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A variational inequality involving nonlocal elliptic operators

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Abstract

In this paper, we study the existence of solutions to a variational inequality involving nonlocal elliptic operators, and the problem studied here is not variational in nature. The proof of the main result is based on Schauder's fixed point theorem combined with adequate variational arguments and a penalization technique.

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Keywords: nonlocal elliptic operators; penalization method; Schauder's fixed point theorem; critical point theory

1 Introduction

In the last years, great attention has been devoted to the study of fractional and nonlocal problems. However, the interest in nonlocal elliptic problems goes beyond the mathematical curiosity. Indeed, this type of problems arises in a quite natural way in many different applications, such as continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces and game theory, as they are the typical outcome of stochastic stabilization of Lévy processes, see [1–4] and the references therein. The literature on nonlocal operators and their applications is very interesting and quite large, we refer the interested readers to [5–13].

In [14] the existence of two nontrivial solutions for hemivariational inequalities driven by nonlocal elliptic operators is obtained by using critical point theory for non-differentiable functionals. In [15] the Lewy-Stampacchia type estimates for variational inequalities driven by nonlocal operators are discussed. In [16] the obstacle problem is considered for a linear elliptic operator perturbed by a nonlinearity having at most a quadratic growth on the gradient, and the existence of weak solutions of the variational inequality is obtained by using a penalization method, Schauder's fixed point theorem and *a priori* estimates. In [17], the existence of nontrivial solutions for a semilinear elliptic variational inequalities with gradient-dependent nonlinearity is discussed by using variational methods combined with a penalization technique and an iterative scheme.

Motivated by the above mentioned works, we study a variational inequality involving nonlocal elliptic operators. To introduce our problem precisely, we first give some notation. Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Denote $\mathcal{Q} = \mathbb{R}^{2N}\setminus O$, where $O = \mathcal{C}(\Omega)\times\mathcal{C}(\Omega)\subset\mathbb{R}^{2N}$ and $\mathcal{C}(\Omega) = \mathbb{R}^N\setminus\Omega$. W is a linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function u in W belongs to $L^p(\Omega)$ and $\int_{\mathcal{Q}} |u(x)-u(y)|^p |x-y|^{-N-ps}\,dx\,dy < \infty$. The space W is equipped with the norm

$$\|u\|_W = \|u\|_{L^p(\Omega)} + \left(\int_{\mathcal{Q}} \bigl|u(x)-u(y)\bigr|^p |x-y|^{-N-ps}\,dx\,dy\right)^{1/p}.$$

Then $(W, \|\cdot\|_W)$ is a uniformly convex Banach space, see [18]. We shall work on the closed linear subspace

$$W_0 = \bigl\{u \in W : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\bigr\}.$$

A norm of W_0 is given by $\|u\| = (\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy)^{1/p}$ for all $u \in W_0$. Set $L_0^p(\Omega) = \{u \in L^p(\Omega) : u = 0 \text{ a.e. on } \mathbb{R}^N \setminus \Omega\}$. Obviously, $(L_0^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ is a Banach space.

In the present paper, we adapt the methods in [16] and [17] in order to prove the existence of solutions of the following variational inequality:

$$\begin{aligned} & u \in W_0, \quad u \le \psi \text{ a.e. in } \Omega, \\ & \int_{\mathcal{Q}} \frac{a(x,y,u(x),u(y))|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} \bigl(\varphi(x)-u(x)-\varphi(y)+u(y)\bigr)\,dx\,dy \\ & \quad \ge \int_\Omega f(\varphi - u)\,dx, \quad \forall \varphi \in W_0, \varphi \le \psi \text{ a.e. in } \Omega, \end{aligned} \tag{1.1}$$

where $N > sp$, $1 < p < \infty$, $\psi : \Omega \to [0,\infty)$ is the obstacle with $\psi \in L^p(\Omega)$, and functions a, f satisfy the following assumptions:

(a) $a : \mathcal{Q} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, *i.e.*, for each $(t_1,t_2) \in \mathbb{R} \times \mathbb{R}$, $a(x,y,t_1,t_2)$ is measurable with respect to $(x,y) \in \mathcal{Q}$; for each $(x,y) \in \mathcal{Q}$, $a(x,y,t_1,t_2)$ is continuous with respect to $(t_1,t_2) \in \mathbb{R} \times \mathbb{R}$. There exist two constants $a_0, a_1 > 0$ such that $a_0 \le a(x,y,t_1,t_2) \le a_1$ for all $(x,y,t_1,t_2) \in \mathcal{Q} \times \mathbb{R} \times \mathbb{R}$.

