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C^*-Valued contractive type mappings

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Abstract

In this paper we generalize the notion of C^*-valued contraction mappings, recently introduced by Ma *et al.*, by weakening the contractive condition introduced by them. Using the new notion of C^*-valued contractive type mappings, we establish a fixed point theorem for such mappings. Our result generalizes the result by Ma *et al.* and those contained therein except for the uniqueness.

Keywords: $C*$-algebra; contractions; fixed point theorems; orbits

1 Introduction and preliminaries

Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a contraction if $\exists\alpha \in (0,1)$ such that

$$d(Tx,Ty) \le \alpha d(x,y). \tag{1}$$

One of the most important tools used for the existence of solutions of many nonlinear problems arising in physics and engineering sciences is the Banach fixed point theorem which asserts that every contraction on a complete metric space has a unique fixed point. This theorem is also known as the Banach contraction principle (BCP), and it first appeared in its explicit form in Banach's PhD Thesis [1]. The strength of BCP lies in the fact that the underlying space is a quite general, complete metric space, while the conclusion is very strong, including even error estimates. Note that a mapping $T : X \to X$ satisfying (1) is uniformly continuous on X. Therefore, the Banach contraction condition forces the mapping T to be continuous. Given a mapping $T : X \to X$ and $x \in X$, the sequence of points $\mathcal{O}_T(x) = \{x, Tx, T^2x, \dots\}$ is called the orbit of x with respect to T. Hicks and Rhoades [2] showed that if a mapping $T : X \to X$ satisfies the following contractive condition

$$d\bigl(Ty, T^2y\bigr) \le hd(y,Ty) \tag{2}$$

for some $h \in (0,1)$ and every $y \in \mathcal{O}_T(x)$, then T has a fixed point. Note that the contractive condition (2) is weaker than condition (1). Moreover, condition (2) does not force the mapping T to be continuous [2]. In contrast to the Banach contraction principle, the Hicks and Rhodes theorem [2] does not guarantee the uniqueness of the fixed point of T.

Recently, Ma *et al.* [3] introduced the notion of C^*-valued metric spaces and, analogous to the Banach contraction principle, established a fixed point theorem for C^*-valued contraction mappings. In this paper, we first introduce the notion of continuity in the context of C^*-valued metric spaces and show that a C^*-valued contraction map is continuous with respect to our notion of continuity. Then we introduce a C^*-valued contractive type condition and establish a fixed point theorem analogous to the results presented in [2]. We also show that a C^*-valued contractive type map need not be continuous in the context of C^*-valued metric.

We now recollect some basic definitions, notations, and results that will be used subsequently. For details, we refer to [4, 5]. An algebra $\mathbb{A}$ together with a conjugate linear involution map $* : \mathbb{A} \to \mathbb{A}$, defined by $a \mapsto a^*$ such that for all $a, b \in \mathbb{A}$ we have $(ab)^* = b^*a^*$ and $(a^*)^* = a$, is called a $*$-algebra. Moreover, if $\mathbb{A}$ contains an identity element $1_{\mathbb{A}}$, then the pair $(\mathbb{A}, *)$ is called a unital $*$-algebra. A unital $*$-algebra $(\mathbb{A}, *)$ together with a complete sub multiplicative norm satisfying $\|a^*\| = \|a\|$ for all $a \in \mathbb{A}$ is called a Banach $*$-algebra. A C^*-algebra is a Banach $*$-algebra $(\mathbb{A}, *)$ such that $\|a^*a\| = \|a\|^2$ for all $a \in \mathbb{A}$. An element $a \in \mathbb{A}$ is called a positive element if $a = a^*$ and $\sigma(a) \subset \mathbb{R}_+$, where $\sigma(a) = \{\lambda \in \mathbb{R} : \lambda I - a \text{ is non-invertible}\}$. If $a \in \mathbb{A}$ is positive, we write it as $a \succeq 0_{\mathbb{A}}$. Using positive elements, one can define a partial ordering on $\mathbb{A}$ as follows: $b \succeq a$ if and only if $b - a \succeq 0_{\mathbb{A}}$. Each positive element a of a C^*-algebra $\mathbb{A}$ has a unique positive square root. Subsequently, $\mathbb{A}$ will denote a unital C^*-algebra with the identity element $1_{\mathbb{A}}$. Further, $\mathbb{A}_+$ is the set $\{a \in \mathbb{A} : a \succeq 0_{\mathbb{A}}\}$ of positive elements of $\mathbb{A}$ and $(a^*a)^{1/2} = |a|$. Using the concept of positive elements in $\mathbb{A}$, a C^*-algebra-valued metric space is defined in the following way.

