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Endpoints of multivalued nonexpansive mappings in geodesic spaces

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Abstract

Let X be either a uniformly convex Banach space or a reflexive Banach space having the Opial property. It is shown that a multivalued nonexpansive mapping on a bounded closed convex subset of X has an endpoint if and only if it has the approximate endpoint property. This is the first result regarding the existence of endpoints for such kind of mappings even in Hilbert spaces. The related result in a complete CAT(0) space is also given.

Keywords: endpoint; fixed point; multivalued nonexpansive mapping; Banach space; CAT(0) space

1 Introduction

Let (X,d) be a metric space, $\emptyset \neq E \subseteq X$, and $x \in X$. The *distance* from x to E is defined by

$$\operatorname{dist}(x,E) = \inf\{d(x,y) : y \in E\}.$$

We denote by $\mathcal{CB}(E)$ the family of nonempty closed bounded subsets of E and by $\mathcal{K}(E)$ the family of nonempty compact subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $\mathcal{CB}(E)$, *i.e.*,

$$H(A,B) = \max\Bigl\{\sup_{a\in A}\operatorname{dist}(a,B), \sup_{b\in B}\operatorname{dist}(b,A)\Bigr\}, \quad A,B \in \mathcal{CB}(E).$$

A multivalued mapping $T : E \to \mathcal{CB}(X)$ is said to be *contractive* if there exists a constant $k \in [0,1)$ such that

$$H\bigl(T(x),T(y)\bigr) \leq kd(x,y), \quad x,y \in E. \tag{1}$$

If (1) is valid when $k = 1$, then T is said to be *nonexpansive*. It is clear that every contractive mapping is nonexpansive and, in general, the converse is not true.

A point $x \in E$ is called a *fixed point* of T if $x \in T(x)$. A point $x \in E$ is called an *endpoint* (or *stationary point*) of T if x is a fixed point of T and $T(x) = \{x\}$. We shall denote by $\operatorname{Fix}(T)$ the set of all fixed points of T and by $\operatorname{End}(T)$ the set of all endpoints of T. We see that for each mapping T, $\operatorname{End}(T) \subseteq \operatorname{Fix}(T)$. Thus, the concept of endpoints seems to be more difficult (but more important) than the concept of fixed points. However, both concepts are equivalent when T is a single-valued mapping since, in this case, $\operatorname{End}(T) = \operatorname{Fix}(T)$.

The existence of endpoints for a special kind of contractive mappings was first studied by Aubin and Siegel [1]. They proved that every multivalued dissipative mapping on a complete metric space always has an endpoint. Since then the endpoint results for several kinds of contractive mappings have been rapidly developed and many papers have appeared (see, *e.g.*, [2–20]).

The first result regarding the existence of endpoints for non-contractive type mappings was discovered by Garcia-Falset *et al.* [21]. They proved that every J-type mapping on a weakly compact convex subset of a Banach space with compact faces always has an endpoint. Later on, Garcia-Falset *et al.* [22] introduced the class of (SL)-type mappings and proved that every (SL)-type mapping on a weakly compact convex subset of a Banach space with normal structure always has an endpoint. But, both classes of J-type and (SL)-type mappings are different from the class of nonexpansive mappings (see Remark 2.5, [22], Example 4, [21], Example 27 and [21], p.1260). Summary: there is no result in metric or Banach spaces regarding the existence of endpoints for nonexpansive mappings.

In this article, we give a necessary and sufficient condition for the existence of endpoints for multivalued nonexpansive mappings in uniformly convex Banach spaces and reflexive Banach spaces having the Opial property. We also obtain the related result in a special kind of metric spaces, namely, CAT(0) spaces. Our main discoveries are Theorems 3.1, 3.4 and 4.7.

2 Preliminaries

In this section we collect some geometric properties of Banach spaces. For more details the reader is referred to [23, 24].

Let E be a bounded subset of a metric space (X, d). For $x \in X$, we set

$$
\begin{aligned}
&r_x(E) = \sup\{d(x,y) : y \in E\},\\
&r(E) = \inf\{r_x(E) : x \in E\},\\
&c(E) = \{x \in E : r_x(E) = r(E)\},\\
&\operatorname{diam}(E) = \sup\{d(x,y) : x, y \in E\}.
\end{aligned}
$$

The number $r_x(E)$ is called the *radius* of E relative to x; $r(E)$, $c(E)$ and $\operatorname{diam}(E)$ are called, respectively, the *Chebyshev radius*, *Chebyshev center* and *diameter* of E. A point $x \in E$ is said to be a *diametral point* of E if $r_x(E) = \operatorname{diam}(E)$. A Banach space X is said to have *normal structure* if for each bounded closed convex subset K of X, which contains at least two points, there exists an element of K which is not a diametral point of K.

