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A new hybrid algorithm and its numerical realization for two nonexpansive mappings

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Abstract

In the paper, first, we introduce a new hybrid projection algorithm and present its strong convergence theorem. Next, we analyze different hybrid algorithms in computing and conclude that our proposed algorithm has an advantage. Finally, the numerical experiments validate the efficiency and advantages of the new algorithm.

MSC: 90C47; 49J35

Keywords: nonexpansive mapping; hybrid algorithm; cyclic algorithm; parallel algorithm; strong convergence

1 Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$ and C be a nonempty closed convex subset of H. Recall that a mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. We denote by $\text{Fix}(T)$ the set of fixed points of T, *i.e.*, $\text{Fix}(T) = \{x \in C : Tx = x\}$.

The construction of common fixed points for a finite family of nonlinear mappings is of practical importance. In particular, iteration algorithms for finding common fixed points of a finite family of nonexpansive mappings have received extensive investigation (see [1–3]) since these algorithms have a variety of applications in inverse problem, image recovery, and signal processing (see [4–7]).

Mann's iteration algorithm [8] is often used to find a fixed point of nonexpansive mappings, but it has only weak convergence (see [9] for an example). However, strong convergence is often much more desirable than weak convergence in many problems that arise in infinite dimensional spaces (see [10] and references therein). So, attempts have been made to modify Mann's iteration algorithm so that strong convergence is guaranteed. Let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is a maximal monotone operator [11]. Inspired by Solodov and Svaiter's hybrid method for finding a zero of a maximal monotone operator [12], Nakajo and Takahashi [13] first introduced a hybrid algorithm for a nonexpansive mapping. Thereafter, some hybrid algorithms have been studied extensively since they have strong convergence (see [14–18]).

In this paper, motivated by Eckstein and Svaiter's splitting methods for approximating a zero of the sum of two maximal monotone operators [19], we introduce a new hybrid algorithm. Let $T, S : C \to C$ be two nonexpansive mappings such that $\text{Fix}(T) \cap \text{Fix}(S) \neq \emptyset$. We consider the following algorithm.

Algorithm 1

$$
\begin{cases}
x_0 \in C \text{ chosen arbitrarily,} \\
y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\
z_n = \beta_n [\gamma_n y_n + (1 - \gamma_n) x_n] + (1 - \beta_n) S y_n, \\
C_n = \{z \in C : \sigma \|z_n - z\|^2 + (1 - \sigma) \|y_n - z\|^2 \leq \|x_n - z\|^2\}, \\
Q_n = \{z \in C : \langle x_n - z, x_n - x_0 \rangle \leq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0
\end{cases}
\tag{1}
$$

for each $n \geq 0$, where P_K denotes the metric projection onto the set K and $\alpha_n, \beta_n \in [0, 1 - \delta], \delta \in [0, 1), \gamma_n \in [0, 1], \sigma \in (0, 1)$ with some conditions.

2 Relation to the previous work

In this section, we analyze and compare Algorithm 1 with two important hybrid algorithms which are simple and easily realized. For the purposes of comparison, set $\gamma_n = 1$ in Algorithm 1, which is reasonable from the numerical experiments in Section 5. In the case where $\gamma_n = 1$, Algorithm 1 actually is a modification of the following cyclic algorithm:

$$
\begin{cases}
y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\
x_{n+1} = \beta_n y_n + (1 - \beta_n) S y_n
\end{cases}
\tag{2}
$$

for each $n \geq 0$, where $x_0 \in C$.

In [20], Takahashi *et al.* modified the cyclic algorithm (2) and introduced another hybrid algorithm.

Algorithm 2

$$
\begin{cases}
x_0 \in C \text{ chosen arbitrarily,} \\
y_n = \alpha_n x_n + (1 - \alpha_n) T_{[n]} x_n, \\
C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n = \{z \in C : \langle x_n - z, x_n - x_0 \rangle \leq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0
\end{cases}
$$

for each $n \geq 0$, where $0 \leq \alpha_n \leq \alpha < 1$, $T_{[n]} = T_{n \bmod 2}$ and the mod function takes values in $\{1, 2\}$ and $T_1 = T$ and $T_2 = S$.

