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Strong convergence theorems for equilibrium problems and asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense

Ren-Xing Ni[1,2], Jian-Shuai Jin[2,1] and Ching-Feng Wen[3*]

*Correspondence:
cfwen@kmu.edu.tw
[3]Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung, 807, Taiwan
Full list of author information is available at the end of the article

Abstract

In this paper, we investigate common solutions to a family of mixed equilibrium problems with a relaxed η-α-monotone mapping and a nonlinear operator equation involving an infinite family of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense. Strong convergence theorems of common solutions are established in a strictly convex and uniformly smooth Banach space. These results extend many important recent ones in the literature.

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1 Introduction

It is well known that equilibrium problems and mixed equilibrium problems have been important tools for solving problems arising in the fields of linear or nonlinear programming, complementary problems, optimization problems, variational inequalities, fixed point problems and in certain applications to economics, physics, mechanics and engineering sciences, *etc.* One of the most significant topics in the theory of equilibria is to develop effective and implementable algorithms for solving equilibrium problems and mixed equilibrium problems (see, *e.g.*, [1–13] and the references therein).

The aim of this paper is to present an iterative method for solving solutions of a family of mixed equilibrium problems with a relaxed η-α-monotone mapping and a nonlinear operator equation involving an infinite family of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense.

The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, an iterative algorithm is presented. A strong convergence theorem is established in a reflexive Banach space. Some results in Hilbert spaces are also discussed.

2 Preliminaries

In this paper, without other specifications, let N^+ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively, C be a nonempty, closed, and convex subset of a real reflexive Banach space E with the dual space E^*. The norm and the dual pair between E^* and E are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Recall that the normalized duality mapping J from E to 2^{E^*} is defined by $Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$.

Recall that E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is said to be smooth provided $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. It is said to be uniformly smooth if the limit is attained uniformly for $x, y \in E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E and E is uniformly smooth if and only if E^* is uniformly convex.

In the paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. Recall that a Banach space E enjoys the Kadec-Klee property if for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$ (see, *e.g.*, [14] and the references therein). It is well known that if E is a uniformly convex Banach space, then E enjoys the Kadec-Klee property.

As we all know, if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, recently, Alber [15] introduced a generalized projection operator Π_C in a Banach space E which is an analog of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Let $\phi : E \times E \to \mathbb{R}$ be a function defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.1}$$

Observe that, in a Hilbert space H, (2.1) is reduced to $\phi(x,y) = \|x - y\|^2$ for all $x, y \in H$. Recall that the generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$; that is, $\Pi_C x = \hat{x}$, where $\hat{x}$ is a solution to the minimization problem

$$\phi(\hat{x}, x) = \inf_{y\in C} \phi(y, x), \tag{2.2}$$

the existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J (see, *e.g.*, [14, 15]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$\big(\|y\| - \|x\|\big)^2 \le \phi(y,x) \le \big(\|y\| + \|x\|\big)^2, \quad \forall x, y \in E \tag{2.3}$$

and

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x, y, z \in E. \tag{2.4}$$

Remark 2.1 If E is a smooth, strictly convex, and reflexive Banach space, then $\phi(x,y) = 0$ if and only if $x = y$ (see [14, 15] and the references therein).

In [16], Fang and Huang introduced a concept called a relaxed η-α-monotone mapping. A mapping $A : C \to E^*$ is said to be relaxed η-α-monotone if there exists a mapping $\eta : C \times C \to E$ and a function $\alpha : E \to \mathbb{R}$ with $\alpha(tz) = t^p\alpha(z)$ for all $t > 0$ and $z \in E$, where $p > 1$ is a constant, such that

$$\langle Ax - Ay, \eta(x,y)\rangle \geq \alpha(x - y), \quad \forall x, y \in C.$$

Especially, if $\eta(x,y) = x - y$ for all $x, y \in C$ and $\alpha(z) = k\|z\|^p$, where $p > 1$ and $k > 1$ are two constants, then A is said to be p-monotone (see, *e.g.*, [17–19]). They proved that, under some suitable assumptions, the following variational inequality is solvable: find $x \in C$ such that

$$\langle Ax, \eta(y,x)\rangle + f(y) - f(x) \geq 0, \quad \forall y \in C, \tag{2.5}$$

where f is a function from C to $\mathbb{R} \cup \{\infty\}$. They also proved that the variational inequality (2.5) is equivalent to the following: find $x \in C$ such that

$$\langle Ay, \eta(y,x)\rangle + f(y) - f(x) \geq \alpha(y - x), \quad \forall y \in C. \tag{2.6}$$

Recently, in [5], Chen *et al.* studied the following mixed equilibrium problem: find $x \in C$ such that

$$\Theta(x,y) + \langle Ax, \eta(y,x)\rangle + f(y) - f(x) \geq 0, \quad \forall y \in C. \tag{2.7}$$

Here Θ is a bifunction from $C \times C$ to $\mathbb{R}$, $f : C \times C \to \mathbb{R} \cup \{+\infty\}$ is a proper convex function, $A : C \to E^*$ is a relaxed η-α-monotone mapping and η is a mapping from $C \times C$ to E. Denote the set of solutions of the problem (2.7) by $EP(\Theta, A)$, *i.e.*,

$$EP(\Theta, A) = \{x \in C \mid \Theta(x,y) + \langle Ax, \eta(y,x)\rangle + f(y) - f(x) \geq 0, \forall y \in C\}.$$

Special cases: (I) If $A = 0$, then the problem (2.7) is equivalent to find $x \in C$ such that

$$\Theta(x,y) + f(y) - f(x) \geq 0, \quad \forall y \in C. \tag{2.8}$$

This is called the mixed equilibrium problem. Denote the set of solutions of (2.8) by $MEP(\Theta, f)$.

(II) If $A = 0, f = 0$, then the problem (2.7) is equivalent to find $x \in C$ such that

$$\Theta(x,y) \geq 0, \quad \forall y \in C. \tag{2.9}$$

This is called the equilibrium problem. Denote the set of solutions of (2.9) by $EP(\Theta)$.

(III) If $\Theta = 0$, then the problem (2.7) is equivalent to the variational inequality (2.5) and (2.6). Denote the set of solutions of (2.5) and (2.6) by Ω.

In order to solve the equilibrium problem, the bifunction Θ is usually to be assumed that following conditions are satisfied:

(C1) $\Theta(x,x) = 0$ for all $x \in C$;
(C2) Θ is monotone; that is, $\Theta(x,y) + \Theta(y,x) \leq 0$ for all $x,y \in C$;
(C3) for all $x,y,z \in C$, $\limsup_{t\downarrow 0} \Theta(tz + (1-t)x, y) \leq \Theta(x,y)$;
(C4) for all $x \in C$, $\Theta(x,\cdot)$ is convex and lower semicontinuous.

Let C be a nonempty subset of E and let $T: C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the fixed point set of T. T is said to be asymptotically regular on C if for any bounded subset K of C, $\lim_{n\to+\infty} \sup_{x\in K} \|T^{n+1}x - T^n x\| = 0$. T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$.

Recall that a point p in C is said to be an asymptotic fixed point of T [20] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$.

A mapping T is said to be relatively nonexpansive if

$$\widetilde{F}(T) = F(T) \neq \emptyset, \qquad \phi(p, Tx) \leq \phi(p,x), \quad \forall x \in C, p \in F(T).$$

A mapping T is said to be relatively asymptotically nonexpansive if

$$\widetilde{F}(T) = F(T) \neq \emptyset, \qquad \phi\bigl(p, T^n x\bigr) \leq (1+\mu_n)\phi(p,x), \quad \forall x \in C, p \in F(T), n \geq 1,$$

where $\{\mu_n\} \subset [0,\infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

Remark 2.2 The class of relatively asymptotically nonexpansive mappings was first considered in [21] (see also, [22] and the reference therein).

Recall that a mapping T is said to be quasi-ϕ-nonexpansive if

$$F(T) \neq \emptyset, \qquad \phi(p, Tx) \leq \phi(p,x), \quad \forall x \in C, p \in F(T).$$

Recall that a mapping T is said to be asymptotically quasi-ϕ-nonexpansive if there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$F(T) \neq \emptyset, \qquad \phi\bigl(p, T^n x\bigr) \leq (1+\mu_n)\phi(p,x), \quad \forall x \in C, p \in F(T), n \geq 1.$$

Remark 2.3 The class of quasi-ϕ-nonexpansive mappings was first considered in [23]. The class of asymptotically quasi-ϕ-nonexpansive mappings that was studied in [24] and [25] includes the class of quasi-ϕ-nonexpansive mappings as a special cases.