(f) $f : \Omega \to \mathbb{R}$ is a measurable function with $f \in L^{p'}(\Omega)$ and $p' = p/(p-1)$.

Remark 1.1 If $a \equiv 1$, the variational inequality (1.1) is governed by the following fractional p-Laplacian problem:

$$\begin{aligned} & (-\Delta)_p^s u = f \quad \text{in } \Omega, \\ & u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned}$$

where $N > ps$ with $s \in (0,1)$, and $(-\Delta)_p^s$ is the fractional p-Laplacian defined for each $x \in \mathbb{R}^N$ as

$$(-\Delta)_p^s v(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy,$$

along any $v \in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x)$ denotes the ball in $\mathbb{R}^N$ with radius $\varepsilon > 0$ centered at $x \in \mathbb{R}^N$. Actually, we can define the fractional p-Laplacian as follows:

$$\bigl\langle(-\Delta)_p^s u, v\bigr\rangle = \iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy$$

for all $u, v \in W_0$. For more details about the fractional p-Laplacian, we refer to [12] and [18]. When $a \not\equiv 1$, it is natural to consider the nonlocal operator $\mathcal{L}$ defined as

$$\langle \mathcal{L}u, v\rangle = \int_{Q} \frac{a(x,y,u(x),u(y))|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy$$

for all $u, v \in W_0$. Hence, the variational inequality (1.1) can be written as

$$\langle \mathcal{L}u, \varphi - u\rangle \geq \int_{\Omega} f(\varphi - u)\,dx.$$

Note that the function a depends on the state variable u. Generally, problem (1.1) is not variational.

Firstly, we 'freeze' the state variable u on the function a, that is, we fix $v \in L_0^p(\Omega)$. Let $\varepsilon \in (0,1)$ be fixed and consider the following penalization problem:

$$\begin{aligned}&\int_{Q} \frac{a(x,y,v(x),v(y))|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\big(\varphi(x)-\varphi(y)\big)\,dx\,dy\\&\quad+\frac{1}{\varepsilon}\int_{\Omega}\big|(u-\psi)^{+}\big|^{p-1}\varphi\,dx = \int_{\Omega} f\varphi\,dx, \quad \forall \varphi \in W_0,\end{aligned}\tag{1.2}$$

where $u^{+} = \max\{u, 0\}$. To get the solutions of problem (1.2), we apply variational methods to the energy functional

$$I(u) = \frac{1}{p}\iint_{Q} \frac{a(x,y,v(x),v(y))|u(x)-u(y)|^{p}}{|x-y|^{N+ps}}\,dx\,dy + \frac{1}{\varepsilon p}\int_{\Omega}\big((u-\psi)^{+}\big)^{p}\,dx - \int_{\mathbb{R}^N} fu\,dx,$$

associated to (1.2), for all $u \in W_0$. Note that the condition $f \in L^{p'}(\Omega)$ implies that $I \in C^1(W_0, \mathbb{R})$ and

$$\begin{aligned}\big\langle I'(u), \varphi\big\rangle = {}&\iint_{Q} \frac{a(x,y,v(x),v(y))|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\big(\varphi(x)-\varphi(y)\big)\,dx\,dy\\&+\frac{1}{\varepsilon}\int_{\Omega}\big((u-\psi)^{+}\big)^{p-1}\varphi\,dx - \int_{\Omega} f\varphi\,dx\end{aligned}$$

for all $\varphi \in W_0$. Hence a critical point of functional I is a solution of penalization problem (1.2). Note that such penalization technique can be found in [19], and we study the penalized problem (1.2) through variational methods, see [20], Theorem 1.21.