The computational complexity of Algorithm 1 on every step is one computation of a metric projection and two values of S and T, while the computational complexity of Algorithm 2 on every step is one computation of a metric projection and one value of S or T. In general, the computational cost of metric projection is larger than that of operators.

By modifying the parallel algorithm (see [4]), it is easy to obtain the following algorithm (see [21] for details).

Algorithm 3

$$
\begin{cases}
x_0 \in C \text{ chosen arbitrarily,} \\
y_n = \alpha_n x_n + \beta_n T x_n + \gamma_n S x_n, \\
C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n = \{z \in C : \langle x_n - z, x_n - x_0 \rangle \leq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0
\end{cases}
$$

for each $n \geq 0$, where $\alpha_n, \beta_n, \gamma_n \in [0,1]$, $\alpha_n + \beta_n + \gamma_n = 1$ and $\limsup_{n \to \infty} \alpha_n < 1$, $\liminf_{n \to \infty} \beta_n \gamma_n > 0$.

In the sense of computational complexity Algorithm 3 is similar to Algorithm 1. However, it is generally recognized that the cyclic algorithm (like the Gauss-Seidel iteration) is faster than the parallel algorithm (like Jacob iteration).

3 Preliminaries

We use the following notation:

- $\rightharpoonup$ for weak convergence and $\rightarrow$ for strong convergence;
- $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 3.1 *We have the identity in a real Hilbert space H:*

$$
\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v \rangle
$$

for all $u, v \in H$.

Lemma 3.2 (Goebel and Kirk [22]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$. If a sequence $\{x_n\}$ in C is such that $x_n \rightharpoonup z$ and $x_n - T x_n \to 0$, then $z = Tz$.*

Lemma 3.3 *Let C be a nonempty closed convex subset of real Hilbert space H and let P_C be the (metric or nearest point) projection from H onto C (i.e., for any $x \in H$, $P_C x$ is the only point in C such that $\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}$). Then, for any $x \in H$ and $z \in C$, $z = P_C x$ if and only if*

$$
\langle x - z, y - z \rangle \leq 0
$$

for all $y \in C$.

Lemma 3.4 (Martinez-Yanes and Xu [23]) *Let C be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w\{x_n\} \subset C$ and*

satisfies the condition

$$\|x_n - u\| \le \|u - q\|$$

for all $n \ge 1$, then $x_n \to q$.

Lemma 3.5 *Let H be a real Hilbert space. Let C be a nonempty closed convex subset of H and let $x, y, z \in H$. For any real number $\sigma \in [0,1]$, the set*

$$D := \left\{ v \in C : \sigma \|z - v\|^2 + (1 - \sigma)\|y - v\|^2 \le \|x - v\|^2 \right\}$$

is convex (*and closed*).

Proof In fact, the defining inequality in D is equivalent to the inequality

$$\langle \sigma(z - x) + (1 - \sigma)(y - x), x - v \rangle \le -\frac{1}{2}\left(\sigma \|z - x\|^2 + (1 - \sigma)\|y - x\|^2 \right).$$

This inequality is affine in v and hence the set D is convex. □

4 Main results

In this section, we first present a strong convergence theorem and its proof for Algorithm 1. Then we extend it to a finite family of nonexpansive mappings.

Theorem 4.1 *Let C be a nonempty closed convex subset of a Hilbert space H and $T, S : C \to C$ be two nonexpansive mappings such that $\mathrm{Fix}(T) \cap \mathrm{Fix}(S) \ne \emptyset$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are the sequences in $[0,1]$ such that $\alpha_n, \beta_n \le 1 - \delta$ for some $\delta \in (0,1]$. Assume $\sigma \in (0,1)$. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges in norm to $P_{\mathrm{Fix}(T) \cap \mathrm{Fix}(S)} x_0$.*