Let $\{x_n\}$ be a bounded sequence in X and $\emptyset \neq E \subseteq X$. The *asymptotic radius* of $\{x_n\}$ in E is defined by

$$
r(E, \{x_n\}) = \inf\Big\{\limsup_{n\to\infty} d(x_n, x) : x \in E\Big\}.
$$

The *asymptotic center* of $\{x_n\}$ with respect to E is defined by

$$
A(E, \{x_n\}) = \Big\{x \in E : \limsup_{n\to\infty} d(x_n, x) = r(E, \{x_n\})\Big\}.
$$

If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$.

The sequence $\{x_n\}$ is called *regular* relative to E if $r(E,\{x_n\}) = r(E,\{x_{n_k}\})$ for all subsequences $\{x_{n_k}\}$ of $\{x_n\}$. It is known that there always exists a subsequence of $\{x_n\}$ which is regular relative to E (see, *e.g.*, [25, 26]).

A Banach space X is said to be *uniformly convex* if for each $\varepsilon \in (0,2]$ there exists $\delta > 0$ such that for any $x, y \in X$ the conditions $\|x\| \leq 1$, $\|y\| \leq 1$, $\|x - y\| \geq \varepsilon$ imply

$$\frac{1}{2}\|x + y\| \leq 1 - \delta.$$

It is well known that if E is a bounded closed convex subset of a uniformly convex Banach space, then $A(E,\{x_n\})$ consists of exactly one point (see, *e.g.*, [24], p.223).

A Banach space X is said to have the *Opial property* if given whenever $\{x_n\}$ converges weakly to $x \in X$,

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \quad \text{for each } y \in X \text{ with } y \neq x.$$

From now on, we will use the notation '$x_n \rightharpoonup x$' for a sequence $\{x_n\}$ converging weakly to a point x.

Proposition 2.1 *The following statements hold.*

(1) *Every Hilbert space is a uniformly convex Banach space.*
(2) *Every Hilbert space is a reflexive Banach space having the Opial property.*
(3) *Every uniformly convex Banach space has normal structure.*
(4) *Every reflexive Banach space with the Opial property has normal structure.*

Let E be a nonempty subset of a metric space X and $T : E \to \mathcal{CB}(X)$ be a mapping. A sequence $\{x_n\}$ in E is called an *approximate fixed point sequence* for T (*a.f.p.s.* in short) if $\lim_{n\to\infty} \operatorname{dist}(x_n, T(x_n)) = 0$. The mapping T is said to have the *approximate fixed point property* if it has an a.f.p.s. in E (or, equivalently, $\inf_{x\in E} \operatorname{dist}(x, T(x)) = 0$). The mapping T is said to have the *approximate endpoint property* [11] if $\inf_{x\in E} r_x(T(x)) = 0$.

Proposition 2.2 *The following statements hold.*

(1) *If T has the approximate endpoint property, then T has the approximate fixed point property.*
(2) *If T is a single-valued mapping, then T has the approximate endpoint property if and only if T has the approximate fixed point property.*

The following example shows that the converse of (1) in Proposition 2.2 may not be true if T is a multivalued mapping.

Example 2.3 Let $E = [0,1]$ and $T : E \to \mathcal{CB}(E)$ be defined by

$$T(x) = [0, 1 - x] \quad \text{for all } x \in E.$$

Since T has a fixed point, it is immediately clear that T has the approximate fixed point property. Next, we consider the values of $r_x(T(x))$ in the following cases.

Case 1. $0 \leq x < 1/4$. We have $r_x(T(x)) = \sup\{|x - y| : y \in [0, 1 - x]\} \geq |x - (1 - x)| = 1 - 2x > 1/2$.

Case 2. $x \geq 1/4$. We have $r_x(T(x)) = \sup\{|x-y| : y \in [0, 1-x]\} \geq |x-0| = x \geq 1/4$.

Thus, $\inf_{x\in E} r_x(T(x)) \geq 1/4$. Therefore, T does not have the approximate endpoint property.