Secondly, for $v \in L_0^p(\Omega)$ fixed, we deduce from (1.2) that there exists $u_v \in W_0$ with $u_v \leq \psi$ a.e. in Ω such that

$$\begin{aligned}&\int_{Q} \frac{a(x,y,v(x),v(y))|u_v(x)-u_v(y)|^{p-2}(u_v(x)-u_v(y))}{|x-y|^{N+ps}}\big(\varphi(x)-u_v(x)-\varphi(y)+u_v(y)\big)\,dx\,dy\\&\quad\geq \int_{\Omega} f(\varphi - u_v)\,dx, \quad \forall \varphi \in W_0, \varphi \leq \psi.\end{aligned}\tag{1.3}$$

Finally, starting from (1.3) and using Schauder's fixed point theorem (see [21], Theorem 3.21), we get our main result.

Theorem 1.1 *Let $\psi \in L^p(\Omega)$, (a) and (f) hold. Then problem (1.1) has a solution in W_0.*

2 The associated problem (1.3)

We start by proving some auxiliary results which will be useful in establishing the existence result of problem (1.3). In the following, we shortly denote by $\|\cdot\|_q$ the norm of $L^q(\Omega)$.

Lemma 2.1 *For each $v \in L_0^p(\Omega)$ and $\varepsilon \in (0,1)$, problem* (1.2) *has a solution in W_0.*

Proof Fix $v \in L_0^p(\Omega)$ and $\varepsilon \in (0,1)$. By (a), the Hölder inequality and the continuous embedding from W_0 to $L^p(\Omega)$ (see [18], Lemma 2.3), it follows that for each $u \in W_0$ we have

$$
\begin{aligned}
I(u) &\geq \frac{a_0}{p}\iint_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \int_\Omega fu\,dx \\
&\geq \frac{a_0}{p}\iint_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \|f\|_{p'}\|u\|_p \geq \frac{a_0}{p}\|u\|^p - C^*\|f\|_{p'}\|u\|,
\end{aligned}
$$

where $C^* > 0$ is the embedding constant of $W_0 \hookrightarrow L^p(\Omega)$. The above estimate shows that I is coercive on W_0, being $1 < p < \infty$.

On the other hand, it is easy to verify that I is convex and, consequently, weakly lower semi-continuous. Hence, we conclude via the direct method of variational methods (see [20], Theorem 1.2) that there exists a global minimum point of I in W_0 and, consequently, a solution of problem (1.2). □

Lemma 2.2 *Let $v \in L_0^p(\Omega)$ and let u_v^ε be the solution of* (1.2) *given by Lemma* 2.1. *If $\psi \in L^p(\Omega)$, then the family $\{u_v^\varepsilon\}_\varepsilon$ is bounded in W_0, i.e., there exists a positive constant C independent of ε and v such that $\|u_v^\varepsilon\| \leq C$ and $\|(u_v^\varepsilon - \psi)^+\|_p \leq C\varepsilon^{1/p}$ for all $\varepsilon > 0$.*

Proof Taking $\varphi = u_v^\varepsilon$ as a test function in (1.2), we have

$$
\iint_Q \frac{a(x,y,v(x),v(y))|u_v^\varepsilon(x)-u_v^\varepsilon(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \frac{1}{\varepsilon}\int_\Omega \bigl|\bigl(u_v^\varepsilon - \psi\bigr)^+\bigr|^p\,dx = \int_\Omega fu_v^\varepsilon\,dx.
$$

It follows from the Hölder inequality, the embedding $W_0 \hookrightarrow L^p(\Omega)$ and (a) that

$$
a_0\iint_Q \frac{|u_v^\varepsilon(x)-u_v^\varepsilon(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \frac{1}{\varepsilon}\int_\Omega \bigl|\bigl(u_v^\varepsilon - \psi\bigr)^+\bigr|^p\,dx \leq \|f\|_{p'}\bigl\|u_v^\varepsilon\bigr\|_p \leq C^*\|f\|_{p'}\bigl\|u_v^\varepsilon\bigr\|,
$$

this implies that

$$
\iint_Q \frac{|u_v^\varepsilon(x)-u_v^\varepsilon(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \frac{1}{\varepsilon}\int_\Omega \bigl|\bigl(u_v^\varepsilon - \psi\bigr)^+\bigr|^p\,dx \leq C,
$$

where $C > 0$ denotes various constants independent of ε and v. Hence $\|u_v^\varepsilon\| \leq C$ and $\|(u_v^\varepsilon - \psi)^+\|_p \leq C\varepsilon^{1/p}$. □