Proof First, observe that C_n is convex by Lemma 3.5. Next, we show that $\mathrm{Fix}(T) \cap \mathrm{Fix}(S) \subset C_n$ for all $n \ge 0$. To observe this, arbitrarily take $p \in \mathrm{Fix}(T) \cap \mathrm{Fix}(S)$, and we have

$$\|y_n - p\| \le \alpha_n \|x_n - p\| + (1 - \alpha_n)\|Tx_n - p\| \le \|x_n - p\|$$

and

$$\begin{aligned} \|z_n - p\| &\le \beta_n\big[\gamma_n\|y_n - p\| + (1 - \gamma_n)\|x_n - p\|\big] + (1 - \beta_n)\|Sy_n - p\| \\ &\le \beta_n\|x_n - p\| + (1 - \beta_n)\|y_n - p\| \\ &\le \|x_n - p\|. \end{aligned}$$

Combining the above inequalities, it follows that, for any $\sigma \in (0,1)$,

$$\sigma\|z_n - p\|^2 + (1 - \sigma)\|y_n - p\|^2 \le \|x_n - p\|^2,$$

and so $p \in C_n$ for all $n \ge 0$.

Next, we show that

$$\text{Fix}(T) \cap \text{Fix}(S) \subset Q_n \tag{3}$$

for all $n \geq 0$ by induction. For $n = 0$, we have $\text{Fix}(T) \cap \text{Fix}(S) \subset C = Q_0$. Assume that $\text{Fix}(T) \cap \text{Fix}(S) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Lemma 3.3, we have

$$\langle x_{n+1} - z, x_{n+1} - x_0 \rangle \leq 0$$

for all $z \in C_n \cap Q_n$. Since $\text{Fix}(T) \cap \text{Fix}(S) \subset C_n \cap Q_n$, by the induction assumption, the last inequality holds, in particular, for all $z \in \text{Fix}(T) \cap \text{Fix}(S)$. This together with the definition of Q_{n+1} implies that $\text{Fix}(T) \cap \text{Fix}(S) \subset Q_{n+1}$. Hence (3) holds for all $n \geq 0$.

Now, since $x_n = P_{Q_n} x_0$ (by the definition of Q_n) and $\text{Fix}(T) \cap \text{Fix}(S) \subset Q_n$, we have

$$\|x_n - x_0\| \leq \|p - x_0\|$$

for all $p \in \text{Fix}(T) \cap \text{Fix}(S)$. In particular, $\{x_n\}$ is bounded and

$$\|x_n - x_0\| \leq \|q - x_0\|, \tag{4}$$

where $q = P_{\text{Fix}(T) \cap \text{Fix}(S)} x_0$. The fact that $x_{n+1} \in Q_n$ implies that $\langle x_{n+1} - x_n, x_n - x_0 \rangle \geq 0$. This together with Lemma 3.1 implies that

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \left\| (x_{n+1} - x_0) - (x_n - x_0) \right\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\
&\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2,
\end{aligned}$$

which implies

$$\|x_{n+1} - x_n\| \to 0 \tag{5}$$

as $n \to \infty$. Observe that $x_{n+1} \in C_n$ implies that

$$\sigma \|z_n - x_{n+1}\|^2 + (1-\sigma)\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2.$$

Due to $\sigma \in (0,1)$, we have

$$\|z_n - x_{n+1}\| \to 0, \qquad \|y_n - x_{n+1}\| \to 0 \tag{6}$$

as $n \to \infty$, which yields

$$\|z_n - y_n\| \to 0. \tag{7}$$

Combining (5) and (6), we obtain

$$\|z_n - x_n\| \to 0, \qquad \|y_n - x_n\| \to 0 \tag{8}$$

as $n \to \infty$. Noting that $(1 - \alpha_n)(Tx_n - x_n) = y_n - x_n$, we obtain

$$\|Tx_n - x_n\| = \frac{1}{1 - \alpha_n}\|y_n - x_n\|.$$

Since $\alpha_n \leq 1 - \delta$, by (8), we have

$$\|Tx_n - x_n\| \to 0 \tag{9}$$

as $n \to \infty$. From $z_n = \beta_n[\gamma_n y_n + (1 - \gamma_n)x_n] + (1 - \beta_n)Sy_n$, we have

$$\|Sy_n - z_n\| = \frac{\beta_n}{1 - \beta_n}\big\|\gamma_n(y_n - z_n) + (1 - \gamma_n)(x_n - z_n)\big\|$$