However, the converse of (1) in Proposition 2.2 is true under some additional conditions.

Proposition 2.4 *Let E be a nonempty subset of a metric space (X, d), $\{x_n\}$ be a sequence in E, and $T : E \to \mathcal{K}(X)$ be a mapping. Then $r_{x_n}(T(x_n)) \to 0$ if and only if* $\operatorname{dist}(x_n, T(x_n)) \to 0$ *and* $\operatorname{diam}(T(x_n)) \to 0$.

Proof Suppose that $r_{x_n}(T(x_n)) \to 0$. Then $\operatorname{dist}(x_n, T(x_n)) \leq r_{x_n}(T(x_n)) \to 0$. To show that $\operatorname{diam}(T(x_n)) \to 0$, we let $u, v \in T(x_n)$. Then

$$d(u, v) \leq d(u, x_n) + d(x_n, v) \leq 2r_{x_n}\big(T(x_n)\big).$$

This implies that $\operatorname{diam}(T(x_n)) \leq 2r_{x_n}(T(x_n)) \to 0$.

Conversely, we suppose that $\operatorname{dist}(x_n, T(x_n)) \to 0$ and $\operatorname{diam}(T(x_n)) \to 0$. Since $T(x_n)$ is compact, for each $n \in \mathbb{N}$, there exist y_n and u_n in $T(x_n)$ such that

$$d(x_n, y_n) = \operatorname{dist}\big(x_n, T(x_n)\big) \quad \text{and} \quad d(x_n, u_n) = r_{x_n}\big(T(x_n)\big).$$

Thus $r_{x_n}(T(x_n)) = d(x_n, u_n) \leq d(x_n, y_n) + d(y_n, u_n) \leq \operatorname{dist}(x_n, T(x_n)) + \operatorname{diam}(T(x_n)) \to 0$. □

Recall that a multivalued mapping $T : E \to \mathcal{CB}(E)$ is said to be an *(SL)-type mapping* on E [22] if the following statements hold:

(1) There exists an a.f.p.s. for T in each nonempty closed convex and T-invariant subset D of E. Here, T-invariant means $T(x) \subseteq D$ for all $x \in D$.
(2) For any a.f.p.s. $\{x_n\}$ of T in E and each $x \in E$, one has

$$\limsup_{n\to\infty} H\big(\{x_n\}, T(x)\big) \leq \limsup_{n\to\infty} \|x_n - x\|.$$

Remark 2.5 The mapping T in Example 2.3 is nonexpansive but is not (SL)-type.

Proof We first show that T is nonexpansive. Let $x, y \in E$. Then

$$\begin{aligned} H\big(T(x), T(y)\big) &= H\big([0, 1-x], [0, 1-y]\big) \\ &= \big|(1-x) - (1-y)\big| \\ &= |x-y|. \end{aligned}$$

Next, we show that T is not (SL)-type. For each $n \in \mathbb{N}$, let $x_n = 1/n$. Then $\operatorname{dist}(x_n, T(x_n)) \to 0$, but

$$\begin{aligned} \lim_{n\to\infty} |x_n - 0| = 0 < 1 &= \lim_{n\to\infty} (1 - 1/n) \\ &= \lim_{n\to\infty} H\big(\{1/n\}, [0, 1]\big) \\ &= \lim_{n\to\infty} H\big(\{x_n\}, T(0)\big). \end{aligned}$$

□

3 Main results

We begin this section by proving a result in uniformly convex Banach spaces.

Theorem 3.1 *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, E be a nonempty bounded closed convex subset of X, and $T : E \to \mathcal{K}(E)$ be a nonexpansive mapping. Then T has an endpoint if and only if T has the approximate endpoint property.*

Proof It is clear that if T has an endpoint, then T has the approximate endpoint property. Conversely, suppose that T has the approximate endpoint property. Then there exists a sequence $\{x_n\}$ in E such that $r_{x_n}(T(x_n)) \to 0$. It follows from Proposition 2.4 that

$$\operatorname{dist}(x_n, T(x_n)) \to 0 \quad \text{and} \quad \operatorname{diam}(T(x_n)) \to 0.$$