Remark 2.4 The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are more general than the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings which require the strong restriction that $\widetilde{F}(T) = F(T)$.

Remark 2.5 The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are generalizations of the class of quasi-nonexpansive mappings and the class of asymptotically quasi-nonexpansive mappings in Banach spaces.

Recall that T is said to be asymptotically quasi-ϕ-nonexpansive in the intermediate sense if $F(T) \neq \emptyset$ and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{p\in F(T),x\in C} \bigl(\phi\bigl(p,T^n x\bigr) - \phi(p,x)\bigr) \leq 0. \tag{2.10}$$

Putting

$$\xi_n = \max\Bigl\{0, \sup_{p\in F(T),x\in C} \bigl(\phi\bigl(p,T^n x\bigr) - \phi(p,x)\bigr)\Bigr\},$$

it follows that $\xi_n \to 0$ as $n \to \infty$. Then (2.10) is reduced to the following:

$$\phi\bigl(p,T^n x\bigr) \leq \phi(p,x)) + \xi_n, \quad \forall p \in F(T), x \in C. \tag{2.11}$$

Remark 2.6 The class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense was first considered by Qin and Wang in [26].

The following Example 2.1 and Example 2.2 show that there is an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense with the nonempty fixed point set which is not ϕ-nonexpansive mapping.

Example 2.1 Let $E = R^1 = \{x \mid |x| < +\infty\}$ and $C = [0,1]$. Define the following mapping $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{2}x, & x \in [0,\frac{1}{2}], \\ 0, & x \in (\frac{1}{2},1]. \end{cases}$$

Then $F(T) = \{0\}$ ($\neq \emptyset$), E is a Hilbert space and $\phi(\cdot,\cdot)$ is reduced to $\phi(x,y) = |x-y|^2$ for all $x,y \in E$. We also have the following:

$$\begin{aligned}
&\limsup_{n\to+\infty} \sup_{p\in F(T),y\in C} \bigl[\phi\bigl(p,T^n y\bigr) - \phi(p,y)\bigr] \\
&\quad = \limsup_{n\to+\infty} \sup_{y\in C} \bigl[\phi\bigl(0,T^n y\bigr) - \phi(0,y)\bigr] \\
&\quad = \limsup_{n\to+\infty} \sup_{y\in C} \bigl(\bigl|T^n y\bigr|^2 - |y|^2\bigr) \leq \limsup_{n\to+\infty} \max\biggl[\biggl(\frac{1}{2^{2n}} - 1\biggr) \inf_{y\in[0,\frac{1}{2}]} |y|^2, - \inf_{y\in(\frac{1}{2},1]} |y|^2\biggr] \\
&\quad = \limsup_{n\to+\infty} \max\biggl(0, -\frac{1}{4}\biggr) = \limsup_{n\to+\infty} 0 = 0.
\end{aligned}$$

Let $x_0 = \frac{1}{2}$, $y_0 = \frac{1}{2} + \frac{1}{2^5} \in C = [0,1]$, then

$$\begin{aligned}
\phi(Tx_0,Ty_0) = |Tx_0 - Ty_0|^2 &= \biggl|\frac{1}{2}x_0 - 0\biggr|^2 = \frac{1}{16} > |x_0 - y_0|^2 \\
&= \biggl|\frac{1}{2} - \biggl(\frac{1}{2} + \frac{1}{2^5}\biggr)\biggr|^2 = \frac{1}{1{,}024} = \phi(x_0,y_0).
\end{aligned}$$

These imply that T is an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense which is not ϕ-nonexpansive mapping. In fact, we may prove that T is an asymptotically ϕ-nonexpansive mapping in the intermediate sense.

Example 2.2 Let $E = l^2$ and $C = \{x \in l^2 \mid \|x\| \leq 1\}$, where $l^2 = \{\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots) \mid \sum_{n=1}^{+\infty} |\sigma_n|^2 < +\infty\}$. $\|\sigma\| = (\sum_{n=1}^{+\infty} |\sigma_n|^2)^{\frac{1}{2}}$, $\forall \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots) \in l^2$; $\langle \sigma, \eta \rangle = \sum_{n=1}^{+\infty} \sigma_n \eta_n$, $\forall \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots)$, $\eta = (\eta_1, \eta_2, \ldots, \eta_n, \ldots) \in l^2$.

Let $T : C \to C$ be a mapping defined by

$$T(x_1, x_2, x_3, \ldots) = \bigl(0, x_1^2, a_2 x_2, a_3 x_3, \ldots\bigr), \quad \forall (x_1, x_2, x_3, \ldots) \in C,$$

where $\{a_j\}$ is a sequence in $(0, 1)$ such that $\prod_{j=2}^{+\infty} a_j = \frac{1}{2}$.

It is proved in Goebel and Kirk [27] that

(i) $\|Tx - Ty\| \leq 2\|x - y\|$, $\forall x, y \in C$;

(ii) $\|T^n x - T^n y\| \leq (2 \prod_{j=2}^{n} a_j)\|x - y\|$, $\forall x, y \in C$, $\forall n \geq 2$.

It is clear that $F(T) = \{0\}$ $(\neq \emptyset)$, E is a Hilbert space, $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$, and from (i) and (ii), we have

$$\|T^n y\|^2 = \|T^n y - T^n 0\|^2 \leq \left(2 \prod_{j=2}^{n} a_j\right)^2 \|y - 0\|^2 = \left(2 \prod_{j=2}^{n} a_j\right)^2 \|y\|^2, \quad \forall y \in C, \forall n \geq 2,$$

and

$$\begin{aligned}
&\limsup_{n \to +\infty} \sup_{p \in F(T), y \in C} \bigl[\phi\bigl(p, T^n y\bigr) - \phi(p, y)\bigr] \\
&\quad = \limsup_{n \to +\infty} \sup_{y \in C} \bigl[\phi\bigl(0, T^n y\bigr) - \phi(0, y)\bigr] \\
&\quad = \limsup_{n \to +\infty} \sup_{y \in C} \bigl(\|T^n y\|^2 - \|y\|^2\bigr) \leq \left\{\limsup_{n \to +\infty} \left[\left(2 \prod_{j=2}^{n} a_j\right)^2 - 1\right]\right\} \cdot \Bigl(\inf_{y \in C} \|y\|^2\Bigr) \\
&\quad = \left[\left(2 \prod_{j=2}^{n} a_j\right)^2 - 1\right] \cdot 0 = 0.
\end{aligned}$$

Let $x_0 = (1, 0, 0, \ldots)$, $y_0 = (\frac{1}{2}, 0, 0, \ldots)$, and $z_0 = (-\frac{1}{2}, 0, 0, \ldots) \in C$, then

$$\begin{aligned}
\phi(Tx_0, Ty_0) &= \|Tx_0 - Ty_0\|^2 = \left\|\bigl(0, 1^2, 0, \ldots\bigr) - \left(0, \frac{1}{4}, 0, \ldots\right)\right\|^2 = \left(1 - \frac{1}{4}\right)^2 = \frac{9}{16} \\
&> \|x_0 - y_0\|^2 = \left\|(1, 0, 0, \ldots) - \left(\frac{1}{2}, 0, 0, \ldots\right)\right\|^2 = \left(1 - \frac{1}{2}\right)^2 = \frac{4}{16} = \phi(x_0, y_0)
\end{aligned}$$

and

$$\begin{aligned}
T(x_0 + z_0) &= \left(0, \frac{1}{4}, 0, 0, \ldots\right) \neq \left(0, \frac{5}{4}, 0, 0, \ldots\right) = (0, 1, 0, 0, \ldots) + \left(0, \frac{1}{4}, 0, 0, \ldots\right) \\
&= Tx_0 + Tz_0;
\end{aligned}$$

These imply that $T : C \to C$ is an asymptotically quasi-ϕ-nonexpansive nonlinear mapping in the intermediate sense with the nonempty fixed point set which is not a ϕ-nonexpansive nonlinear mapping.

Remark 2.7 The class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense is a generalization of the class of asymptotically quasi-nonexpansive mappings in the intermediate sense, which was considered by Kirk [28], in the framework of Banach spaces.

The following lemmas are needed for the proof of our main results in next section.