Theorem 2.1 *Let $v \in L_0^p(\Omega)$ be fixed and let conditions* (a) *and* (f) *hold true. Furthermore, let $\psi \in L^p(\Omega)$. Then the variational inequality* (1.3) *has a unique solution $u_v \in W_0$.*

Proof By Lemma 2.2, there exist a subsequence of $\{u_v^\varepsilon\}$ still denoted by $\{u_v^\varepsilon\}$ and $u_v \in W_0$ such that $u_v^\varepsilon \rightharpoonup u_v$ in W_0 as $\varepsilon \to 0$. It follows from $\|(u_v^\varepsilon - \psi)^+\|_p \leq C\varepsilon^{1/p}$ that $(u_v^\varepsilon - \psi)^+ \to 0$ in $L^p(\Omega)$ as $\varepsilon \to 0$. By the compact embedding $W_0 \hookrightarrow\hookrightarrow L^p(\Omega)$ (see [18], Lemma 2.5), we obtain that $u_v^\varepsilon \to u_v$ strongly in $L^p(\Omega)$ as $\varepsilon \to 0$. Furthermore, $(u_v^\varepsilon - \psi)^+ \to (u_v - \psi)^+$ in $L^p(\Omega)$ as $\varepsilon \to 0$. Hence $(u_v - \psi)^+ = 0$ a.e. in Ω, that is, $u_v \leq \psi$ a.e. in Ω.

Now we show that $u_\nu^\varepsilon \to u_\nu$ strongly in W_0 as $\varepsilon \to 0$. For each fixed $\omega \in W_0$, we introduce a linear operator $F_a(\omega, \cdot) : W_0 \to \mathbb{R}$ defined by

$$F_a(\omega, \varphi) = \iint_{\mathcal{Q}} \frac{a(x,y,\nu(x),\nu(y))|\omega(x) - \omega(y)|^{p-2}(\omega(x) - \omega(y))}{|x-y|^{N+ps}} \big(\varphi(x) - \varphi(y)\big)\,dx\,dy$$

for each $\varphi \in W_0$. Obviously, $F_a(\omega, \cdot)$ is a continuous linear functional on W_0 by the following inequality:

$$\big|F_a(\omega, \varphi)\big| \le a_1 \|\omega\|^{p-1}\|\varphi\|, \quad \forall \varphi \in W_0,$$

thanks to assumption (a) and the Hölder inequality. Hence $u_\nu^\varepsilon \rightharpoonup u_\nu$ in W_0 as $\varepsilon \to 0$ implies that

$$\lim_{\varepsilon\to 0} F_a\big(u_\nu, u_\nu^\varepsilon - u_\nu\big) = 0. \tag{2.1}$$

Taking $u_\nu^\varepsilon - u_\nu$ as a test function in (1.2), we obtain

$$F_a\big(u_\nu^\varepsilon, u_\nu^\varepsilon - u_\nu\big) + \frac{1}{\varepsilon}\int_\Omega \big|\big(u_\nu^\varepsilon - \psi\big)^+\big|^{p-1}\big(u_\nu^\varepsilon - u_\nu\big)\,dx = \int_\Omega f\big(u_\nu^\varepsilon - u_\nu\big)\,dx. \tag{2.2}$$

Note that

$$\begin{aligned}&\frac{1}{\varepsilon}\int_\Omega \big|\big(u_\nu^\varepsilon - \psi\big)^+\big|^{p-1}\big(u_\nu^\varepsilon - u_\nu\big)\,dx\\ &\quad = \frac{1}{\varepsilon}\int_\Omega \big|\big(u_\nu^\varepsilon - \psi\big)^+\big|^{p}\,dx + \frac{1}{\varepsilon}\int_\Omega \big|\big(u_\nu^\varepsilon - \psi\big)^+\big|^{p-1}(\psi - u_\nu)\,dx \ge 0,\end{aligned}$$

by $u_\nu \le \psi$ a.e. in Ω. Putting this into (2.2), we have

$$F_a\big(u_\nu^\varepsilon, u_\nu^\varepsilon - u_\nu\big) \le \int_\Omega f\big(u_\nu^\varepsilon - u_\nu\big)\,dx. \tag{2.3}$$