By passing through a subsequence, we may assume that $\{x_n\}$ is regular relative to E. Let $A(E, \{x_n\}) = \{x\}$ and $r = r(E, \{x_n\})$. For each $n \in \mathbb{N}$, select $y_n \in T(x_n)$ and $z_n \in T(x)$ so that

$$\|x_n - y_n\| = \operatorname{dist}(x_n, T(x_n)) \quad \text{and} \quad \|y_n - z_n\| = \operatorname{dist}(y_n, T(x)).$$

Since $T(x)$ is compact, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \to w \in T(x)$. Thus

$$\begin{aligned} \|x_{n_k} - w\| &\le \|x_{n_k} - y_{n_k}\| + \|y_{n_k} - z_{n_k}\| + \|z_{n_k} - w\| \\ &\le \|x_{n_k} - y_{n_k}\| + H(T(x_{n_k}), T(x)) + \|z_{n_k} - w\| \\ &\le \operatorname{dist}(x_{n_k}, T(x_{n_k})) + \|x_{n_k} - x\| + \|z_{n_k} - w\|. \end{aligned}$$

This implies by the regularity of $\{x_n\}$ that $\limsup_{k\to\infty} \|x_{n_k} - w\| \le \limsup_{k\to\infty} \|x_{n_k} - x\| = r$. Hence $w \in A(E, \{x_{n_k}\}) = \{x\}$. Therefore $x = w \in T(x)$. Next, we show that $T(x) = \{x\}$. Take any point $v \in T(x)$ and choose $u_n \in T(x_n)$ so that $\|v - u_n\| = \operatorname{dist}(v, T(x_n))$. Thus

$$\begin{aligned} \|x_n - v\| &\le \|x_n - y_n\| + \|y_n - u_n\| + \|u_n - v\| \\ &\le \|x_n - y_n\| + \operatorname{diam}(T(x_n)) + H(T(x), T(x_n)) \\ &\le \operatorname{dist}(x_n, T(x_n)) + \operatorname{diam}(T(x_n)) + \|x_n - x\|. \end{aligned}$$

This implies that $\limsup_{n\to\infty} \|x_n - v\| \le \limsup_{n\to\infty} \|x_n - x\| = r$. Hence $v \in A(E, \{x_n\}) = \{x\}$, and so $v = x$ for all $v \in T(x)$. That is, $T(x) = \{x\}$. Therefore $x \in \operatorname{End}(T)$. □

We observe that if X has the Opial property, then the assumption that $T : E \to \mathcal{K}(E)$ in Theorem 3.1 can be weakened to $T : E \to \mathcal{K}(X)$. For this, we need the following fact which is known as the demiclosed principle.

Proposition 3.2 *Let $(X, \|\cdot\|)$ be a Banach space having the Opial property, E be a nonempty closed convex subset of X, and $T : E \to \mathcal{K}(X)$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence in E and $x \in E$, then the conditions $x_n \rightharpoonup x$, $\operatorname{dist}(x_n, T(x_n)) \to 0$, and $\operatorname{diam}(T(x_n)) \to 0$ imply $x \in \operatorname{End}(T)$.*

Proof Since E is weakly closed, $x \in E$. For each $n \in \mathbb{N}$, we can choose $y_n \in T(x_n)$ and $z_n \in T(x)$ so that

$$\|x_n - y_n\| = \operatorname{dist}\bigl(x_n, T(x_n)\bigr) \quad \text{and} \quad \|y_n - z_n\| = \operatorname{dist}\bigl(y_n, T(x)\bigr).$$

Since $T(x)$ is compact, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \to w \in T(x)$. As in the proof of Theorem 3.1, we can obtain

$$\limsup_{k\to\infty} \|x_{n_k} - w\| \le \limsup_{k\to\infty} \|x_{n_k} - x\|.$$

The Opial property of X implies that $x = w \in T(x)$. Next, we show that $T(x) = \{x\}$. Take any point $v \in T(x)$ and choose $u_n \in T(x_n)$ so that $\|v - u_n\| = \operatorname{dist}(v, T(x_n))$. As in the proof of Theorem 3.1, we can obtain

$$\limsup_{n\to\infty} \|x_n - v\| \le \limsup_{n\to\infty} \|x_n - x\|.$$

The Opial property of X implies that $v = x$ and hence $T(x) = \{x\}$. Therefore $x \in \operatorname{End}(T)$. □

The following fact is an immediate consequence of Propositions 2.4 and 3.2.

