980508798	1.13188130791299	0.95198821855406	1.13027756377320	0.95262315543817
x_3	1.01072751268322	1.00000000000000	1.04396043597100	1.00000000000000	1.04342585459107	1.00000000000000
x_4	1.00268187817080	1.00000000000000	1.01099010899275	1.00000000000000	1.01085646364777	1.00000000000000
x_5	1.00053637563416	1.00000000000000	1.00219802179855	1.00000000000000	1.00217129272955	1.00000000000000
x_6	1.00008939593903	1.00000000000000	1.00036633696643	1.00000000000000	1.00036188212159	1.00000000000000
x_7	1.00001277084843	1.00000000000000	1.00005233385235	1.00000000000000	1.00005169744594	1.00000000000000
x_8	1.00000159635605	1.00000000000000	1.00000654173154	1.00000000000000	1.00000646218074	1.00000000000000
x_9	1.00000017737289	1.00000000000000	1.00000072685906	1.00000000000000	1.00000071802008	1.00000000000000
x_{10}	1.00000001773729	1.00000000000000	1.00000007268591	1.00000000000000	1.00000007180201	1.00000000000000
x_{11}	1.00000000161248	1.00000000000000	1.00000000660781	1.00000000000000	1.00000000652746	1.00000000000000
x_{12}	1.00000000013437	1.00000000000000	1.00000000055065	1.00000000000000	1.00000000054395	1.00000000000000
x_{13}	1.00000000001034	1.00000000000000	1.00000000004236	1.00000000000000	1.00000000004184	1.00000000000000
x_{14}	1.00000000000074	1.00000000000000	1.00000000000303	1.00000000000000	1.00000000000299	1.00000000000000
x_{15}	1.00000000000005	1.00000000000000	1.00000000000020	1.00000000000000	1.00000000000020	1.00000000000000
x_{16}	1.00000000000000	1.00000000000000	1.00000000000001	1.00000000000000	1.00000000000001	1.00000000000000
x_{17}	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000

Table 2 Comparison of fastness for different control conditions

α_n	Least number of iterate to reach the fixed point 1					
	$x_1 = 1.1$		$x_1 = 1.5$		$x_1 = 1.9$	
	Iteration (1.1)	Iteration (1.3)	Iteration (1.1)	Iteration (1.3)	Iteration (1.1)	Iteration (1.3)
0.3	x_{87}	x_3	x_{91}	x_3	x_{92}	x_3
0.5	x_{45}	x_3	x_{47}	x_3	x_{47}	x_3
0.84	x_{19}	x_2	x_{21}	x_2	x_{21}	x_2
0.95	x_{13}	x_2	x_{14}	x_2	x_{14}	x_2
$\frac{n}{n+1}$	x_{16}	x_3	x_{17}	x_3	x_{17}	x_3
$1-\frac{1}{\sqrt{n+1}}$	x_{26}	x_3	x_{27}	x_3	x_{28}	x_3
$\frac{1}{\sqrt{n+1}}-\frac{1}{(n+1)^2}$	x_{235}	x_3	x_{258}	x_3	x_{261}	x_3
$\frac{1}{\sqrt{n+5}}$	x_{265}	x_3	x_{288}	x_3	x_{295}	x_3
$\frac{1}{\sqrt{2n+5}}$	x_{492}	x_3	x_{537}	x_3	x_{555}	x_3

Set $\alpha_n = \frac{n}{n+1}$ for all $n \geq 1$. Using MATLAB, we computed the iterates of (1.1) and (1.3) for three different initial points $x_1 = 1.1$, $x_1 = 1.5$ and $x_1 = 1.9$. Both iterations converge to the fixed point 1, the detailed observation is given in Table 1.

Next we examine the fastness of both iterations for different control conditions. Summary of the findings is given in Table 2, where the least number of iterates required to obtain the fixed point for different initial conditions is given under various control conditions.