Definition 1.1 [3] Let X be a nonempty set. A C^*-algebra-valued metric on X is a mapping $d : X \times X \to \mathbb{A}$ satisfying the following conditions:

(i) $0_{\mathbb{A}} \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_{\mathbb{A}} \Leftrightarrow x = y$,

(ii) $d(x, y) = d(y, x)$ $\forall x, y \in X$,

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ $\forall x, y, z \in X$.

The triplet $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued metric space.

A sequence $\{x_n\}$ in $(X, \mathbb{A}, d)$ is said to converge to $x \in X$ with respect to $\mathbb{A}$ if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|d(x_n, x)\| < \epsilon$ for all $n > N$. We write it as $\lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ is called a Cauchy sequence with respect to $\mathbb{A}$ if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|d(x_n, x_m)\| < \epsilon$ for all $n, m > N$. The triplet $(X, \mathbb{A}, d)$ is said to be a complete C^*-valued metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent. Now we state the definition and result from [3], for convenience.

Definition 1.2 [3] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space. A mapping $T : X \to X$ is said to be a C^*-algebra-valued contraction mapping on X if there exists $a \in \mathbb{A}$ with $\|a\| < 1$ such that

$$d(Tx, Ty) \preceq a^* d(x, y) a \quad \text{for all } x, y \in X. \tag{3}$$

Theorem 1.3 [3] *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued complete metric space and $T : X \to X$ satisfy* (3), *then T has a unique fixed point in X.*

From now on, we call a C^*-algebra-valued metric and a C^*-algebra-valued metric space simply a C^*-valued metric and a C^*-valued metric space, respectively.

2 Main results

We begin this section by introducing the notion of continuity in the context of C^*-valued metric spaces.

Definition 2.1 Let $(X,\mathbb{A},d)$ be a C^*-valued metric space. A mapping $T:X\to X$ is said to be continuous at x_0 with respect to $\mathbb{A}$ if given any $\epsilon > 0$ there exists $\delta > 0$ such that $\|d(Tx,Tx_0)\| < \epsilon$ whenever $\|d(x,x_0)\| < \delta$. T is said to be continuous on X with respect to $\mathbb{A}$ if it is continuous for every $x\in X$.

Example 2.2 Let $\mathbb{A}=\mathbb{R}^2$, then $\mathbb{A}$ is a C^*-algebra with pointwise operations of addition, multiplication, and scaler multiplication. The norm on $\mathbb{A}$ is defined by

$$\bigl\|(x,y)\bigr\| = \max\bigl(|x|,|y|\bigr), \tag{4}$$

where ordering on $\mathbb{A}$ is given by

$$(a,b)\preceq(c,d) \quad\Leftrightarrow\quad a\leq c \text{ and } b\leq d. \tag{5}$$

Let $X=[0,1]$, define a C^*-valued metric $d:X\times X\to\mathbb{A}$ on X by

$$d(x,y)=\bigl(|x-y|,0\bigr). \tag{6}$$

Then $T:X\to X$, given by $T(x)=\frac{x}{3}$, is continuous with respect to $\mathbb{A}$ since

$$\bigl\|d(Tx,Ty)\bigr\| = \left\|d\left(\frac{x}{3},\frac{y}{3}\right)\right\| = \left\|\frac{x}{3}-\frac{y}{3}\right\| < \epsilon \quad \text{whenever } \|x-y\| < \delta = 3\epsilon.$$

Remark 2.3 Note that every continuous self-map is continuous with respect $\mathbb{A}=\mathbb{R}$ and a C^*-valued contraction map is continuous with respect to the C^*-algebra $\mathbb{A}$.