Lemma 2.1 [15] *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.2 [15] *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty closed convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.3 [5] *Let E be a strictly convex, uniformly smooth Banach space with the dual space E^* and let C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be an η-hemicontinuous and relaxed η-α-monotone mapping, let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (C1), (C2), *and* (C4), *and let f be a proper convex function from $C \times C$ to $\mathbb{R} \cup \{+\infty\}$. Let $r > 0$ and define a mapping $T_r : E \to C$ as follows*:

$$T_r(x) = \left\{ z \in C : \Theta(z, y) + \langle Az, \eta(y, z) \rangle + f(y) - f(z) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\},$$

for all $x \in E$. Assume that

(i) $\eta(x, y) + \eta(y, x) = 0$, *for all $x, y \in C$*;
(ii) *for any fixed $u, v \in C$, the mapping $x \mapsto \langle Av, \eta(x, u) \rangle$ is convex and lower semicontinuous*;
(iii) $\alpha : E \to \mathbb{R}$ *is weakly lower semicontinuous*; *that is, for any net $\{x_\beta\}$, x_β converges to x in $\sigma(E, E^*)$ implying that $\alpha(x) \leq \liminf \alpha(x_\beta)$*;
(iv) *for any $x, y \in C$, $\alpha(x - y) + \alpha(y - x) \geq 0$*;
(v) $\langle A(tz_1 + (1 - t)z_2), \eta(y, tz_1 + (1 - t)z_2) \rangle \geq t \langle Az_1, \eta(y, z_1) \rangle + (1 - t) \langle Az_2, \eta(y, z_2) \rangle$, *for any $z_1, z_2, y \in C$ and $t \in [0, 1]$.*

Then the following hold:

(1) *T_r is single-valued*;
(2) *T_r is a firmly nonexpansive-type mapping*; *that is, for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle;$$

(3) $F(T_r) = EP(\Theta, A)$;
(4) *T_r is quasi-ϕ-nonexpansive satisfying $\phi(w, T_r x) + \phi(T_r x, x) \leq \phi(w, x)$ for all $w \in F(T_r)$ and $x \in E$*;
(5) *$EP(\Theta, A)$ is closed and convex.*

Lemma 2.4 [29] *Let E be an uniformly convex Banach space, and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0,2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$\left\| \sum_{i=1}^{\infty} \alpha_i x_i \right\|^2 \leq \sum_{i=1}^{\infty} \alpha_i \|x_i\|^2 - \alpha_i \alpha_j g\big(\|x_i - x_j\|\big),$$

for all $x_1, x_2, \ldots, x_N, \ldots \in B_r := \{x \in E : \|x\| \leq r\}$ and $\alpha_1, \alpha_2, \ldots, \alpha_N, \ldots \in [0,1]$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$.

3 Main results

Theorem 3.1 *Let E be a strictly convex, and uniformly smooth Banach space such that E has the Kadec-Klee property. Let C be a nonempty closed and convex subset of E and let $\triangle$ be an index set. Let $A : C \to E^*$ be an η-hemicontinuous and relaxed η-α-monotone mapping, let Θ_j be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (C1)-(C4) *for every $j \in \triangle$, and let f be a proper convex and lower semicontinuous function from $C \times C$ to $\mathbb{R} \cup \{+\infty\}$. Let $T_i : C \to C$ be an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense for every $i \geq 1$. Assume that T_i is closed asymptotically regular on C and $F = \bigcap_{i=1}^{\infty} F(T_i) \cap \bigcap_{j\in\triangle} EP(\Theta_j, A)$ is nonempty and bounded. Also assume that the conditions* (i)-(v) *in Lemma* 2.3 *and the following condition hold*:

(vi) *for all $x, y, z, w \in C$,*

$$\limsup_{t\downarrow 0} \big\langle Az, \eta\big(x, ty + (1-t)w\big)\big\rangle \leq \big\langle Az, \eta(x, w)\big\rangle.$$

Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_{1,j} = C, \\ C_1 = \bigcap_{j\in\triangle} C_{1,j}, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J T_i^n x_n), \\ u_{n,j} \in C \text{ such that} \\ \quad \Theta_j(u_{n,j}, y) + \langle A u_{n,j}, \eta(y, u_{n,j})\rangle + f(y) - f(u_{n,j}) \\ \qquad + \frac{1}{r_{n,j}} \langle y - u_{n,j}, J u_{n,j} - J y_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1,j} = \{z \in C_n : \phi(z, u_{n,j}) \leq \phi(z, x_n) + \xi_n\}, \\ C_{n+1} = \bigcap_{j\in\triangle} C_{n+1,j}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases} \tag{3.1}$$

where $\xi_n = \sup_{i\in N^+}\{0, \sup_{p\in F(T_i), x\in C}(\phi(p, T_i^n x) - \phi(p, x))\}$, $\{\alpha_{n,i}\}$ is a real number sequence in $(0,1)$ for every $i \geq 1$, $\{r_{n,j}\}$ is a real number sequence in $[k, \infty)$, where k is some positive real number. Assume that $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ for every $i \geq 1$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_1$, where Π_F is the generalized projection from E onto F.

Proof The proof is split into the following six steps.

Step 1. We first show that F is closed and convex.

From Theorem 3.1 in [26], we see that $\bigcap_{i=1}^{\infty} F(T_i)$ is closed and convex, which combines with Lemma 2.3 shows that the common element set F is closed and convex.

Step 2. Next, we show that C_n is closed and convex for each $n \geq 1$.

It suffices to show, for any fixed but arbitrary $j \in \triangle$, that $C_{n,j}$ is closed and convex. This can be proved by induction on n. It is obvious that $C_{1,i} = C$ is closed and convex. Assume that $C_{h,j}$ is closed and convex for some $h \geq 1$. We next prove that $C_{h+1,j}$ is closed and convex for the same h. This completes the proof that C_n is closed and convex. It is clear that $C_{h+1,j}$ is closed. We only prove the convexity. Indeed, $\forall a_1, a_2 \in C_{h+1,j}$, we see that $a_1, a_2 \in C_{h,j}$, and

$$\phi(a_1, u_{h,j}) \leq \phi(a_1, x_h) + \xi_h,$$

and

$$\phi(a_2, u_{h,j}) \leq \phi(a_2, x_h) + \xi_h.$$

Notice that the two inequalities above are equivalent to the following inequalities, respectively:

$$2\langle a_1, Jx_h - Ju_{h,j}\rangle \leq \|x_h\|^2 - \|u_{h,j}\|^2 + \xi_h,$$

and

$$2\langle a_2, Jx_h - Ju_{h,j}\rangle \leq \|x_h\|^2 - \|u_{h,j}\|^2 + \xi_h.$$

These imply that

$$2\bigl\langle ta_1 + (1-t)a_2, Jx_h - Ju_{h,j}\bigr\rangle \leq \|x_h\|^2 - \|u_{h,j}\|^2 + \xi_h, \quad \forall t \in (0,1).$$

Since $C_{h,j}$ is convex, we see that $ta_1 + (1-t)a_2 \in C_{h,j}$. Notice that the above inequality is equivalent to

$$\phi\bigl(ta_1 + (1-t)a_2, u_{h,j}\bigr) \leq \phi\bigl(ta_1 + (1-t)a_2, x_h\bigr) + \xi_h.$$

It follows that $C_{h+1,j}$ is convex. This in turn implies that C_n is closed and convex for all $n \geq 1$.

Step 3. We prove that $F \subset C_n$ for each $n \geq 1$.

It suffices to claim that $F \subset C_{n,j}$ for every $j \in \triangle$. In fact, it is obvious that $F \subset C_{1,j} = C$. Suppose that $F \subset C_{h,j}$ for some $h \geq 1$ and for every $j \in \triangle$. On the other hand, since $T_{r_{n,j}}$ is quasi-ϕ-nonexpansive, according to Lemma 2.3(4), we have, for any $w \in F \subset C_{h,j}$,

$$\begin{aligned} \phi(w, u_{h,j}) &= \phi(w, T_{r_{h,j}} y_h) \\ &\leq \phi(w, y_h) \\ &= \phi\left(w, J^{-1}\left(\alpha_{h,0} Jx_h + \sum_{i=1}^{\infty} \alpha_{h,i} JT_i^h x_h\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= \|w\|^2 - 2\left\langle w, \alpha_{h,0}Jx_h + \sum_{i=1}^{\infty}\alpha_{h,i}JT_i^h x_h\right\rangle + \left\|\alpha_{h,0}Jx_h + \sum_{i=1}^{\infty}\alpha_{h,i}JT_i^h x_h\right\|^2 \\
&\leq \|w\|^2 - 2\alpha_{h,0}\langle w, Jx_h\rangle - 2\sum_{i=1}^{\infty}\alpha_{h,i}\langle w, JT_i^h x_h\rangle + \alpha_{h,0}\|x_h\|^2 + \sum_{i=1}^{\infty}\alpha_{h,i}\|T_i^h x_h\|^2 \\
&= \alpha_{h,0}\phi(w,x_h) + \sum_{i=1}^{\infty}\alpha_{h,i}\phi(w, T_i^h x_h) \\
&\leq \alpha_{h,0}\phi(w,x_h) + \sum_{i=1}^{\infty}\alpha_{h,i}\phi(w,x_h) + \sum_{i=1}^{\infty}\alpha_{h,i}\xi_h \\
&= \phi(w,x_h) + \sum_{i=1}^{\infty}\alpha_{h,i}\xi_h \\
&\leq \phi(w,x_h) + \xi_h,
\end{aligned} \tag{3.2}
$$

which shows that $w \in C_{h+1,j}$. This implies that $F \subset C_{n,j}$ for all $n \geq 1$ and for every $j \in \triangle$. Therefore we obtain $F \subset C_n$. This in turn shows that the sequence $\{x_n\}$ generated by the algorithm (3.1) is well defined.