Proposition 3.3 *Let $(X, \|\cdot\|)$ be a Banach space having the Opial property, E be a nonempty closed convex subset of X, and $T : E \to \mathcal{K}(X)$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence in E such that $x_n \rightharpoonup x \in E$ and $r_{x_n}(T(x_n)) \to 0$, then $x \in \operatorname{End}(T)$.*

Theorem 3.4 *Let $(X, \|\cdot\|)$ be a reflexive Banach space having the Opial property, E be a nonempty bounded closed convex subset of X, and $T : E \to \mathcal{K}(X)$ be a nonexpansive mapping. Then T has an endpoint if and only if T has the approximate endpoint property.*

Proof The necessity is clear. For the sufficiency, we suppose that T has the approximate endpoint property. Then there exists a sequence $\{x_n\}$ in E such that $r_{x_n}(T(x_n)) \to 0$. Since $\{x_n\}$ is bounded, by the reflexivity of X, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x \in E$. The conclusion follows from Proposition 3.3. □

As an immediate consequence of Theorem 3.4 and Proposition 2.2, we can obtain the following.

Corollary 3.5 *Let $(X, \|\cdot\|)$ be a reflexive Banach space having the Opial property, E be a nonempty bounded closed convex subset of X, and $f : E \to X$ be a single-valued nonexpansive mapping. Then f has a fixed point if and only if f has the approximate fixed point property.*

4 CAT(0) spaces

Let $[0, l]$ be a closed interval in $\mathbb{R}$ and x, y be two points in a metric space (X, d). A *geodesic* joining x to y is a map $\xi : [0, l] \to X$ such that $\xi(0) = x$, $\xi(l) = y$, and $d(\xi(s), \xi(t)) = |s - t|$ for all $s, t \in [0, l]$. The image of ξ is called a *geodesic segment* joining x and y which when unique is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points in X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset E of X is said to be *convex* if every pair of points $x, y \in E$ can be joined by a geodesic in X and the image of every such geodesic is contained in E.

A *geodesic triangle* $\triangle(p,q,r)$ in a geodesic space (X,d) consists of three points p, q, r in X and a choice of three geodesic segments $[p,q]$, $[q,r]$, $[r,p]$ joining them. A *comparison triangle* for geodesic triangle $\triangle(p,q,r)$ in X is a triangle $\overline{\triangle}(\bar{p},\bar{q},\bar{r})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{p},\bar{q}) = d(p,q)$, $d_{\mathbb{R}^2}(\bar{q},\bar{r}) = d(q,r)$, and $d_{\mathbb{R}^2}(\bar{r},\bar{p}) = d(r,p)$. A point $\bar{u} \in [\bar{p},\bar{q}]$ is called a *comparison point* for $u \in [p,q]$ if $d(p,u) = d_{\mathbb{R}^2}(\bar{p},\bar{u})$. Comparison points on $[\bar{q},\bar{r}]$ and $[\bar{r},\bar{p}]$ are defined in the same way.

Definition 4.1 A geodesic triangle $\triangle(p,q,r)$ in (X,d) is said to satisfy the *CAT(0) inequality* if for any $u, v \in \triangle(p,q,r)$ and for their comparison points $\bar{u}, \bar{v} \in \overline{\triangle}(\bar{p},\bar{q},\bar{r})$, one has

$$d(u,v) \le d_{\mathbb{R}^2}(\bar{u},\bar{v}).$$

A geodesic space X is said to be a *CAT(0) space* if all of its geodesic triangles satisfy the CAT(0) inequality. For other equivalent definitions and basic properties of CAT(0) spaces, we refer the reader to standard texts such as [27, 28]. It is well known that every CAT(0) space is uniquely geodesic. Notice also that pre-Hilbert spaces, $\mathbb{R}$-trees, and Euclidean buildings are examples of CAT(0) spaces (see [27, 29]).

It is known from Proposition 7 of [30] that if $\{x_n\}$ is a bounded sequence in a complete CAT(0) space X, then its asymptotic center $A(\{x_n\})$ consists of exactly one point.

We now give the concept of Δ-convergence and collect some of its basic properties.

Definition 4.2 ([31]) A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-*converge* to $x \in X$ if x is the unique asymptotic center of $\{x_{n_k}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case we write $x_n \xrightarrow{\Delta} x$ and call x the Δ-*limit* of $\{x_n\}$.

Lemma 4.3 ([31]) *Every bounded sequence in a complete CAT*(0) *space always has a* Δ-*convergent subsequence*.