$$\leq \frac{1}{1 - \beta_n}\big(\gamma_n\|y_n - z_n\| + (1 - \gamma_n)\|x_n - z_n\|\big),$$

which, from (7)-(8), yields

$$\|Sy_n - z_n\| \to 0$$

as $n \to \infty$ and so

$$\|Sx_n - x_n\| \leq \|Sx_n - Sy_n\| + \|Sy_n - z_n\| + \|z_n - x_n\|$$

$$\leq \|x_n - y_n\| + \|Sy_n - z_n\| + \|z_n - x_n\|$$

$$\to 0 \tag{10}$$

as $n \to \infty$. Therefore, by (9)-(10) and Lemma 3.2, we obtain

$$\omega_w(x_n) \subset \mathrm{Fix}(T) \cap \mathrm{Fix}(S).$$

This, together with (4) and Lemma 3.4, guarantees the strong convergence of $\{x_n\}$ to $P_{\mathrm{Fix}(T) \cap \mathrm{Fix}(S)}x_0$. This completes the proof. $\quad\square$

Let $S = T$ in Algorithm 1, we have a hybrid algorithm for a nonexpansive mapping T:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ z_n = \beta_n[\gamma_n y_n + (1 - \gamma_n)x_n] + (1 - \beta_n)Ty_n, \\ C_n = \{z \in C : \sigma\|z_n - z\|^2 + (1 - \sigma)\|y_n - z\|^2 \leq \|x_n - z\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x_0 \end{cases} \tag{11}$$

for each $n \geq 0$, where $\alpha_n, \beta_n \in [0, 1 - \delta]$, $\delta \in [0, 1)$, $\gamma_n \in [0, 1]$, $\sigma \in (0, 1)$.

When $\gamma_n = 0$, the algorithm (11) is actually a modification of the Ishikawa iteration ([24]). In 2006, Martinez-Yanes and Xu [23] modified the Ishikawa iteration and introduced the

following algorithm:

$$
\begin{cases}
x_0 \in C \text{ chosen arbitrarily,} \\
y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\
z_n = \beta_n x_n + (1 - \beta_n) T y_n, \\
C_n = \{z \in C : \|z_n - z\|^2 \leq \beta_n \|x_n - z\|^2 + (1 - \beta_n)\|y_n - z\|^2\}, \\
Q_n = \{z \in C : \langle x_n - z, x_n - x_0 \rangle \leq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0
\end{cases}
\tag{12}
$$

for each $n \geq 0$, where $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences in $[0,1]$ such that $\alpha_n \to 1$ and $\beta_n \leq 1 - \delta$ for some $\delta \in (0,1]$.

It is easily observed that the algorithm (11) with $\gamma_n = 0$ is different from the algorithm (12) in the definitions of the sets C_n and the conditions on α_n.

From Theorem 4.1, we get directly the following result.

Corollary 4.1 *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are the sequences in $[0,1]$ such that $\alpha_n, \beta_n \leq 1 - \delta$ for some $\delta \in (0,1]$. Assume $\sigma \in (0,1)$. Then the sequence $\{x_n\}$ generated by the algorithm (11) converges in norm to $P_{\mathrm{Fix}(T)} x_0$.*

Letting $\gamma_n = 1$ in Algorithm 1, from Theorem 4.1, we obtain the following result.

Corollary 4.2 *Let C be a nonempty closed convex subset of a Hilbert space H and $T, S : C \to C$ be two nonexpansive mappings such that $\mathrm{Fix}(T) \cap \mathrm{Fix}(S) \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $[0,1]$ such that $\alpha_n, \beta_n \leq 1 - \delta$ for some $\delta \in (0,1]$. Assume $\sigma \in (0,1)$. Then the sequence $\{x_n\}$ generated by the following algorithm:*

$$
\begin{cases}
x_0 \in C \text{ chosen arbitrarily,} \\
y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\
z_n = \beta_n y_n + (1 - \beta_n) S y_n, \\
C_n = \{z \in C : \sigma \|z_n - z\|^2 + (1 - \sigma)\|y_n - z\|^2 \leq \|x_n - z\|^2\}, \\
Q_n = \{z \in C : \langle x_n - z, x_n - x_0 \rangle \leq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0 \quad \text{for each } n \geq 0
\end{cases}
\tag{13}
$$

converges in norm to $P_{\mathrm{Fix}(T) \cap \mathrm{Fix}(S)} x_0$.