From Table 1 and Table 2, it is clear that for a total asymptotically nonexpansive mapping, modified Mann iteration (1.1) is very much sensitive about the choice of initial point and control condition α_n, whereas the behavior of proposed iteration (1.3) is consistent.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References
1. Halpern, B: Fixed points of nonexpanding maps. Bull. Am. Math. Soc. **73**, 957-961 (1967)
2. Mann, WR: Mean value methods in iteration. Proc. Am. Math. Soc. **4**, 506-510 (1953)
3. Reich, S: Weak convergence theorems for nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. **67**, 274-276 (1979)
4. Shioji, N, Takahashi, W: Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces. Proc. Am. Math. Soc. **125**, 3641-3645 (1997)
5. Takahashi, W, Kim, GE: Approximating fixed points of nonexpansive mappings in Banach spaces. Math. Jpn. **48**, 1-9 (1998)
6. Wittmann, R: Approximation of fixed points of nonexpansive mappings. Arch. Math. (Basel) **58**, 486-491 (1992)
7. Goebel, K, Kirk, WA: A fixed point theorem for asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **35**, 171-174 (1972)
8. Kirk, WA: Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type. Isr. J. Math. **17**, 339-346 (1974)
9. Bruck, RE, Kuczumow, Y, Reich, S: Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property. Colloq. Math. **65**(2), 169-179 (1993)
10. Alber, YI, Chidume, CE, Zegeye, H: Approximating fixed points of total asymptotically nonexpansive mappings. Fixed Point Theory Appl. **2006**, Article ID 10673 (2006)
11. Schu, J: Iterative construction of fixed points of asymptotically nonexpansive mappings. J. Math. Anal. Appl. **158**(2), 407-413 (1991)
12. Chidume, CE, Ofoedu, EU: Approximation of common fixed points for finite families of total asymptotically nonexpansive mappings. J. Math. Anal. Appl. **333**(1), 128-141 (2007)
13. Alber, Y, Espinola, R, Lorenzo, P: Strongly convergent approximations to fixed points of total asymptotically nonexpansive mappings. Acta Math. Sin. Engl. Ser. **24**(6), 1005-1022 (2008)
14. Mukhamedov, F, Saburov, M: Strong convergence of an explicit iteration process for a totally asymptotically I-nonexpansive mapping in Banach spaces. Appl. Math. Lett. **23**(12), 1473-1478 (2010)
15. Qin, X, Cho, SY, Kang, SM: A weak convergence theorem for total asymptotically pseudocontractive mappings in Hilbert spaces. Fixed Point Theory Appl. **2011**, Article ID 859795 (2011)
16. Khan, SH: A Picard-Mann hybrid iterative process. Fixed Point Theory Appl. **2013**, Article ID 69 (2013)
17. Kirk, WA: Geodesic geometry and fixed point theory. In: Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003). Colecc. Abierta, vol. 64, pp. 195-225. Univ. Sevilla Secr. Publ., Seville (2003)
18. Kirk, WA: Geodesic geometry and fixed point theory II. In: International Conference in Fixed Point Theory and Applications, pp. 113-142. Yokohama Publishers, Yokohama (2004)
19. Chaoha, P, Phon-on, A: A note on fixed point sets in CAT(0) spaces. J. Math. Anal. Appl. **320**(2), 983-987 (2006)
20. Dhompongsa, S, Kaewkhao, A, Panyanak, B: Lim's theorems for multivalued mappings in CAT(0) spaces. J. Math. Anal. Appl. **312**(2), 478-487 (2005)

21. Dhompongsa, S, Kaewkhao, A, Panyanak, B: On Kirk's strong convergence theorem for multivalued nonexpansive mappings on CAT(0) spaces. Nonlinear Anal. **75**(2), 459-468 (2012)
22. Dhompongsa, S, Kirk, WA, Sims, B: Fixed points of uniformly Lipschitzian mappings. Nonlinear Anal., Theory Methods Appl. **65**(4), 762-772 (2006)
23. Dhompongsa, S, Panyanak, B: On Δ-convergence theorems in CAT(0) spaces. Comput. Math. Appl. **56**(10), 2572-2579 (2008)
24. Kirk, WA, Panyanak, B: A concept of convergence in geodesic spaces. Nonlinear Anal., Theory Methods Appl. **68**(12), 3689-3696 (2008)
25. Nanjaras, B, Panyanak, B: Demiclosed principle for asymptotically nonexpansive mappings in CAT(0) spaces. Fixed Point Theory Appl. **2010**, Article ID 268780 (2010)
26. Abbas, M, Thakur, BS, Thakur, D: Fixed points of asymptotically nonexpansive mappings in the intermediate sense in CAT(0) spaces. Commun. Korean Math. Soc. **28**(1), 107-121 (2013)
27. Tang, JF, Chang, SS, Lee, HWJ, Chan, CK: Iterative algorithm and Δ-convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces. Abstr. Appl. Anal. **2012**, Article ID 965751 (2012)
28. Chang, SS, Wang, L, Lee, HWJ, Chan, CK, Yang, L: Demiclosed principle and Δ-convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces. Appl. Math. Comput. **219**(5), 2611-2617 (2012)
29. Dhompongsa, S, Kirk, WA, Panyanak, B: Nonexpansive set-valued mappings in metric and Banach spaces. J. Nonlinear Convex Anal. **8**(1), 35-45 (2007)
30. Qihou, L: Iterative sequences for asymptotically quasi-nonexpansive mappings with error member. J. Math. Anal. Appl. **259**(1), 18-24 (2001)
31. Panyanak, B: On total asymptotically nonexpansive mappings in $CAT(\kappa)$ spaces. J. Inequal. Appl. **2014**, Article ID 336 (2014)
32. Senter, HF, Dotson, WG: Approximating fixed points of nonexpansive mappings. Proc. Am. Math. Soc. **44**(2), 375-380 (1974)
33. Kim, GE: Strong convergence to fixed point of a total asymptotically nonexpansive mapping. Fixed Point Theory Appl. **2013**, Article ID 302 (2013)