Definition 2.4 A function $f:X\to\mathbb{A}$ is said to be T-orbitally lower semicontinuous at ξ with respect to $\mathbb{A}$ if there exist a mapping $T:X\to X$ and a sequence $\{x_n\}$ in $\mathcal{O}_T(x_0)$, for some $x_0\in X$, such that $\lim_{n\to\infty}x_n=\xi$ with respect to $\mathbb{A}$ implies

$$\bigl\|f(\xi)\bigr\| \leq \liminf\bigl\|f(x_n)\bigr\|. \tag{7}$$

Remark 2.5 If $\mathbb{A}=\mathbb{R}$, then our definition coincides with the usual definition of T-orbitally lower semicontinuous as defined by [2].

Example 2.6 Consider the C^*-algebra $\mathbb{A}=\mathbb{R}^2$ as defined in Example 2.2. Let $X=[-1,1]$ and define $f:X\to\mathbb{A}$ by

$$f(x)=\begin{cases}(\frac{x}{2},0) & \text{if } x\geq 0,\\ (|x-1|,0) & \text{if } x<0.\end{cases}$$

By taking $T:X\to X$, $Tx=\frac{x^2}{2}$, we see that, for $\frac{1}{2}\in[-1,1]$, we have

$$\mathcal{O}_T\left(\frac{1}{2}\right)=\left\{\frac{1}{2},\frac{1}{2^3},\frac{1}{2^7},\frac{1}{2^{15}},\dots\right\},$$

and any sequence $\{x_n\}$ in X converges to 0. Further,

$$\|f(0)\| = \|(0,0)\| = \liminf \|f(x_n)\|.$$

Thus f is T-orbitally lower semicontinuous at $x = 0$.

Definition 2.7 Let $(X, \mathbb{A}, d)$ be a C^*-valued metric space. A mapping $T : X \to X$ is said to be a C^*-valued contractive type mapping if $\exists x \in X$ and $a \in \mathbb{A}$ such that

$$d(Ty, T^2y) \preceq a^* d(y, Ty)a \quad \text{with } \|a\| < 1 \text{ for every } y \in \mathcal{O}_T(x). \tag{8}$$

Remark 2.8 A C^*-valued contraction mapping is a C^*-valued contractive type mapping, but the converse is not true as shown in the following example.

Example 2.9 Let $X = [-1, 1]$ and $\mathbb{A} = M_{2\times 2}(\mathbb{R})$ with $\|A\| = \max\{|a_1|, |a_2|, |a_3|, |a_4|\}$, where a_i's are the entries of the matrix $A \in M_{2\times 2}(\mathbb{R})$. Then $(X, \mathbb{A}, d)$ is a C^*-algebra-valued metric space, where

$$d(x,y) = \begin{bmatrix} |x-y| & 0 \\ 0 & |x-y| \end{bmatrix},$$

and partial ordering on $\mathbb{A}$ is given as

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \succeq \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \quad \Leftrightarrow \quad a_i \geq b_i \quad \text{for } i = 1, 2, 3, 4.$$

Define a mapping $T : X \to X$ by

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \geq 0, \\ 1 & \text{if } x < 0. \end{cases}$$

Then, for $y \in \mathcal{O}_T(x)$, $x \geq 0$,

$$\begin{aligned} d(Ty, T^2y) &= \begin{bmatrix} |\frac{y}{2} - \frac{y}{4}| & 0 \\ 0 & |\frac{y}{2} - \frac{y}{4}| \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} |\frac{y}{2}| & 0 \\ 0 & |\frac{y}{2}| \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= a^* d(y, Ty)a, \\ &\qquad \text{where } a = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \|a\| = \frac{1}{\sqrt{2}}. \end{aligned}$$

Thus T is a C^*-valued contractive type mapping. Note that T is not continuous with respect to the C^*-algebra $\mathbb{A}$ and hence not a C^*-valued contraction mapping.

Before giving our main result, we prove the following lemma which is essentially extracted from the proof of Theorem 1.3.