Step 4. Next, we prove that the sequence $\{x_n\}$ is bounded.

Observe that $x_n = \Pi_{C_n}x_1$, we find from Lemma 2.1 that $\langle x_n - z, Jx_1 - Jx_n\rangle \geq 0$ for each $z \in C_n$. Since $F \subset C_n$, we know that

$$
\langle x_n - w, Jx_1 - Jx_n\rangle \geq 0, \quad \forall w \in F. \tag{3.3}
$$

It then follows from Lemma 2.2 that

$$
\phi(x_n, x_1) = \phi(\Pi_{C_n}x_1, x_1) \leq \phi(\Pi_F x_1, x_1) - \phi(\Pi_F x_1, x_n) \leq \phi(\Pi_F x_1, x_1).
$$

This shows that the sequence $\{\phi(x_n, x_1)\}$ is bounded. We can know from (2.3) that the sequence $\{x_n\}$ is also bounded.

Step 5. Now we show that $x_n \to x^*$, where $x^* \in F$ as $n \to \infty$.

Note that E is an uniformly smooth Banach space, it follows from the uniformly convexity of E^* that the space E is reflexive. Since $\{x_n\}$ is bounded, we may assume that $x_n \rightharpoonup x^*$. Since C_n is closed and convex, we see that $x^* \in C_n$. On the other hand, we see from the weakly lower semicontinuity of $\|\cdot\|$ that

$$
\begin{aligned}
\phi(x^*, x_1) &= \|x^*\|^2 - 2\langle x^*, Jx_1\rangle + \|x_1\|^2 \\
&\leq \liminf_{n\to\infty}\left(\|x_n\|^2 - 2\langle x_n, Jx_1\rangle + \|x_1\|^2\right) \\
&= \liminf_{n\to\infty}\phi(x_n, x_1) \\
&\leq \limsup_{n\to\infty}\phi(x_n, x_1) \\
&\leq \phi(x^*, x_1),
\end{aligned}
$$

from which it follows that $\lim_{n\to\infty}\phi(x_n,x_1)=\phi(x^*,x_1)$. Hence, we obtain

$$\begin{aligned}\lim_{n\to\infty}\|x_n\| &= \sqrt{\lim_{n\to\infty}\|x_n\|^2} = \sqrt{\lim_{n\to\infty}\bigl\{\bigl[\phi(x_n,x_1)-\phi\bigl(x^*,x_1\bigr)\bigr]+\bigl\|x^*\bigr\|^2+2\bigl\langle x_n-x^*,Jx_1\bigr\rangle\bigr\}}\\ &= \sqrt{\bigl\|x^*\bigr\|^2} = \bigl\|x^*\bigr\|.\end{aligned}$$

In view of the Kadec-Klee property of E, we see that $x_n\to x^*$ as $n\to\infty$. Next, we show that $x^*\in F$.

(a) First we prove that $x^*\in\bigcap_{i=1}^{\infty}F(T_i)$.

Since $x_n=\Pi_{C_n}x_1$ and $x_{n+1}=\Pi_{C_{n+1}}x_1\in C_{n+1}\subset C_n$, we find that $\phi(x_n,x_1)\le\phi(x_{n+1},x_1)$, which shows that $\{\phi(x_n,x_1)\}$ is nondecreasing. From the boundedness, $\lim_{n\to\infty}\phi(x_n,x_1)$ exists. By $x_{n+1}=\Pi_{C_{n+1}}x_1\in C_{n+1}\subset C_n$, we have

$$\phi(x_{n+1},x_n)=\phi(x_{n+1},\Pi_{C_n}x_1)\le\phi(x_{n+1},x_1)-\phi(\Pi_{C_n}x_1,x_1)=\phi(x_{n+1},x_1)-\phi(x_n,x_1).$$

It follows that

$$\lim_{n\to\infty}\phi(x_{n+1},x_n)=0. \tag{3.4}$$

In the light of $x_{n+1}=\Pi_{C_{n+1}}x_1\in C_{n+1}$, we arrive at

$$\phi(x_{n+1},u_{n,j})\le\phi(x_{n+1},x_n)+\xi_n.$$

This in turn implies from (3.4) that

$$\lim_{n\to\infty}\phi(x_{n+1},u_{n,j})=0.$$

From (2.3), we see that $\lim_{n\to\infty}(\|x_{n+1}\|-\|u_{n,j}\|)=0$. This in turn implies that

$$\lim_{n\to\infty}\|u_{n,j}\|=\bigl\|x^*\bigr\|.$$

It follows that

$$\lim_{n\to\infty}\|Ju_{n,j}\|=\lim_{n\to\infty}\|u_{n,j}\|=\bigl\|Jx^*\bigr\|. \tag{3.5}$$

This implied that $\{Ju_{n,j}\}$ is bounded. Since both E and E^* are reflexive, we may assume that $Ju_{n,j}\rightharpoonup u^{*,j}\in E^*$. In view of the reflexivity of E, we see that $J(E)=E^*$. This implies that there exists an element $u^j\in E$ such that $Ju^j=u^{*,j}$. It follows that

$$\phi(x_{n+1},u_{n,j})=\|x_{n+1}\|^2-2\langle x_{n+1},Ju_{n,j}\rangle+\|u_{n,j}\|^2=\|x_{n+1}\|^2-2\langle x_{n+1},Ju_{n,j}\rangle+\|Ju_{n,j}\|^2.$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above yields

$$\begin{aligned}0&\ge\bigl\|x^*\bigr\|^2-2\bigl\langle x^*,u^{*,j}\bigr\rangle+\bigl\|u^{*,j}\bigr\|^2=\bigl\|x^*\bigr\|^2-2\bigl\langle x^*,Ju^j\bigr\rangle+\bigl\|Ju^j\bigr\|^2\\ &=\bigl\|x^*\bigr\|^2-2\bigl\langle x^*,Ju^j\bigr\rangle+\bigl\|u^j\bigr\|^2=\phi\bigl(x^*,u^j\bigr).\end{aligned}$$

That is, $x^* = u^j$, which shows that $u^{*,j} = Jx^*$. It follows that $Ju_{n,j} \rightharpoonup Jx^* \in E^*$. In view of the Kadec-Klee property of E^*, we have from (3.5) that $\lim_{n\to\infty} Ju_{n,j} = Jx^*$. In view of the demicontinuity of $J^{-1} : E^* \to E$ and the Kadec-Klee property of E, we have $u_{n,j} \to x^*$, as $n \to \infty$. Note that

$$\|x_n - u_{n,j}\| \le \|x_n - x^*\| + \|x^* - u_{n,j}\|.$$

It follows that

$$\lim_{n\to\infty} \|x_n - u_{n,j}\| = 0. \tag{3.6}$$

Since J is uniformly norm-to-norm continuous on any bounded set, we have

$$\lim_{n\to\infty} \|Jx_n - Ju_{n,j}\| = 0. \tag{3.7}$$

On the other hand, we have

$$\begin{aligned} \phi(w,x_n) - \phi(w,u_{n,j}) &= \|x_n\|^2 - \|u_{n,j}\|^2 - 2\langle w, Jx_n - Ju_{n,j}\rangle \\ &\le \|x_n - u_{n,j}\| \big(\|x_n\| + \|u_{n,j}\|\big) + 2\|w\| \|Jx_n - Ju_{n,j}\|. \end{aligned}$$