Let $\{T_k : C \to C, k = 1, 2, \ldots, N\}$ be a finite family of nonexpansive mappings such that $\bigcap_{k=1}^N \mathrm{Fix}(T_k) \neq \emptyset$. The algorithm (13) can be extended to a finite family of nonexpansive

mappings as follows:

$$
\begin{cases}
x_0 \in C \text{ chosen arbitrarily,} \\
y_{n,1} = \alpha_{n,1} x_n + (1 - \alpha_{n,1}) T_1 x_n, \\
y_{n,k} = \alpha_{n,k} y_{n,k-1} + (1 - \alpha_{n,k}) T_k y_{n,k-1}, \\
C_n = \{z \in C : \sum_{k=1}^{N} \sigma_k \|y_{n,k} - z\|^2 \leq \|x_n - z\|^2\}, \\
Q_n = \{z \in C : \langle x_n - z, x_n - x_0 \rangle \leq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0
\end{cases}
\tag{14}
$$

for each $n \geq 0$ and $k = 2, \ldots, N$, where $\alpha_{n,k} \in [0, 1 - \delta]$, $\delta \in (0, 1]$, $\sigma_k \in (0, 1)$ for each $k = 1, 2, \ldots, N$, and $\sum_{k=1}^{N} \sigma_k = 1$. Note that the algorithm (14) is different from the cyclic and parallel algorithms [20, 21] and other algorithms for finding a common fixed point of a finite family of nonexpansive mappings.

Extending Corollary 4.2 to a finite family of nonexpansive mappings, we can easily obtain the strong convergence for the algorithm (14), whose proof is similar to Theorem 4.1 and omitted here.

Corollary 4.3 *Let C be a nonempty closed convex subset of a Hilbert space H and $\{T_k : C \to C, k = 1, 2, \ldots, N\}$ be a finite family of nonexpansive mappings such that $\bigcap_{k=1}^{N} \mathrm{Fix}(T_k) \neq \emptyset$. Assume that $\{\alpha_{n_k}\}$ is the sequence in $[0, 1]$ such that $\alpha_{n_k} \leq 1 - \delta$ for each $k = 1, 2, \ldots, N$ for some $\delta \in (0, 1]$. Assume that $\sigma_k \in (0, 1)$ for each $k = 1, 2, \ldots, N$, and $\sum_{k=1}^{N} \sigma_k = 1$. Then the sequence $\{x_n\}$ generated by the algorithm (14) converges in norm to $P_{\bigcap_{k=1}^{N} \mathrm{Fix}(T_k)} x_0$.*

5 Numerical experiments

Many authors studied hybrid algorithms and analyzed strong convergence of the algorithms, however, as far as we know, the results of the realization for the algorithms are very limited (see, for example, [25, 26]).

Recently, He *et al.* [27] pointed out that it is difficult to realize the hybrid algorithm in actual computing programs because the specific expression of $P_{C_n \cap Q_n} x_0$ cannot be got, in general. For the special case $C = H$, where C_n and Q_n are two half-spaces, they obtained the specific expression of $P_{C_n \cap Q_n} x_0$ and thus easily realized the hybrid algorithm proposed by Nakajo and Takahashi [13].

In the case $C = H$, following some ideas of He *et al.* [27], we obtain the specific expression of $P_{C_n \cap Q_n} x_0$ of Algorithm 1 as follows:

$$
\begin{cases}
x_0 \in H \text{ chosen arbitrarily,} \\
x_{n+1} = x_n \quad \text{if } Tx_n = Sx_n = x_n, \\
\left.\begin{aligned}
&y_n = \alpha_n x_n + (1 - \alpha_n) Tx_n, \\
&z_n = \beta_n [\gamma_n y_n + (1 - \gamma_n) x_n] + (1 - \beta_n) Sy_n, \\
\end{aligned}\right\} \quad \text{if } Tx_n \neq x_n \text{ or } Sx_n \neq x_n, \\
u_n = \sigma z_n + (1 - \sigma) y_n - x_n, \\
v_n = \frac{-(\sigma \|z_n - x_n\|^2 + (1-\sigma)\|y_n - x_n\|^2)}{2}, \\
C_n = \{z \in H : \langle u_n, x_n - z \rangle \leq v_n\}, \\
Q_n = \{z \in H : \langle x_n - z, x_n - x_0 \rangle \leq 0\}, \\
x_{n+1} = p_n, \quad \text{if } p_n \in Q_n, \\
x_{n+1} = q_n, \quad \text{if } p_n \notin Q_n
\end{cases}
\tag{15}
$$

for each $n \geq 0$, where

$$
p_n = x_0 - \frac{\langle u_n, x_0 - x_n \rangle + v_n}{\|u_n\|^2} u_n,
$$

$$
q_n = \left(1 - \frac{\langle x_0 - x_n, x_n - p_n \rangle}{\langle x_0 - x_n, w_n - p_n \rangle}\right) p_n + \frac{\langle x_0 - x_n, x_n - p_n \rangle}{\langle x_0 - x_n, w_n - p_n \rangle} w_n,
$$

$$
w_n = x_n - \frac{v_n}{\|u_n\|^2} u_n.
$$

Let $\mathbb{R}^2$ be a 2-dimensional Euclidean space with the usual inner product $\langle v^{(1)}, v^{(2)} \rangle = v_1^{(1)} v_1^{(2)} + v_2^{(1)} v_2^{(2)}$ for all $v^{(1)} = (v_1^{(1)}, v_2^{(1)})^T, v^{(2)} = (v_1^{(2)}, v_2^{(2)})^T \in \mathbb{R}^2$, and the norm $\|v\| = \sqrt{v_1^2 + v_2^2}$ ($v = (v_1, v_2)^T \in \mathbb{R}^2$). He *et al.* [27] defined a mapping:

$$
T : v = (v_1, v_2)^T \mapsto \left(\sin \frac{v_1 + v_2}{\sqrt{2}}, \cos \frac{v_1 + v_2}{\sqrt{2}} \right)^T
$$

and showed that T is nonexpansive. It is easily to observe that T has a fixed point in the unit disk. Define a mapping $S : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$
S(v) := P_K(v)
$$

for all $v \in \mathbb{R}^2$, where $K = \{v \in \mathbb{R}^2 : \|v\| \leq 1\}$. It is well known that S is nonexpansive (actually firmly nonexpansive) and $\text{Fix}(S) = K$. Thus we get $\text{Fix}(T) \cap \text{Fix}(S) = \text{Fix}(T) \neq \emptyset$.

Denote by $E(x) = \frac{\|x - Tx\| + \|x - Sx\|}{\|x\|}$ the relative rate of convergence of the algorithms since we do not know the exact value of the projection of x_0 onto common fixed points set of S and T.

In the numerical results listed in the following tables, *Iter.* and *Sec.* denote the number of iterations and the cpu time in seconds, respectively. We took $E(x) < \varepsilon$ as the stopping criterion and $\varepsilon = 10^{-4}$ unless specified otherwise. We chose different x_0 as initial point. The algorithms were coded in Matlab 7.1 and run on a personal computer.

We firstly investigated the choice of the parameters of Algorithm 1.

Table 1 Algorithm 1 with $\alpha_n = 0.1$, $\beta_n = 0.1$, $\gamma_n = 1.0$

x_0	σ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$(0,0)$	Iter.	372	472	404	538	539	417	461	452	523
$(2,7)$	Iter.	167	760	249	897	385	321	615	355	1,182
$(-5,2)$	Iter.	915	1,424	2,047	1,597	972	1,077	1,672	2,309	2,101
$(-3,-4)$	Iter.	1,032	669	1,930	1,166	934	766	753	1,181	530