Lemma 2.10 *Let $\mathbb{A}$ be a C^*-algebra with the identity element $1_{\mathbb{A}}$ and x be a positive element of $\mathbb{A}$. If $a \in \mathbb{A}$ is such that $\|a\| < 1$, then for $m < n$ we have*

$$\lim_{n\to\infty} \sum_{k=m}^{n} (a^*)^k x a^k = 1_{\mathbb{A}} \|(x)^{1/2}\|^2 \left(\frac{\|a\|^m}{1-\|a\|}\right) \tag{9}$$

and

$$\sum_{k=m}^{n} (a^*)^k x a^k \longrightarrow 0_{\mathbb{A}} \quad \text{as } m \longrightarrow \infty. \tag{10}$$

Proof Since x is a positive element of $\mathbb{A}$, we have

$$\begin{aligned}
\sum_{k=m}^{n} (a^*)^k x a^k &= \sum_{k=m}^{n} (a^*)^k (x)^{1/2}(x)^{1/2} a^k \\
&= \sum_{k=m}^{n} \big((x)^{1/2}a^k\big)^* \big((x)^{1/2}a^k\big) \\
&= \sum_{k=m}^{n} \big|(x)^{1/2}a^k\big|^2 \\
&\preceq 1_{\mathbb{A}} \left\| \sum_{k=m}^{n} \big|(x)^{1/2}a^k\big|^2 \right\| \\
&\preceq 1_{\mathbb{A}} \sum_{k=m}^{n} \|(x)^{1/2}\|^2 \|a^k\|^2 \\
&= 1_{\mathbb{A}} \|(x)^{1/2}\|^2 \sum_{k=m}^{n} \|a^2\|^k.
\end{aligned}$$

Since $\|a\| < 1$ and $m < n$, therefore $m \longrightarrow \infty$ implies that $n \longrightarrow \infty$. The proof of (9) follows from the fact that $\sum_{k=m}^{n} \|a^2\|^k$ is a geometric series. Moreover, $m \longrightarrow \infty \Rightarrow \|a\|^m \longrightarrow 0$ and hence (10) follows from (9). □

We are now ready to state and prove our main result.

Theorem 2.11 *Let $(X, \mathbb{A}, d)$ be a complete C^*-valued metric space and $T : X \to X$ be a C^*-valued contractive type mapping. Then*

(A1) *$\exists x_0 \in X$ such that the sequence $T^n x$ converges to x_0,*
(A2) *$d(T^n x, x_0) \preceq \frac{\|a\|^{2n}}{1-\|a\|} \|d(x, Tx)^{\frac{1}{2}}\|^2 1_{\mathbb{A}}$,*
(A3) *x_0 is a fixed point of T if and only if $G(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at x_0 with respect to $\mathbb{A}$.*

Proof If $\mathbb{A} = \{0_{\mathbb{A}}\}$, then there is nothing to prove. Assume that $\mathbb{A} \neq \{0_{\mathbb{A}}\}$.

(A1): Let $x \in X$ and consider the orbit $\mathcal{O}_T(x)$. Since condition (8) holds for each element of $\mathcal{O}_T(x)$ and $\|a\| < 1$, it follows that

$$\begin{aligned} d(T^2x, T^3x) &= d(T(Tx), T^2(Tx)) \\ &\preceq a^* d(Tx, T(Tx))a \\ &= a^* d(Tx, T^2x)a \\ &\preceq a^* a^* d(x, Tx)aa \\ &= (a^*)^2 d(x, Tx)a^2. \end{aligned}$$

Continuing in this way, one can show that

$$d(T^n x, T^{n+1}x) \preceq (a^*)^n d(x, Tx)a^n. \tag{11}$$

Let $\{T^n x\}$ be a sequence in $\mathcal{O}_T(x)$. Then, for $m < n$, from the triangle inequality and (11) we have

$$\begin{aligned} d(T^{n+1}x, T^m x) &\preceq d(T^m x, T^{m+1}x) + d(T^{m+1}x, T^{m+2}x) + \cdots + d(T^n x, T^{n+1}x) \\ &\preceq (a^*)^m d(x, Tx)a^m + (a^*)^{m+1} d(x, Tx)a^{m+1} + \cdots + (a^*)^n d(x, Tx)a^n \\ &= \sum_{k=m}^{n} (a^*)^k d(x, Tx)a^k \longrightarrow 0_{\mathbb{A}} \quad \text{as } m \longrightarrow \infty \end{aligned}$$

using (10) of Lemma 2.10. This shows that $\{T^n x\}$ is a Cauchy sequence in $\mathcal{O}_T(x) \subset X$ with respect to $\mathbb{A}$. Since $(X, \mathbb{A}, d)$ is a complete C^*-valued metric space, there exists $x_0 \in X$ such that $T^n x \longrightarrow x_0$. This completes the proof of (A1).