Lemma 4.4 ([32]) *If E is a closed convex subset of a complete CAT*(0) *space and if $\{x_n\}$ is a bounded sequence in E, then the asymptotic center of $\{x_n\}$ is in E.*

Let $x, y \in X$, by Lemma 2.1 of [33] for each $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that

$$d(x,z) = (1-t)d(x,y) \quad \text{and} \quad d(y,z) = td(x,y). \tag{2}$$

We use the notation $tx \oplus (1-t)y$ for the unique point z satisfying (2).

Lemma 4.5 ([33]) *If (X,d) is a CAT*(0) *space, then*

$$d\big(z, tx \oplus (1-t)y\big)^2 \le td(z,x)^2 + (1-t)d(z,y)^2 - t(1-t)d(x,y)^2 \tag{3}$$

for all $t \in [0,1]$ and $x, y, z \in X$.

Inequality (3) is known as the (CN) inequality of Bruhat and Tits [34]. The following lemma is an analog of Proposition 3.2. It can be viewed as an extension of Proposition 3.7 in [31].

Lemma 4.6 *Let E be a nonempty closed convex subset of a complete CAT(0) space (X,d) and $T : E \to \mathcal{K}(X)$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence in E and $x \in E$, then the conditions $x_n \xrightarrow{\Delta} x$, $\operatorname{dist}(x_n, T(x_n)) \to 0$, and $\operatorname{diam}(T(x_n)) \to 0$ imply $x \in \operatorname{End}(T)$.*

Proof By Lemma 4.4, $x \in E$. For each $n \in \mathbb{N}$, we can choose $y_n \in T(x_n)$ and $z_n \in T(x)$ such that

$$d(x_n, y_n) = \operatorname{dist}(x_n, T(x_n)) \quad \text{and} \quad d(y_n, z_n) = \operatorname{dist}(y_n, T(x)).$$

Since $T(x)$ is compact, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \to w \in T(x)$. As in the proof of Theorem 3.1, we can obtain that $\limsup_{k\to\infty} d(x_{n_k}, w) \le \limsup_{k\to\infty} d(x_{n_k}, x)$. Therefore, $w \in A(\{x_{n_k}\}) = \{x\}$ and hence $x = w \in T(x)$. Next, we show that $T(x) = \{x\}$. Take any point $v \in T(x)$ and choose $u_n \in T(x_n)$ so that $d(v, u_n) = \operatorname{dist}(v, T(x_n))$. Again, as in the proof of Theorem 3.1, we can obtain $\limsup_{n\to\infty} d(x_n, v) \le \limsup_{n\to\infty} d(x_n, x)$. This implies that $v \in A(\{x_n\}) = \{x\}$ and hence $T(x) = \{x\}$. Therefore, $x \in \operatorname{End}(T)$. □

Theorem 4.7 *Let E be a nonempty bounded closed convex subset of a complete CAT(0) space (X,d) and $T : E \to \mathcal{K}(X)$ be a nonexpansive mapping. Then T has an endpoint if and only if T has the approximate endpoint property.*

Proof The necessity is clear. For the sufficiency, we suppose that T has the approximate endpoint property. Then there exists a sequence $\{x_n\}$ in E such that $r_{x_n}(T(x_n)) \to 0$ and hence $\operatorname{dist}(x_n, T(x_n)) \to 0$ and $\operatorname{diam}(T(x_n)) \to 0$ by Proposition 2.4. Since $\{x_n\}$ is bounded, by Lemmas 4.3 and 4.4, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \xrightarrow{\Delta} x \in E$. By Lemma 4.6, x is an endpoint of T. □

As a consequence of Theorem 4.7, we can obtain the following.

Corollary 4.8 ([35], Theorem 21) *Let E be a nonempty bounded closed convex subset of a complete CAT(0) space (X,d) and $f : E \to X$ be a single-valued nonexpansive mapping. Then f has a fixed point if and only if $\inf\{d(x, f(x)) : x \in E\} = 0$.*

If T is a single-valued nonexpansive mapping on a closed convex subset of a complete CAT(0) space, then $\operatorname{Fix}(T)$ is closed and convex (see, *e.g.*, Kirk [36]). The closedness of $\operatorname{Fix}(T)$ can be easily extended to the multivalued case. But the convexity of $\operatorname{Fix}(T)$ cannot be extended (see, *e.g.*, [37, 38]). However, if T is a multivalued nonexpansive mapping, then $\operatorname{End}(T)$ is always closed and convex as the following result.