From (3.6) and (3.7), we obtain

$$\lim_{n\to\infty} \big(\phi(w,x_n) - \phi(w,u_{n,j})\big) = 0. \tag{3.8}$$

Since E is uniformly smooth, we see that E^* is uniformly convex. We find from Lemma 2.4 that

$$\begin{aligned} \phi(w,u_{n,j}) &= \phi(w, T_{r_{n,j}} y_n) \\ &\le \phi(w,y_n) \\ &= \phi\left(w, J^{-1}\left(\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JT_i^n x_n\right)\right) \\ &= \|w\|^2 - 2\left\langle w, \alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JT_i^n x_n \right\rangle + \left\| \alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JT_i^n x_n \right\|^2 \\ &\le \|w\|^2 - 2\alpha_{n,0}\langle w, Jx_n\rangle - 2\sum_{i=1}^{\infty} \alpha_{n,i} \langle w, JT_i^n x_n\rangle + \alpha_{n,0}\|x_n\|^2 + \sum_{i=1}^{\infty} \alpha_{n,i} \|T_i^n x_n\|^2 \\ &\quad - \alpha_{n,0}\alpha_{n,i} g\big(\|Jx_n - JT_i^n x_n\|\big) \\ &= \alpha_{n,0}\phi(w,x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} \phi\big(w, T_i^n x_n\big) - \alpha_{n,0}\alpha_{n,i} g\big(\|Jx_n - JT_i^n x_n\|\big) \\ &\le \alpha_{n,0}\phi(w,x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}\phi(w,x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}\xi_n - \alpha_{n,0}\alpha_{n,i} g\big(\|Jx_n - JT_i^n x_n\|\big) \\ &\le \phi(w,x_n) + \xi_n - \alpha_{n,0}\alpha_{n,i} g\big(\|Jx_n - JT_i^n x_n\|\big). \end{aligned}$$

It follows that

$$\alpha_{n,0}\alpha_{n,i}g\big(\|Jx_n - JT_i^n x_n\|\big) \le \phi(w,x_n) - \phi(w,u_{n,j}) + \xi_n.$$

By the restriction on the sequences, we find from (3.8) that

$$\lim_{n\to\infty} g\big(\|Jx_n - JT_i^n x_n\|\big) = 0.$$

It follows that

$$\lim_{n\to\infty} \|Jx_n - JT_i^n x_n\| = 0.$$

Notice that $\|JT_i^n x_n - Jx^*\| \le \|JT_i^n x_n - Jx_n\| + \|Jx_n - Jx^*\|$. It follows that

$$\lim_{n\to\infty} \|JT_i^n x_n - Jx^*\| = 0. \tag{3.9}$$

In view of the demicontinuity of $J^{-1}: E^* \to E$, we have $T_i^n x_n \rightharpoonup x^*$. Note that

$$\big|\|T_i^n x_n\| - \|x^*\|\big| = \big|\|JT_i^n x_n\| - \|Jx^*\|\big| \le \|JT_i^n x_n - Jx^*\|.$$

It follows from (3.9) that $\lim_{n\to\infty}\|T_i^n x_n\| = \|x^*\|$. In view of the Kadec-Klee property of E, we have $\lim_{n\to\infty}\|T_i^n x_n - x^*\| = 0$. On the other hand, we have

$$\|T_i^{n+1} x_n - x^*\| \le \|T_i^{n+1} x_n - T_i^n x_n\| + \|T_i^n x_n - x^*\|.$$

It follows from the uniformly asymptotic regularity of T_i that

$$\lim_{n\to\infty} \|T_i^{n+1} x_n - x^*\| = 0.$$

That is, $T_i T_i^n x_n \to x^*$. In view of the closedness of T_i, we obtain $x^* = T_i x^*$ for each $i \ge 1$. This proves that $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$.

(b) Next, we show that $x^* \in \bigcap_{j\in\triangle} EP(\Theta_j, A)$.

From (3.2), we arrived at $\phi(w,y_n) \le \phi(w,x_n) + \xi_n$. From $u_{n,j} = T_{r_{n,j}} y_n$ and Lemma 2.3(4), one has

$$\begin{aligned}
\phi(u_{n,j},y_n) &= \phi(T_{r_{n,j}} y_n, y_n) \\
&\le \phi(w,y_n) - \phi(w, T_{r_{n,j}} y_n) \\
&\le \phi(w,x_n) + \xi_n - \phi(w,u_{n,j}).
\end{aligned}$$

Thus, it follows from (3.8) that

$$\phi(u_{n,j},y_n) \to 0 \quad \text{as } n \to \infty.$$

From (2.3), we see that $\lim_{n\to\infty}(\|u_{n,j}\| - \|y_n\|) = 0$. This in turn implies from (3.6) that

$$\lim_{n\to\infty} \|y_n\| = \|x^*\|.$$

It follows that

$$\lim_{n\to\infty} \|Jy_n\| = \lim_{n\to\infty} \|y_n\| = \|Jx^*\|. \tag{3.10}$$

This implied that $\{Jy_n\}$ is bounded. Since E^* are reflexive, we may assume that $Jy_n \rightharpoonup v^* \in E^*$. In view of $J(E) = E^*$, we see that there exists an element $v \in E$ such that $Jv = v^*$. It follows that

$$\begin{aligned}\phi(u_{n,j}, y_n) &= \|u_{n,j}\|^2 - 2\langle u_{n,j}, Jy_n\rangle + \|y_n\|^2 \\ &= \|u_{n,j}\|^2 - 2\langle u_{n,j}, Jy_n\rangle + \|Jy_n\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above yields

$$\begin{aligned}0 &\geq \|x^*\|^2 - 2\langle x^*, v^*\rangle + \|v^*\|^2 \\ &= \|x^*\|^2 - 2\langle x^*, Jv\rangle + \|Jv\|^2 \\ &= \|x^*\|^2 - 2\langle x^*, Jv\rangle + \|v\|^2 \\ &= \phi(x^*, v).\end{aligned}$$

That is, $x^* = v$, which shows that $v^* = Jx^*$. It follows that $Jy_n \rightharpoonup Jx^* \in E^*$. In view of the Kadec-Klee property of E^*, we have from (3.10) that $\lim_{n\to\infty} Jy_n = Jx^*$. In view of the demicontinuity of $J^{-1} : E^* \to E$ and the Kadec-Klee property of E, we have $y_n \to x^*$, as $n \to \infty$. Note that

$$\|u_{n,j} - y_n\| \leq \|u_{n,j} - x^*\| + \|x^* - y_n\|.$$

It follows that

$$\lim_{n\to\infty} \|u_{n,j} - y_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on any bounded set, we have

$$\lim_{n\to\infty} \|Ju_{n,j} - Jy_n\| = 0. \tag{3.11}$$

Note that

$$\Theta_j(u_{n,j}, y) + \langle Au_{n,j}, \eta(y, u_{n,j})\rangle + f(y) - f(u_{n,j}) + \frac{1}{r_{n,j}}\langle y - u_{n,j}, Ju_{n,j} - Jy_n\rangle \geq 0, \quad \forall y \in C.$$

From (C2) and (i), it follows that

$$\begin{aligned}\|y - u_{n,j}\|\frac{\|Ju_{n,j} - Jy_n\|}{r_{n,j}} &\geq \langle Au_{n,j}, \eta(u_{n,j}, y)\rangle + f(u_{n,j}) - f(y) - \Theta_j(u_{n,j}, y) \\ &\geq \langle Au_{n,j}, \eta(u_{n,j}, y)\rangle + f(u_{n,j}) - f(y) + \Theta_j(y, u_{n,j}), \quad \forall y \in C. \end{aligned}\tag{3.12}$$

Noticing that $r_{n,j} \geq k > 0$ for all $n \geq 1$, it follows from (C4), (ii), (3.11), and (3.12) that

$$0 \geq \langle Ax^*, \eta(x^*, y)\rangle + f(x^*) - f(y) + \Theta_j(y, x^*), \quad \forall y \in C.$$

For all $0 < t_j \leq 1$ and $y \in C$, define $y_{t_j} = t_j y + (1 - t_j)x^*$. Noticing that $x^*, y \in C$, one obtains $y_{t_j} \in C$, which yields

$$0 \geq \langle Ax^*, \eta(x^*, y_{t_j})\rangle + f(x^*) - f(y_{t_j}) + \Theta_j(y_{t_j}, x^*). \tag{3.13}$$

It follows from (C1), (C4), (i), (ii), the convexity of f, and (3.13) that

$$\begin{aligned} 0 &= \Theta_j(y_{t_j}, y_{t_j}) + \langle Ax^*, \eta(y_{t_j}, y_{t_j})\rangle + f(y_{t_j}) - f(y_{t_j}) \\ &\leq t_j\big[\Theta_j(y_{t_j}, y) + \langle Ax^*, \eta(y, y_{t_j})\rangle + f(y) - f(y_{t_j})\big] + (1 - t_j)\big[\Theta_j(y_{t_j}, x^*) \\ &\quad + \langle Ax^*, \eta(x^*, y_{t_j})\rangle + f(x^*) - f(y_{t_j})\big] \\ &\leq t_j\big[\Theta_j(y_{t_j}, y) + \langle Ax^*, \eta(y, y_{t_j})\rangle + f(y) - f(y_{t_j})\big]. \end{aligned}$$

That is,

$$\Theta_j(y_{t_j}, y) + \langle Ax^*, \eta(y, y_{t_j})\rangle + f(y) - f(y_{t_j}) \geq 0.$$

Letting $t \downarrow 0$, it follows from (C3), (vi), and the lower semicontinuity of f that

$$\Theta_j(x^*, y) + \langle Ax^*, \eta(y, x^*)\rangle + f(y) - f(x^*) \geq 0, \quad \forall y \in C.$$

This implies that $x^* \in \bigcap_{j\in\triangle} EP(\Theta_j, A)$.