(A2): It follows again from the triangle inequality and (11) that

$$\begin{aligned} d(T^n x, T^{n+m}x) &\preceq d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x) + \cdots + d(T^{n+m-1}x, T^{n+m}x) \\ &\preceq (a^*)^n d(x, Tx)a^n + (a^*)^{n+1} d(x, Tx)a^{n+1} + \cdots \\ &\quad + (a^*)^{n+m-1} d(x, Tx)a^{n+m-1} \\ &= \sum_{k=n}^{n+m-1} (a^*)^k d(x, Tx)a^k. \end{aligned}$$

Since $d(x, Tx)$ is a positive element of $\mathbb{A}$, using (9) of Lemma 2.10 and letting $m \longrightarrow \infty$, we conclude (A2).

(A3): To prove (A3), if $Tx_0 = x_0$ and $\{x_n\}$ is a sequence in $\mathcal{O}_T(x)$ with $x_n \longrightarrow x_0$ with respect to $\mathbb{A}$, then $\|G(x_0)\| = \|d(Tx_0, x_0)\| = 0 \le \liminf \|G(x_n)\|$. Conversely, if G is T-orbitally lower semicontinuous at x_0, then

$$\begin{aligned} \|G(x_0)\| &= \|d(x_0, Tx_0)\| \le \liminf \|G(T^n x)\| \\ &= \liminf \|d(T^n x, T^{n+1}x)\| \\ &\le \liminf \frac{\|a\|^{2n}}{1 - \|a\|} \|d(x, Tx)^{\frac{1}{2}}\|^2 = 0. \end{aligned}$$

This implies that $d(x_0, Tx_0) = 0_{\mathbb{A}}$, *i.e.*, $x_0 = Tx_0$. □

Remark 2.12 Note that:

(1) By taking $\mathbb{A} = \mathbb{R}$, we see that the main result of [2] follows immediately from Theorem 2.11.

(2) Theorem 1.3 is a special case of Theorem 2.11 except for the uniqueness of a fixed point of the mapping involved.

The following example shows that our result properly generalizes Theorem 1.3.

Example 2.13 Consider the C^*-algebra $\mathbb{A} = \mathbb{R}^2$ with component-wise operations where norm and ordering are given by (4) and (5), respectively. Let $X = [-1,1] \times [-1,1]$ and define the C^*-valued metric $d : X \times X \to \mathbb{R}^2$ by $d(x,y) = (|x_1 - y_1|, |x_2 - y_2|)$ for all $x = (x_1, x_2), y = (y_1, y_2) \in X$. Define $T : X \to X$ by

$$T(x_1,x_2) = \begin{cases} (\frac{x_1}{2}, \frac{x_2}{2}) & \text{if } x_1, x_2 \geq 0, \\ (1,0) & \text{otherwise.} \end{cases}$$

Taking $(u_1, u_2) \in X$ such that $0 < u_1, u_2 < 1$, we have

$$\mathcal{O}_T\bigl((u_1,u_2)\bigr) = \biggl\{(u_1,u_2), \biggl(\frac{u_1}{2}, \frac{u_2}{2}\biggr), \biggl(\frac{u_1}{4}, \frac{u_2}{4}\biggr), \ldots\biggr\}.$$

For any $u_n = (\frac{u_1}{2^{n-1}}, \frac{u_2}{2^{n-1}}) \in \mathcal{O}_T((u_1, u_2))$, we have

$$d\bigl(Tu_n, T^2u_n\bigr) = a^* d(u_n, Tu_n)a,$$

where $a = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Note that $u_n \to (0,0)$. Further, $G : X \to \mathbb{A}$ defined by $G(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at $(0,0)$. Therefore, all conditions of Theorem 2.11 are satisfied and $(0,0)$ is the fixed point of T. Note that Theorem 1.3 is not applicable here since T is not continuous at $(0,0)$ with respect to $\mathbb{A}$.