Table 2 Algorithm 1 with $\alpha_n = 0.1$, $\beta_n = 0.1$, $\sigma = 0.1$

x_0	γ_n	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$(0,0)$	Iter.	539	626	393	441	408	303	576	759	469	422	372
$(2,7)$	Iter.	205	825	242	488	1,278	214	301	362	174	750	167
$(-5,2)$	Iter.	1,237	1,373	1,853	1,627	2,275	2,173	1,609	2,036	1,316	2,633	915
$(-3,-4)$	Iter.	919	780	1,647	958	875	957	1,411	663	1,146	573	1,032

Table 3 Algorithm 1 with $\beta_n = 0.1$, $\gamma_n = 1.0$, $\sigma = 0.1$

x_0	α_n	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$(0,0)$	Iter.	372	524	853	1,080	1,398	–	–	–	–
$(2,7)$	Iter.	167	482	625	1,479	2,186	2,046	3,115	4,111	14,440
$(-5,2)$	Iter.	915	1,418	2,950	3,998	6,098	6,827	1,330	20,114	68,421
$(-3,-4)$	Iter.	1,032	2,306	2,262	1,975	4,341	2,882	6,321	11,761	38,330

In Algorithm 1, there are four parameters, σ, α_n, β_n, γ_n (indeed three nonnegative sequences). Thus we need to set them before performing the algorithm.

Tables 1 and 2 show the effect of different choice of σ and γ_n and illustrated that the number of iteration was relatively small as $\sigma = 0.1$ and $\gamma_n = 1.0$ for the initial points $(0,0)$, $(2,7)$, and $(-5,2)$.

Table 3 lists the impact of α_n on the efficiency of the algorithm and showed that the number of iteration was increasing with α_n, with few exceptions. For $\alpha_n \geq 0.6$, the stopping criterion could not even be satisfied at initial point $(0,0)$. The impact of β_n was similar to that of α_n and we do not list it here.

Next, we discuss the choice of parameters in Algorithms 2 and 3. We took $\varepsilon = 5 \times 10^{-4}$, since for Algorithms 2 and 3, it was difficult for the relative rate of convergence to run up to 10^{-4}. For Algorithm 2, we tested $\alpha_n = [0.1, 0.9]$ and the numerical results are reported in Figure 1, which showed that the number of iterations was less for small α_n. Since the number of iterations for $\alpha_n > 0.6$ was generally larger than those for $\alpha_n \leq 0.6$, we only report the results for $0.1 \leq \alpha_n \leq 0.6$. For Algorithm 3, Figure 2 reports the number of iterations for $0.1 \leq \alpha_n \leq 0.6$, where $\beta_n = \gamma_n = (1 - \alpha_n)/2$. The conclusion was similar to Algorithm 3, that is, small α_n was better.

Finally, we compare Algorithm 1 with Algorithms 2 and 3. We took $\alpha_n = 0.1$, $\beta_n = 0.1$, $\gamma_n = 1.0$, $\sigma = 0.1$ for Algorithm 1, $\alpha_n = 0.1$ in Algorithm 2 and $\alpha_n = 0.1$, $\beta_n = \gamma_n = 0.45$ in Algorithm 3. The stopping criterion was $E(x) < \varepsilon = 5 \times 10^{-5}$. Table 4 illustrates the efficiency of the Algorithm 1, both from the points of view of number of iterations and cpu time.