Theorem 4.9 *Let E be a nonempty closed convex subset of a complete CAT(0) space (X,d) and $T : E \to \mathcal{CB}(X)$ be a nonexpansive mapping with $\operatorname{End}(T) \ne \emptyset$. Then $\operatorname{End}(T)$ is closed and convex.*

Proof Let $\{x_n\}$ be a sequence in $\operatorname{End}(T)$ such that $x_n \to x \in E$. We will show that $x \in \operatorname{End}(T)$. Since T is nonexpansive, we have

$$\begin{aligned}\operatorname{dist}\bigl(x, T(x)\bigr) &\leq d(x,x_n) + \operatorname{dist}\bigl(x_n, T(x)\bigr)\\ &\leq d(x,x_n) + H\bigl(T(x_n), T(x)\bigr)\\ &\leq 2d(x,x_n) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

This implies that $x \in T(x)$. Next, we show that $T(x) = \{x\}$. Take any point $v \in T(x)$. Since $x_n \in \operatorname{End}(T)$, we have

$$\begin{aligned}d(v,x) &\leq d(v,x_n) + d(x_n,x)\\ &= \operatorname{dist}\bigl(v, T(x_n)\bigr) + d(x_n,x)\\ &\leq H\bigl(T(x), T(x_n)\bigr) + d(x,x_n)\\ &\leq 2d(x,x_n) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence $v = x$. Since $v \in T(x)$ is arbitrary, $T(x) = \{x\}$. Therefore, $\operatorname{End}(T)$ is closed. Next, we show that $\operatorname{End}(T)$ is convex. Let $p, q \in \operatorname{End}(T)$ and $w = tp \oplus (1-t)q$ for some $t \in (0,1)$. We will show that $T(w) = \{w\}$. Take any point $v \in T(w)$. By the (CN) inequality, we have

$$\begin{aligned}d^2(v,w) &\leq td^2(v,p) + (1-t)d^2(v,q) - t(1-t)d^2(p,q)\\ &\leq tH^2\bigl(T(w), T(p)\bigr) + (1-t)H^2\bigl(T(w), T(q)\bigr) - t(1-t)d^2(p,q)\\ &\leq td^2(w,p) + (1-t)d^2(w,q) - t(1-t)d^2(p,q)\\ &= t(1-t)^2d^2(p,q) + (1-t)t^2d^2(p,q) - t(1-t)d^2(p,q)\\ &= 0.\end{aligned}$$

This implies that $v = w$. Since $v \in T(w)$ is arbitrary, we have $T(w) = \{w\}$. Therefore, $\operatorname{End}(T)$ is convex. □

5 Concluding remarks and open questions

Remark 5.1 As we have observed from Proposition 2.1, every Hilbert space is a uniformly convex Banach space and is a reflexive Banach space having the Opial property and is even a CAT(0) space. Thus, all results in this article also hold in Hilbert spaces.

In view of Theorems 3.4 and 4.7, we do not know if Theorem 3.1 can be extended to nonself-mappings. Therefore, the following question remains open.

Question 5.2 Let X be a uniformly convex Banach space, E be a nonempty bounded closed convex subset of X, and $T : E \to \mathcal{K}(X)$ be a nonexpansive mapping. If T has the approximate endpoint property, then does T have an endpoint?

In view of Theorems 3.1 and 3.4, along with Proposition 2.1, the following question should be of interest.

Question 5.3 Let X be a reflexive Banach space with normal structure, E be a nonempty bounded closed convex subset of X, and $T : E \to \mathcal{K}(E)$ be a nonexpansive mapping. If T has the approximate endpoint property, then does T have an endpoint?

One may observe that the (CN) inequality is a key tool in the proof of Theorem 4.9 and there is an inequality in uniformly convex Banach spaces similar to it (see [39], p.1133). However, Theorem 4.9 for uniformly convex Banach spaces is unknown. Therefore, the following question remains open.

Question 5.4 Let X be a uniformly convex Banach space, E be a nonempty closed convex subset of X, and $T : E \to \mathcal{CB}(X)$ be a nonexpansive mapping. Is $\mathrm{End}(T)$ convex?

Competing interests
The author declares that there is no conflict of interests regarding the publication of this article.

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