Using $u_\nu^\varepsilon \to u_\nu$ strongly in $L^p(\Omega)$, (2.1) and (2.3), we deduce

$$\limsup_{\varepsilon\to 0}\big[F_a\big(u_\nu^\varepsilon, u_\nu^\varepsilon - u_\nu\big) - F_a\big(u_\nu, u_\nu^\varepsilon - u_\nu\big)\big] \le 0. \tag{2.4}$$

On the other hand, by the following well-known inequalities (see [18]):

$$|\xi - \eta|^p \le \begin{cases} C_p(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi - \eta) & \text{for } p \ge 2,\\ \widetilde{C}_p[(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi-\eta)]^{p/2}(|\xi|^p + |\eta|^p)^{(2-p)/2} & \text{for } 1 < p < 2\end{cases}$$

for all $\xi, \eta \in \mathbb{R}^N$, where C_p and $\widetilde{C}_p$ are positive constants depending only on p, we get

$$\big[F_a\big(u_\nu^\varepsilon, u_\nu^\varepsilon - u_\nu\big) - F_a\big(u_\nu, u_\nu^\varepsilon - u_\nu\big)\big] \ge 0.$$

Hence, we conclude from (2.4) that $\lim_{\varepsilon\to 0}[F_a(u_\nu^\varepsilon, u_\nu^\varepsilon - u_\nu) - F_a(u_\nu, u_\nu^\varepsilon - u_\nu)] = 0$. A similar discussion as in [18] gives that $u_\nu^\varepsilon \to u_\nu$ strongly in W_0 as $\varepsilon \to 0$.

Now we are ready to show that u_ν is a solution of (1.3). Taking $\varphi - u_\nu^\varepsilon$, with $\varphi \in W_0$ and $\varphi \le \psi$ a.e. in Ω, as a test function in (1.2), we have

$$F_a\left(u_\nu^\varepsilon, \varphi - u_\nu^\varepsilon\right) + \frac{1}{\varepsilon}\int_\Omega \left|\left(u_\nu^\varepsilon - \psi\right)^+\right|^{p-1}\left(\varphi - u_\nu^\varepsilon\right)dx = \int_\Omega f\left(\varphi - u_\nu^\varepsilon\right)dx. \tag{2.5}$$

Thanks to the choice of φ, one has $\frac{1}{\varepsilon}\int_\Omega |(u_\nu^\varepsilon - \psi)^+|^{p-1}(\varphi - u_\nu^\varepsilon)\,dx \le 0$, so that we deduce from (2.5) that

$$F_a\left(u_\nu^\varepsilon, \varphi - u_\nu^\varepsilon\right) \ge \int_\Omega f\left(\varphi - u_\nu^\varepsilon\right)dx, \quad \forall \varphi \in W_0, \varphi \le \psi. \tag{2.6}$$

By the boundedness of a and the stronger convergence of u_ν^ε in W_0 and $L^p(\Omega)$, we conclude from (2.6) that

$$F_a(u_\nu, \varphi - u_\nu) \ge \int_\Omega f(\varphi - u_\nu)\,dx, \quad \forall \varphi \in W_0, \varphi \le \psi.$$

This means that u_ν is a solution of (1.3).

It remains to prove the uniqueness of u_ν. Now we suppose that there exist u_ν^1 and u_ν^2 such that (1.3) holds, that is,

$$F_a\left(u_\nu^1, \varphi - u_\nu^1\right) \ge \int_\Omega f\left(\varphi - u_\nu^1\right)dx \tag{2.7}$$

and

$$F_a\left(u_\nu^2, \varphi - u_\nu^2\right) \ge \int_\Omega f\left(\varphi - u_\nu^2\right)dx \tag{2.8}$$

for all $\varphi \in W_0$ with $\varphi \le \psi$ a.e. in Ω. Taking $\varphi = u_\nu^2$ and $\varphi = u_\nu^1$ in (2.7) and (2.8), respectively, we have $F_a(u_\nu^1, u_\nu^2 - u_\nu^1) \ge \int_\Omega f(u_\nu^2 - u_\nu^1)\,dx$ and $F_a(u_\nu^2, u_\nu^1 - u_\nu^2) \ge \int_\Omega f(u_\nu^1 - u_\nu^2)\,dx$. From these two inequalities, we get $F_a(u_\nu^1, u_\nu^1 - u_\nu^2) - F_a(u_\nu^2, u_\nu^1 - u_\nu^2) \le 0$, this implies that $u_\nu^1 = u_\nu^2$ a.e. in Ω. Thus we complete the proof. □