Step 6. Finally, we prove $x^* = \Pi_F x_1$.

Letting $n \to \infty$ in (3.3), we obtain

$$\langle x^* - w, Jx_1 - Jx^*\rangle \geq 0, \quad \forall w \in F.$$

In view of Lemma 2.1, we have $x^* = \Pi_F x_1$. This completes the proof. □

Remark 3.1 Theorem 3.1 improves and generalizes the main theorem in Chen *et al.* [5] in the following aspects.

(1) From a quasi-ϕ-nonexpansive mapping to an infinite family of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense.
(2) From a mixed equilibrium problem to a finite family of mixed equilibrium problems.
(3) From a uniformly smooth and uniformly convex Banach space to a uniformly smooth and strictly convex Banach space such that the space has the Kadec-Klee property.

The space in Theorem 3.1 can be applicable to L^P, $P > 1$. Now, we give Example 3.1 in order to support Theorem 3.1.

Example 3.1 Let $E = l^2$ and $C = \{x \in l^2 \mid \|x\| \leq 1\}$, where $l^2 = \{\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n, \dots) \mid \sum_{n=1}^{+\infty} |\sigma_n|^2 < +\infty\}$. $\|\sigma\| = (\sum_{n=1}^{+\infty} |\sigma_n|^2)^{\frac{1}{2}}$, $\forall \sigma = (\sigma_1, \sigma_2, \dots, \sigma_n, \dots) \in l^2$; $\langle \sigma, \eta \rangle = \sum_{n=1}^{+\infty} \sigma_n \eta_n$, $\forall \sigma = (\sigma_1, \sigma_2, \dots, \sigma_n, \dots)$, $\eta = (\eta_1, \eta_2, \dots, \eta_n, \dots) \in l^2$. Set $x_0 \in C$ and $\|x_0\| = 1$. Define the following a countable family of mappings $T_i : C \to C$ by

$$T_i x = \begin{cases} \frac{1}{2}x, & x = \frac{x_0}{2^n}, \\ -\frac{1}{i+1}x, & x \neq \frac{x_0}{2^n} \text{ and } x \in C, \end{cases}$$

for all $i \in N^+$ and $n \in N^+$.

It is clear that $F(T_i) = \{0\}$ for all $i \in N^+$, E is a Hilbert space, $\phi(x,y) = \|x - y\|^2$ for all $x, y \in E$, and

$$T_i^n x = \begin{cases} \frac{1}{2^n}x, & x = \frac{x_0}{2^n}, \\ \frac{(-1)^n}{(i+1)^n}x, & x \neq \frac{x_0}{2^n} \text{ and } x \in C. \end{cases}$$

Choose $i \in N^+$, for any $n \in N^+$, we set $x_n = \frac{x_0}{2^{n+1}}$, then $x_n \in C$, $x_n \to 0 \in F(T_i) = \{0\}$ as $n \to +\infty$, and

$$\begin{aligned} &\limsup_{n\to+\infty} \sup_{p \in F(T_i), y \in C} \bigl[\phi\bigl(p, T_i^n y\bigr) - \phi(p,y)\bigr] \\ &\quad = \limsup_{n\to+\infty} \sup_{y\in C} \bigl[\phi\bigl(0, T_i^n y\bigr) - \phi(0,y)\bigr] = \limsup_{n\to+\infty} \sup_{y\in C} \bigl(\bigl\|T_i^n y\bigr\|^2 - \|y\|^2\bigr) \\ &\quad \leq \limsup_{n\to+\infty} \max\biggl[\biggl(\frac{1}{2^{2n}} - 1\biggr) \inf_{y=\frac{x_0}{2^n}} \|y\|^2, \biggl(\frac{1}{(i+1)^{2n}} - 1\biggr) \inf_{y \in C\setminus\{\frac{x_0}{2^n}\}} \|y\|^2\biggr] \\ &\quad \leq \limsup_{n\to+\infty} \max\biggl[\biggl(\frac{1}{2^{2n}} - 1\biggr) \cdot \frac{1}{2^{2n}}, \biggl(\frac{1}{(i+1)^{2n}} - 1\biggr) \cdot 0\biggr] = \limsup_{n\to+\infty} \biggl(\frac{1}{2^{2n}} - 1\biggr) \frac{1}{2^{2n}} = 0. \end{aligned}$$

This implies that $T_i : C \to C$ is an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense for every $i \in N^+$. Next, we claim that T_i is not ϕ-nonexpansive mapping for all $i \in N^+$. Indeed, let $p = \frac{x_0}{2}$, $q = \frac{3x_0}{5} \in C$, then

$$\begin{aligned} \phi(T_i p, T_i q) = \|T_i p - T_i q\|^2 &= \biggl\|\frac{1}{2}p - \biggl(-\frac{1}{i+1}q\biggr)\biggr\|^2 = \biggl\|\frac{1}{4}x_0 - \biggl(-\frac{1}{i+1} \cdot \frac{3}{5}x_0\biggr)\biggr\|^2 \\ &= \biggl\|\biggl(\frac{1}{4} + \frac{3}{5i+5}\biggr)x_0\biggr\|^2 = \biggl(\frac{1}{10}\biggr)^2 \cdot \biggl(\frac{5i+17}{2i+2}\biggr)^2 > \biggl(\frac{1}{10}\biggr)^2 = \biggl\|\frac{x_0}{2} - \frac{3x_0}{5}\biggr\|^2 \\ &= \|p - q\|^2 = \phi(p,q) \end{aligned}$$

for all $i \in N^+$.

For any bounded subset K of C, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to+\infty} \sup_{y\in K} \bigl\|T_i^{n+1} y - T_i^n y\bigr\| \\ &\leq \limsup_{n\to+\infty} \max\biggl(\sup_{y=\frac{x_0}{2^n}} \biggl\|\frac{1}{2^{n+1}}y - \frac{1}{2^n}y\biggr\|, \sup_{y\in K\setminus\{\frac{x_0}{2^n}\}} \biggl\|\frac{(-1)^{n+1}}{(i+1)^{n+1}}y - \frac{(-1)^n}{(i+1)^n}y\biggr\|\biggr) \\ &= \limsup_{n\to+\infty} \max\biggl(\frac{1}{2^{2n+1}}, \sup_{y\in K\setminus\{\frac{x_0}{2^n}\}} \frac{i+2}{(i+1)^{n+1}} \|y\|\biggr) \end{aligned}$$

$$\begin{aligned}
&\leq \limsup_{n\to+\infty} \max\left(\frac{1}{2^{2n+1}}, \sup_{y\in C\setminus\{\frac{x_0}{2^n}\}} \frac{i+2}{(i+1)^{n+1}}\|y\|\right) \\
&\leq \limsup_{n\to+\infty} \max\left(\frac{1}{2^{2n+1}}, \frac{i+2}{(i+1)^{n+1}}\cdot 1\right) \leq \limsup_{n\to+\infty} \max\left(\frac{1}{2^{2n+1}}, \frac{2}{(i+1)^{n}}\right) \\
&\leq \limsup_{n\to+\infty} \max\left(\frac{1}{2^{2n+1}}, \frac{1}{2^{n-1}}\right) = \limsup_{n\to+\infty} \frac{1}{2^{n-1}} = 0.
\end{aligned}$$

This implies that $\limsup_{n\to+\infty} \sup_{y\in K} \|T_i^{n+1}y - T_i^n y\| = 0$, *i.e.*, T_i is an asymptotically regular on C.