6 Conclusions

Hybrid algorithms for nonexpansive mappings have extensively been studied over the past decade. In this paper, we introduced a new hybrid algorithm and, for the first time in the
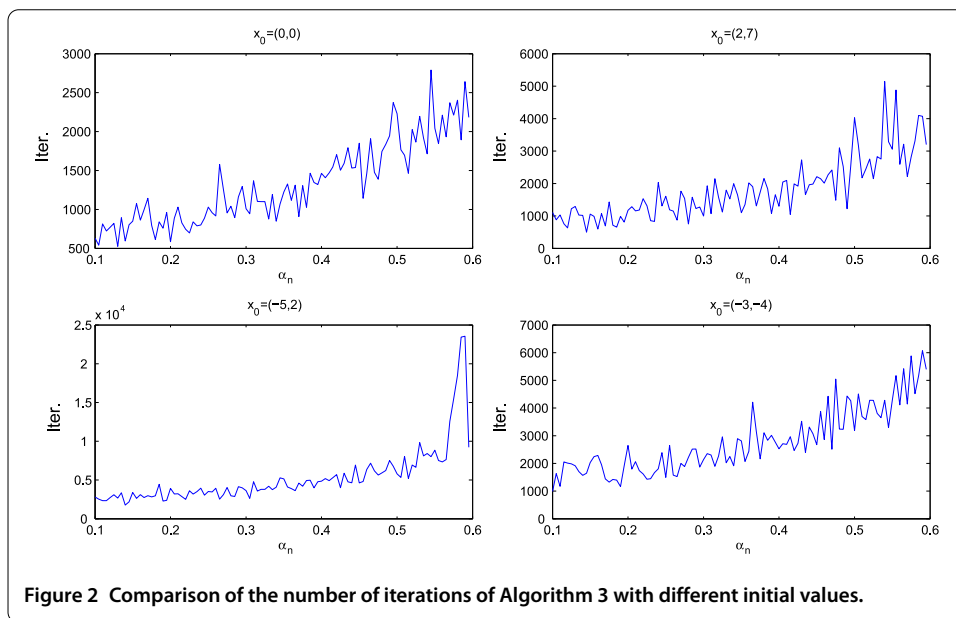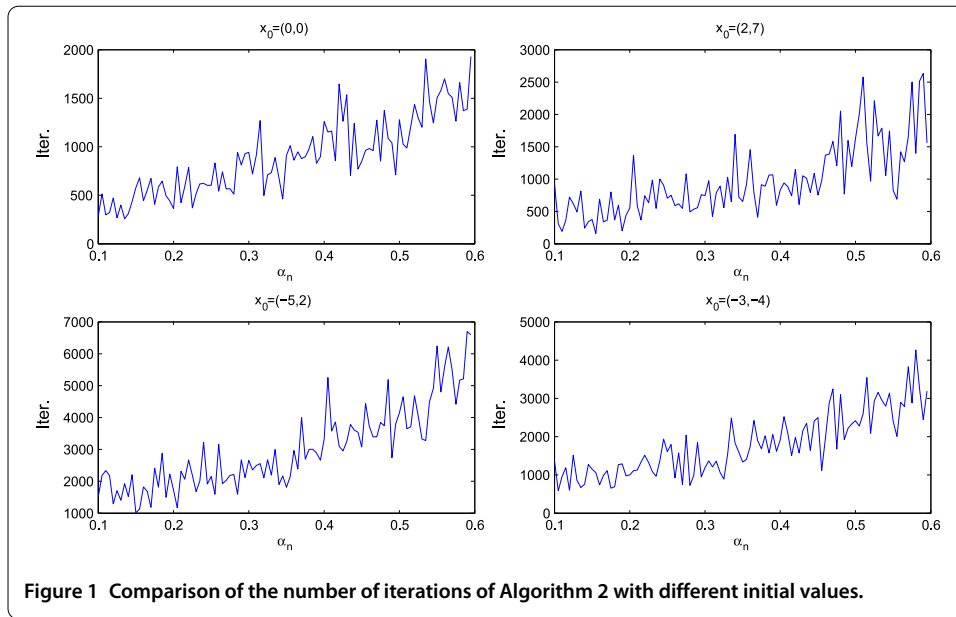
Figure 1 Comparison of the number of iterations of Algorithm 2 with different initial values.

Figure 2 Comparison of the number of iterations of Algorithm 3 with different initial values.

Table 4 Comparison of Algorithm 1 with Algorithms 2 and 3

x_0	Algorithm 1		Algorithm 2		Algorithm 3	
	Iter.	Sec.	Iter.	Sec.	Iter.	Sec.
$(0,0)$	238	0.0156	290	0.0156	610	0.0625
$(2,7)$	162	0.0313	922	0.0781	792	0.0781
$(-5,2)$	532	0.0469	1,524	0.1094	2,305	0.3594
$(-3,-4)$	511	0.0625	1,356	0.0938	1,034	0.1250

literature, compared the different hybrid algorithms in computing. Numerical examples were provided, which showed the advantages of the new algorithm.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the manuscript.

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