3 The proof of main result

For each $\nu \in L_0^p(\Omega)$, let $u = T(\nu) \in W_0$ be the weak solution of problem (1.3) given by Theorem 2.1. Thus, we can actually introduce an application $T : L_0^p(\Omega) \to W_0$ associating to each $\nu \in L_0^p(\Omega)$, the solution of problem (1.3), $T(\nu) \in W_0$.

Lemma 3.1 *There exists a constant $C > 0$ such that*

$$\int_Q \frac{|T(\nu(x)) - T(\nu(y))|^p}{|x-y|^{N+ps}}\,dx\,dy \le C, \quad \forall \nu \in L_0^p(\Omega).$$

Proof For each $\nu \in L_0^p(\Omega)$, $T(\nu)$ is a solution of problem (1.3). Let u_ν^ε be the solution of problem (1.2). By Lemma 2.2 and Theorem 2.1, we have $\|u_\nu^\varepsilon\|^p \le C$ and $u_\nu^\varepsilon \to T(\nu)$ strongly in W_0 as $\varepsilon \to 0$, where $C > 0$ is a constant independent of ε and ν. Hence $\|T(\nu)\|^p \le C$. □

Remark 3.1 By Lemma 3.1 and the fractional Sobolev embedding $W_0 \hookrightarrow L^p(\Omega)$, it clearly follows that there exists $C_1 > 0$ such that $\int_\Omega |T(\nu)|^p\,dx \le C_1$, $\forall \nu \in L_0^p(\Omega)$.

Lemma 3.2 *The map $T : L_0^p(\Omega) \to W_0$ is continuous.*

Proof Let $\{v_n\}_n \subset L_0^p(\Omega)$ and $v \in L_0^p(\Omega)$ be such that v_n converges to v in $L_0^p(\Omega)$ as $n \to \infty$. Without loss of generality, we assume that $v_n \to v$ a.e. in Ω. Set $u_n := T(v_n)$ for all $n \geq 1$. By Lemma 3.1, we have

$$\iint_{Q} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+ps}}\, dx\, dy = \int_{Q} \frac{|T(v_n(x)) - T(v_n(y))|^p}{|x - y|^{N+ps}}\, dx\, dy \leq C, \quad \forall n \geq 1,$$

i.e., $\{u_n\}_n$ is bounded in W_0. It follows that up to a subsequence we can deduce that u_n converges weakly to some u in W_0. By the compact embedding $W_0 \hookrightarrow L^p(\Omega)$, up to a subsequence, we have $u_n \to u$ strongly in $L^p(\Omega)$ and a.e. in Ω. Obviously, we also have u_n converges to u strongly in $L_0^p(\Omega)$. Since u_n ($n \geq 1$) is a solution of problem (1.3), we have $u_n \leq \psi$ a.e. in Ω. Hence, $u \leq \psi$ a.e. in Ω.

On the other hand, for each n, we have

$$\begin{aligned}
&\iint_{Q} \frac{a(x, y, v_n(x), v_n(y))|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x - y|^{N+ps}} \\
&\quad \times \bigl(\varphi(x) - u_n(x) - \varphi(y) + u_n(y)\bigr)\, dx\, dy \\
&\quad \geq \int_{\Omega} f(\varphi - u_n)\, dx
\end{aligned} \tag{3.1}$$

for all $\varphi \in W_0$ with $\varphi \leq \psi$. Taking $\varphi = u$ in (3.1), we obtain

$$\begin{aligned}
&\iint_{Q} \frac{a(x, y, v_n(x), v_n(y))|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x - y|^{N+ps}} \\
&\quad \times \bigl(u_n(x) - u(x) - u_n(y) + u(y)\bigr)\, dx\, dy \\
&\quad \leq \int_{\Omega} f(u_n - u)\, dx.
\end{aligned} \tag{3.2}$$