For any sequence $\{y_n\} \subseteq C$ such that $\lim_{n\to+\infty} y_n = x^0$ and $\lim_{n\to+\infty} T_i y_n = y^0$, we consider the following two cases:

(1) If the sequence $y_n = \frac{x_0}{2^n}$ and $\lim_{n\to+\infty} y_n = x^0$, then we have $x^0 = 0$ and

$$\begin{aligned}
0 &= \lim_{n\to+\infty} \|T_i y_n - y^0\| = \lim_{n\to+\infty} \left\|\frac{y_n}{2} - y^0\right\| = \lim_{n\to+\infty} \left\|\frac{x_0}{2^{n+1}} - y^0\right\| \\
&\geq \limsup_{n\to+\infty} \left| \|y^0\| - \left\|\frac{x_0}{2^{n+1}}\right\| \right| = \limsup_{n\to+\infty} \left| \|y^0\| - \frac{1}{2^{n+1}} \right| = \|y^0\| \geq 0,
\end{aligned}$$

this implies that $y^0 = 0$ and $T_i x^0 = -\frac{x^0}{i+1} = 0 = y^0$.

(2) If $y_n \neq \frac{x_0}{2^n}$, $y_n \in C$ and $\lim_{n\to+\infty} y_n = x^0$, then it follows from

$$\begin{aligned}
0 &= \lim_{n\to+\infty} \|T_i y_n - y^0\| = \lim_{n\to+\infty} \left\| -\frac{1}{i+1} y_n - y^0 \right\| \\
&= \lim_{n\to+\infty} \left\| -\frac{1}{i+1}(y_n - x^0) - \left(y^0 + \frac{x^0}{i+1}\right) \right\| \\
&\geq \limsup_{n\to+\infty} \left| \left\| y^0 + \frac{x^0}{i+1} \right\| - \left\| \frac{y_n - x_0}{i+1} \right\| \right| = \left\| y^0 + \frac{x^0}{i+1} \right\| \geq 0
\end{aligned}$$

that $y^0 = -\frac{x^0}{i+1}$, thus $T_i x^0 = -\frac{1}{i+1} x^0 = y^0$.

In summary, we can see that T_i is closed for every $i \in N^+$.

Finally, it is obvious that the family $\{T_i\}_{i\in N^+}$ satisfies all the aspects of the hypothesis of Theorem 3.1.

For a single mapping and bifunction in Theorem 3.1, we have Corollary 3.1.

Corollary 3.1 *Let E be a strictly convex, and uniformly smooth Banach space such that E has the Kadec-Klee property. Let C be a nonempty closed and convex subset of E. Let $A : C \to E^*$ be an η-hemicontinuous and relaxed η-α-monotone mapping, let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (C1)-(C4), *and let f be a proper convex and lower semicontinuous function from $C \times C$ to $\mathbb{R} \cup \{+\infty\}$. Let $T : C \to C$ be an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense. Assume that T is closed asymptotically regular on C and $F = F(T) \cap EP(\Theta, A)$ is nonempty and bounded. Also assume that the conditions* (i)-(v), *Lemma* 2.3, *and the following condition hold*:

(vi) *for all $x, y, z, w \in C$, $\limsup_{t\downarrow 0} \langle Az, \eta(x, ty + (1-t)w)\rangle \leq \langle Az, \eta(x, w)\rangle$.*

Let $\{x_n\}$ *be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n), \\ u_n \in C \text{ such that} \\ \quad \Theta(u_n, y) + \langle A u_n, \eta(y, u_n)\rangle + f(y) - f(u_n) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\xi_n = \max\{0, \sup_{p\in F(T), x\in C}(\phi(p, T^n x) - \phi(p, x))\}$, $\{\alpha_n\}$ *is a real number sequences in* $(0,1)$, $\{r_n\}$ *is a real number sequence in* $[k,\infty)$, *where* k *is some positive real number. Assume that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_F x_1$, *where* Π_F *is the generalized projection from E onto F*.

Remark 3.2 Corollary 3.1 improves and generalizes the main theorem in Chen *et al.* [5] in the following aspects:

(1) From a quasi-ϕ-nonexpansive mapping to an asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense.
(2) From a uniformly smooth and uniformly convex Banach space to a uniformly smooth and strictly convex Banach space such that the space has the Kadec-Klee property.

Setting $A \equiv 0$ in Theorem 3.1, we have Corollary 3.2.

Corollary 3.2 *Let E be a strictly convex, and uniformly smooth Banach space such that E has the Kadec-Klee property. Let C be a nonempty closed and convex subset of E and let* $\triangle$ *be an index set. Let* Θ_j *be a bifunction from* $C\times C$ *to* $\mathbb{R}$ *satisfying* (C1)-(C4) *for every* $j \in \triangle$, *and let* f *be a proper convex and lower semicontinuous function from* $C \times C$ *to* $\mathbb{R}\cup\{+\infty\}$. *Let* $T_i : C \to C$ *be an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense for every* $i \geq 1$. *Assume that* T_i *is closed asymptotically regular on C and* $F = \bigcap_{i=1}^{\infty} F(T_i) \cap \bigcap_{j\in\triangle} MEP(\Theta_j, f)$ *is nonempty and bounded. Let* $\{x_n\}$ *be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_{1,j} = C, \\ C_1 = \bigcap_{j\in\triangle} C_{1,j}, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J T_i^n x_n), \\ u_{n,j} \in C \text{ such that } \Theta_j(u_{n,j}, y) + f(y) - f(u_{n,j}) + \frac{1}{r_{n,j}}\langle y - u_{n,j}, J u_{n,j} - J y_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1,j} = \{z \in C_n : \phi(z, u_{n,j}) \leq \phi(z, x_n) + \xi_n\}, \\ C_{n+1} = \bigcap_{j\in\triangle} C_{n+1,j}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\xi_n = \sup_{i\in N^+}\{0, \sup_{p\in F(T_i), x\in C}(\phi(p, T_i^n x) - \phi(p, x))\}$, $\{\alpha_{n,i}\}$ *is a real number sequence in* $(0,1)$ *for every* $i \geq 1$, $\{r_{n,j}\}$ *is a real number sequence in* $[k,\infty)$, *where* k *is some positive* *real number. Assume that* $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ *and* $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for every* $i \geq 1$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_F x_1$, *where* Π_F *is the generalized projection from* E *onto* F.

Setting $A \equiv 0, f \equiv 0$ in Theorem 3.1, we have Corollary 3.3.

Corollary 3.3 *Let* E *be a strictly convex, and uniformly smooth Banach space such that* E *has the Kadec-Klee property. Let* C *be a nonempty closed and convex subset of* E *and let* $\triangle$ *be an index set. Let* Θ_j *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (C1)-(C4) *for every* $j \in \triangle$. *Let* $T_i : C \to C$ *be an asymptotically quasi-*ϕ*-nonexpansive mapping in the intermediate sense for every* $i \geq 1$. *Assume that* T_i *is closed asymptotically regular on* C *and* $F = \bigcap_{i=1}^{\infty} F(T_i) \cap \bigcap_{j\in\triangle} EP(\Theta_j)$ *is nonempty and bounded. Let* $\{x_n\}$ *be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_{1,j} = C, \\ C_1 = \bigcap_{j\in\triangle} C_{1,j}, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JT_i^n x_n), \\ u_{n,j} \in C \text{ such that } \Theta_j(u_{n,j}, y) + \frac{1}{r_{n,j}} \langle y - u_{n,j}, Ju_{n,j} - Jy_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1,j} = \{z \in C_n : \phi(z, u_{n,j}) \leq \phi(z, x_n) + \xi_n\}, \\ C_{n+1} = \bigcap_{j\in\triangle} C_{n+1,j}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\xi_n = \sup_{i\in N^+}\{0, \sup_{p\in F(T_i), x\in C}(\phi(p, T_i^n x) - \phi(p, x))\}$, $\{\alpha_{n,i}\}$ *is a real number sequence in* $(0, 1)$ *for every* $i \geq 1$, $\{r_{n,j}\}$ *is a real number sequence in* $[k, \infty)$, *where* k *is some positive real number. Assume that* $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ *and* $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for every* $i \geq 1$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_F x_1$, *where* Π_F *is the generalized projection from* E *onto* F.

Remark 3.3 Corollary 3.3 improves the main theorem in Huang and Ma [8] from an equilibrium problem to a family of equilibrium problems.

Setting $\Theta \equiv 0$ in Theorem 3.1, we have Corollary 3.4.