3 Application

In this section we provide the existence result for an integral equation as an application of C^*-valued contractive type mappings on complete C^*-valued metric spaces. Let E be a Lebesgue measurable set, $X = L^\infty(E)$, and $H = L^2(E)$. We denote the set of all bounded linear operators on a Hilbert space H by $L(H)$. With the usual operator norm, $L(H)$ is a C^*-algebra. For $S, T \in X$, define $d : X \times X \to L(H)$ by $d(T,S) = \pi_{|T-S|}$, where $\pi_h : H \to H$ is the multiplication operator given by $\pi_h(\phi) = h \cdot \phi$ for $\phi \in H$. Then $(X, L(H), d)$ is a complete C^*-valued metric space [3].

Example 3.1 Let E, X, H, and the metric d be as above. Suppose that

(1) $K : E \times E \times \mathbb{R} \to \mathbb{R}$, and let T be a self-mapping on X,

(2) there exists a continuous function $\phi : E \times E \to \mathbb{R}$ and $\alpha \in (0,1)$ such that for every $x \in X$, $y \in \mathcal{O}_T(x)$, and $t, s \in E$, we have

$$\bigl|K\bigl(t,s,x(s)\bigr) - K\bigl(t,s,y(s)\bigr)\bigr| \leq \alpha \bigl|\phi(t,s)\bigl(x(s) - y(s)\bigr)\bigr|. \tag{12}$$

(3) $\sup_{t\in E} \int_E |\phi(t,s)|\, ds \leq 1$.

Then the integral equation

$$x(t) = \int_E K\bigl(t, s, x(s)\bigr)\, ds, \quad t \in E$$

has a solution.

Proof Here $(X, L(H), d)$ is a complete C^*-valued metric space with respect to $L(H)$.
Let $T : X \to X$ be

$$Tx(t) = \int_E K\bigl(t, s, x(s)\bigr)\, ds, \quad t \in E.$$

Let $Tx = y$, then

$$\begin{aligned}
\bigl\| d\bigl(Tx, T^2x\bigr)\bigr\| &= \bigl\| d(Tx, Ty)\bigr\| \\
&= \|\pi_{|Tx-Ty|}\| \\
&= \sup_{\|h\|=1} \langle \pi_{|Tx-Ty|} h, h\rangle \quad \text{for any } h \in H \\
&= \sup_{\|h\|=1} \int_E \biggl[\biggl| \int_E \bigl(K\bigl(t,s,x(s)\bigr) - K\bigl(t,s,y(s)\bigr)\bigr)\, ds \biggr| \biggr] h(t)\overline{h(t)}\, dt \\
&\le \sup_{\|h\|=1} \int_E \biggl[\biggl| \int_E \bigl(K\bigl(t,s,x(s)\bigr) - K\bigl(t,s,y(s)\bigr)\bigr)\, ds \biggr| \biggr] \bigl|h(t)\bigr|^2\, dt \\
&\le \sup_{\|h\|=1} \int_E \biggl[\int_E |k\phi(t,s)\bigl(x(s) - y(s)\bigr)|\, ds \biggr] \bigl|h(t)\bigr|^2\, dt \\
&\le k \sup_{\|h\|=1} \int_E \biggl[\int_E \bigl|\phi(t,s)\bigr|\, ds \biggr] \bigl|h(t)\bigr|^2\, dt \cdot \|x - y\|_\infty \\
&\le k \sup_{t\in E} \int_E \bigl|\phi(t,s)\bigr|\, ds \cdot \sup_{\|h\|=1} \int_E \bigl|h(t)\bigr|^2\, dt \cdot \|x - y\|_\infty \\
&\le k\|x - y\|_\infty \\
&= \|a\| \bigl\| d(x,y)\bigr\| = \|a\| \bigl\| d(x,Tx)\bigr\|.
\end{aligned}$$

Setting $a = kI$, we have $a \in L(H)_+$ and $\|a\| = k$. Thus all the conditions of Theorem 2.11 hold and hence the conclusion. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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