It follows from u_n converges to u in $L^p(\Omega)$ and (3.2) that

$$\begin{aligned}
&\limsup_{n\to\infty} \iint_{Q} \frac{a(x, y, v_n(x), v_n(y))|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x - y|^{N+ps}} \\
&\quad \times \bigl(u_n(x) - u(x) - u_n(y) + u(y)\bigr)\, dx\, dy \\
&\quad \leq 0.
\end{aligned} \tag{3.3}$$

Note that

$$\left| a\bigl(x, y, v_n(x), v_n(y)\bigr) \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{(N+ps)/p'}} \right|^{p'} \leq a_1^{p'} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \in L^1(Q, dx\, dy)$$

and

$$\begin{aligned}
&a\bigl(x, y, v_n(x), v_n(y)\bigr) \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{(N+ps)/p'}} \\
&\quad \to a\bigl(x, y, v(x), v(y)\bigr) \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{(N+ps)/p'}}
\end{aligned}$$

a.e. on $\mathcal{Q}$. Hence the Lebesgue dominated convergence theorem implies that

$$\begin{aligned} & a\bigl(x,y,v_n(x),v_n(y)\bigr)\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{(N+ps)/p'}} \\ & \quad \to a\bigl(x,y,v(x),v(y)\bigr)\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{(N+ps)/p'}} \end{aligned}$$

strongly in $L^{p'}(\mathcal{Q},dx\,dy)$. Thus, we deduce from the Hölder inequality that

$$\begin{aligned} & \lim_{n\to\infty}\iint_{\mathcal{Q}}\frac{[a(x,y,v_n(x),v_n(y))-a(x,y,v(x),v(y))]|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} \\ & \quad \times \bigl(u_n(x)-u(x)-u_n(y)+u(y)\bigr)\,dx\,dy=0. \end{aligned} \tag{3.4}$$

Notice that

$$\begin{aligned} & \lim_{n\to\infty}\iint_{\mathcal{Q}}\frac{a(x,y,v(x),v(y))|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} \\ & \quad \times \bigl(u_n(x)-u(x)-u_n(y)+u(y)\bigr)\,dx\,dy=0. \end{aligned}$$

Hence, from (3.4), it yields

$$\begin{aligned} & \lim_{n\to\infty}\iint_{\mathcal{Q}}\frac{a(x,y,v_n(x),v_n(y))|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} \\ & \quad \times \bigl(u_n(x)-u(x)-u_n(y)+u(y)\bigr)\,dx\,dy=0. \end{aligned}$$

Therefore, by (3.3) and (3.4), we get $\lim_{n\to\infty}[F_a(u_n,u_n-u)-F_a(u,u_n-u)]=0$. Further, we can conclude that $u_n\to u$ strongly in W_0. Consequently, $T:L_0^p(\Omega)\to W_0$ is continuous. □

Now we define an operator $L:L_0^p(\Omega)\to L_0^p(\Omega)$, with $L=i\circ T$, where $i:W_0\to L_0^p(\Omega)$ is the inclusion operator. Since W_0 is compactly embedded in $L_0^p(\Omega)$, the inclusion operator i is compact. It follows by Lemma 3.2 that the operator $L=i\circ T$ is compact.

Proof of Theorem 1.1 Let C_1 be the constant given in Remark 3.1, *i.e.*, $\int_\Omega|L(v)|^p\,dx\le C_1$, $\forall v\in L_0^p(\Omega)$. Consider the ball $B_{C_1}(0):=\{v\in L_0^p(\Omega):\int_\Omega|v|^p\,dx\le C_1\}$. Clearly, $B_{C_1}(0)$ is a convex closed subset of $L_0^p(\Omega)$ and $L(B_{C_1}(0))\subset B_{C_1}(0)$. Moreover, $L(B_{C_1}(0))$ is relatively compact in $B_{C_1}(0)$.

Hence, by Lemma 3.2, we have $L:B_{C_1}(0)\to B_{C_1}(0)$ is a continuous map. Therefore, we can apply Schauder's fixed point theorem (see [21], Theorem 3.21) to obtain L with a fixed point. This means that problem (1.1) has a solution, and thus the proof of Theorem 1.1 is complete. □

Competing interests
The author declares that they have no competing interests.

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