Corollary 3.4 *Let* E *be a strictly convex, and uniformly smooth Banach space such that* E *has the Kadec-Klee property. Let* C *be a nonempty closed and convex subset of* E. *Let* $A : C \to E^*$ *be an* η*-hemicontinuous and relaxed* η*-*α*-monotone mapping, and let* f *be a proper convex and lower semicontinuous function from* $C \times C$ *to* $\mathbb{R} \cup \{+\infty\}$. *Let* $T_i : C \to C$ *be an asymptotically quasi-*ϕ*-nonexpansive mapping in the intermediate sense for every* $i \geq 1$. *Assume that* T_i *is closed asymptotically regular on* C *and* $F = \bigcap_{i=1}^{\infty} F(T_i) \cap \Omega$ *is nonempty and bounded. Also assume that the conditions* (i)-(v), *Lemma* 2.3, *and the following condition hold*:

(vi) *for all* $x, y, z, w \in C$, $\limsup_{t\downarrow 0} \langle Az, \eta(x, ty + (1 - t)w)\rangle \leq \langle Az, \eta(x, w)\rangle$.

Let $\{x_n\}$ *be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J T_i^n x_n), \\ u_n \in C \text{ such that } \langle A u_n, \eta(y, u_n) \rangle + f(y) - f(u_n) + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\xi_n = \sup_{i \in N^+}\{0, \sup_{p \in F(T_i), x \in C}(\phi(p, T_i^n x) - \phi(p, x))\}$, $\{\alpha_{n,i}\}$ *is a real number sequence in* $(0,1)$ *for every* $i \geq 1$, $\{r_n\}$ *is a real number sequence in* $[k, \infty)$, *where* k *is some positive real number. Assume that* $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ *and* $\liminf_{n \to \infty} \alpha_{n,0} \alpha_{n,i} > 0$ *for every* $i \geq 1$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_F x_1$, *where* Π_F *is the generalized projection from* E *onto* F.

Remark 3.4 Corollary 3.4 improves Corollary 15 in Chen *et al.* [5] from a quasi-ϕ-nonexpansive mapping to an infinite family of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense, from a mixed equilibrium problem to a family of mixed equilibrium problems, and from a uniformly smooth and uniformly convex Banach space to a uniformly smooth and strictly convex Banach space such that the space has the Kadec-Klee property.

Setting E to be a Hilbert space in Corollary 3.3, we have Corollary 3.5.

Corollary 3.5 *Let* E *be a Hilbert space. Let* C *be a nonempty closed and convex subset of* E *and let* $\triangle$ *be an index set. Let* Θ_j *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (C1)-(C4) *for every* $j \in \triangle$. *Let* $T_i : C \to C$ *be an asymptotically quasi-nonexpansive mapping in the intermediate sense for every* $i \geq 1$. *Assume that* T_i *is closed asymptotically regular on* C *and* $F = \bigcap_{i=1}^{\infty} F(T_i) \cap \bigcap_{j \in \triangle} EP(\Theta_j)$ *is nonempty and bounded. Let* $\{x_n\}$ *be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_{1,j} = C, \\ C_1 = \bigcap_{j \in \triangle} C_{1,j}, \\ x_1 = \mathrm{Proj}_{C_1} x_0, \\ y_n = \alpha_{n,0} x_n + \sum_{i=1}^{\infty} \alpha_{n,i} T_i^n x_n, \\ u_{n,j} \in C \text{ such that } \Theta_j(u_{n,j}, y) + \frac{1}{r_{n,j}} \langle y - u_{n,j}, u_{n,j} - y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1,j} = \{z \in C_n : \|z - u_{n,j}\|^2 \leq \|z - x_n\|^2 + \xi_n\}, \\ C_{n+1} = \bigcap_{j \in \triangle} C_{n+1,j}, \\ x_{n+1} = \mathrm{Proj}_{C_{n+1}} x_1, \end{cases}$$

where $\xi_n = \sup_{i \in N^+}\{0, \sup_{p \in F(T_i), x \in C}(\|p - T_i^n x\|^2 - \|p - x\|^2)\}$, $\{\alpha_{n,i}\}$ *is a real number sequence in* $(0,1)$ *for every* $i \geq 1$, $\{r_{n,j}\}$ *is a real number sequence in* $[k, \infty)$, *where* k *is some positive real number. Assume that* $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ *and* $\liminf_{n \to \infty} \alpha_{n,0} \alpha_{n,i} > 0$ *for every* $i \geq 1$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\mathrm{Proj}_F x_1$, *where* Proj_F *is the metric projection from* E *onto* F.

Proof Note that $\phi(x,y) = \|x-y\|^2$, $J = I$, the identity mapping, and the generalized projection is reduced to the metric projection. In the framework of Hilbert spaces, the class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense is reduced to the class of asymptotically quasi-nonexpansive mappings in the intermediate sense. By Corollary 3.3, we draw the desired conclusion immediately. □

Remark 3.5 Corollary 3.5 improves Theorem 4.1 in Zhang and Wu [9] from asymptotically quasi-nonexpansive mappings to asymptotically quasi-nonexpansive mappings in the intermediate sense.

Setting $T_i = I$ in Corollary 3.5, we have Corollary 3.6.

Corollary 3.6 *Let E be a Hilbert space. Let C be a nonempty closed and convex subset of E and let $\triangle$ be an index set. Let Θ_j be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (C1)-(C4) *for every $j \in \triangle$. Assume that $F = \bigcap_{j\in\triangle} EP(\Theta_j)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_{1,j} = C, \\ C_1 = \bigcap_{j\in\triangle} C_{1,j}, \\ x_1 = \operatorname{Proj}_{C_1} x_0, \\ u_{n,j} \in C \text{ such that } \Theta_j(u_{n,j}, y) + \frac{1}{r_{n,j}}\langle y - u_{n,j}, u_{n,j} - x_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1,j} = \{z \in C_n : \|z - u_{n,j}\| \leq \|z - x_n\|\}, \\ C_{n+1} = \bigcap_{j\in\triangle} C_{n+1,j}, \\ x_{n+1} = \operatorname{Proj}_{C_{n+1}} x_1, \end{cases}$$

where $\{r_{n,j}\}$ is a real number sequence in $[k,\infty)$, where k is some positive real number. Then the sequence $\{x_n\}$ converges strongly to $\operatorname{Proj}_F x_1$, where Proj_F is the metric projection from E onto F.

Setting $\Theta \equiv 0$ in Corollary 3.5, we have Corollary 3.7.

Corollary 3.7 *Let E be a Hilbert space. Let C be a nonempty closed and convex subset of E. Let $T_i : C \to C$ be an asymptotically quasi-nonexpansive mapping in the intermediate sense for every $i \geq 1$. Assume that T_i is closed asymptotically regular on C and $F = \bigcap_{i=1}^{\infty} F(T_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \operatorname{Proj}_{C_1} x_0, \\ y_n = \alpha_{n,0} x_n + \sum_{i=1}^{\infty} \alpha_{n,i} T_i^n x_n, \\ C_{n+1} = \{z \in C_n : \|z - y_n\|^2 \leq \|z - x_n\|^2 + \xi_n\}, \\ x_{n+1} = \operatorname{Proj}_{C_{n+1}} x_1, \end{cases}$$

where $\xi_n = \sup_{i\in N^+}\{0, \sup_{p\in F(T_i), x\in C}(\|p - T_i^n x\|^2 - \|p - x\|^2)\}$, $\{\alpha_{n,i}\}$ is a real number sequence in $(0,1)$ for every $i \geq 1$. Assume that $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ for every $i \geq 1$.

Then the sequence $\{x_n\}$ *converges strongly to* $\mathrm{Proj}_F x_1$, *where* Proj_F *is the metric projection from* E *onto* F.

Remark 3.6 Corollary 3.7 improves Corollary 4.3 in Zhang and Wu [9] from asymptotically quasi-nonexpansive mappings to asymptotically quasi-nonexpansive mappings in the intermediate sense.

From Corollary 3.7, we can obtain Corollary 3.8 easily.

Corollary 3.8 *Let* E *be a Hilbert space. Let* C *be a nonempty, closed, and convex subset of* E. *Let* $T : C \to C$ *be a closed quasi-nonexpansive mapping. Let* $\{x_n\}$ *be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \mathrm{Proj}_{C_1} x_0, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_{n+1} = \{z \in C_n : \|z - y_n\| \le \|z - x_n\|\}, \\ x_{n+1} = \mathrm{Proj}_{C_{n+1}} x_1, \end{cases}$$

where $\{\alpha_n\}$ *is a real number sequence in* $(0,1)$ *such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\mathrm{Proj}_F x_1$, *where* Proj_F *is the metric projection from* E *onto* F.

Remark 3.7 Corollary 3.8 is a shrinking version of the corresponding results in Nakajo and Takahashi [30]. Note that the mapping in our result is quasi-nonexpansive. The restriction of the demiclosed principal is relaxed.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Shaoxing University, Zhejiang, 312000, China. [2]Department of Mathematics, Zhejiang Normal University, Zhejiang, 321004, China. [3]Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung, 807, Taiwan.

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