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Scalar gap functions and error bounds for generalized mixed vector equilibrium problems with applications

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Abstract

It is well known that equilibrium problems are very important mathematical models and are closely related with fixed point problems, variational inequalities, and Nash equilibrium problems. Gap functions and error bounds which play a vital role in algorithms design, are two much-addressed topics of vector equilibrium problems. This paper is devoted to studying the scalar-valued gap functions and error bounds for the generalized mixed vector equilibrium problem (GMVE). First, a scalar gap function for (GMVE) is proposed without any scalarization methods, and then error bounds of (GMVE) are established in terms of the gap function. As applications, error bounds for generalized vector variational inequalities and vector variational inequalities are derived, respectively. The main results obtained are new and improve corresponding results of Charitha and Dutta (*Pac. J. Optim.* 6:497-510, 2010) and Sun and Chai (*Optim. Lett.* 8:1663-1673, 2014).

Keywords: error bound; scalar gap function; generalized mixed vector equilibrium

1 Introduction

Equilibrium problems, which were firstly studied by Blum and Oettli [1], provide an unified framework for fixed point problems, variational inequality, complementarity problems, and optimization problems. It is well known that the vector equilibrium problem is a vital extension of equilibrium problems, which contain vector variational inequality, vector complementarity problems, and vector optimization problems as special cases. In the past decades, various kinds of vector equilibrium problems and their applications have been introduced and studied; see [2–9] and the references therein. Recently, Chang *et al.* [10] and Kumam *et al.* [5] researched a generalized mixed equilibrium problem,

$$F(x, y) + \langle A(x), y - x \rangle + g(y) - g(x) \geq 0, \quad \forall y \in K.$$

It is very general in the sense that it includes fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and equilibrium problems as special cases.

Error bounds, which play a critical role in algorithm design, can be used to measure how much the approximate solution fails to be in the solution set and to analyze the convergence rates of various methods. Recently, kinds of error bounds have been presented

for variational inequalities in [11–18]. Results for error bounds have been established for a weak vector variational inequality (WVVI) in [12–15, 19]. Xu and Li [15] obtained error bounds for a weak vector variational inequality with cone constraints by using a method of image space analysis. By using a scalarization approach of Konnov [20], Li and Mastroeni [13] established error bounds for two kinds of (WVVI) with set-valued mappings. By a regularized gap function and a D-gap function, Charitha and Dutta [12] used a projection operator method to obtain error bounds of (WVVI), respectively. Sun and Chai [14] studied some error bounds for generalized vector variational inequalities in virtue of the regularized gap functions. Very recently, a global error bound of a weak vector variational inequality was established by the nonlinear scalarization method in Li [19].

However, to the best of our knowledge, an error bound of the generalized mixed vector equilibrium problem (GMVE) has never been investigated. In this paper, motivated by ideas in Sun and Chai [14] and Yamashita *et al.* [18], we introduce a scalar gap function for (GMVE). Then an error bound of (GMVE) is presented. As an application of an error bound for (GMVE), we also get error bounds of (GVVI) and (VVI), respectively.

This paper is organized as follows: In Section 2, we first recall some basic definitions. In Section 3, we introduce scalar gap functions for (GMVE), (GVVI), and (VVI). By using these gap functions, we obtain some error bound results for (GMVE), (GVVI), and (VVI), respectively.

2 Mathematical preliminaries

Throughout this paper, let \mathbb{R}^n be the n -dimensional Euclidean space and denote $\mathbb{R}_+^n = \{x = (x_1, x_2, \dots, x_n) : x_j \geq 0, j = 1, 2, \dots, n\}$, the norms of all finite dimensional spaces be denoted by $\|\cdot\|$, the inner products of all finite dimensional spaces be denoted by $\langle \cdot, \cdot \rangle$. Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Let $A_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ($i = 1, 2, \dots, m$) be a vector-valued mapping, $F_i : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$ ($i = 1, 2, \dots, m$) be real-valued functions. For abbreviation, we put

$$A := (A_1, A_2, \dots, A_m), \quad F := (F_1, F_2, \dots, F_m), \quad g := (g_1, g_2, \dots, g_m)$$

and for any $x, v \in \mathbb{R}^n$

$$\langle A(x), v \rangle := (\langle A_1(x), v \rangle, \langle A_2(x), v \rangle, \dots, \langle A_m(x), v \rangle).$$

In this paper, we consider the generalized mixed vector equilibrium problem (GMVE) of finding $x \in K$ such that

$$F(x, y) + \langle A(x), y - x \rangle + g(y) - g(x) \notin -\text{int } \mathbb{R}_+^m, \quad \forall y \in K. \tag{1}$$

Denote by S_{GMVE} the solution set of (GMVE).

If $m = 1$, our problem is to find $x \in K$ such that

$$F(x, y) + \langle A(x), y - x \rangle + g(y) - g(x) \geq 0, \quad \forall y \in K. \tag{2}$$

Then this problem reduces to a generalized mixed equilibrium problem [5].

If $F = 0$, our problem is to find $x \in K$ such that

$$\langle A(x), y - x \rangle + g(y) - g(x) \notin -\text{int } \mathbb{R}_+^m, \quad \forall y \in K. \tag{3}$$

Then this problem reduces to a generalized vector variational inequality problem (GVVI) [14].

In the case of $F = 0$ and $g = 0$, (GMVE) is a vector variational inequality problem (VVI), introduced and studied by Giannessi [21], finding $x \in K$ such that

$$\langle A(x), y - x \rangle \notin -\text{int } \mathbb{R}_+^m, \quad \forall y \in K. \tag{4}$$

In the case of $A \equiv 0$ and $g = 0$, then (GMVE) is a vector equilibrium problem, finding $x \in K$ such that

$$F(x, y) \notin -\text{int } \mathbb{R}_+^m, \quad \forall y \in K. \tag{5}$$

In the case of $A \equiv 0$ and $F = 0$, (GMVE) is a vector optimization problem, finding $x \in K$ such that

$$g(y) - g(x) \notin -\text{int } \mathbb{R}_+^m, \quad \forall y \in K. \tag{6}$$

For $i = 1, 2, \dots, m$, we denote the generalized mixed vector equilibrium problems (GMVE) associated with F_i, A_i and g_i as $(GMVE)^i$, the generalized vector variational inequality problems (GVVI) associated with A_i and g_i as $(GVVI)^i$, and the vector variational inequality problems (VVI) associated with A_i as $(VVI)^i$, respectively. The solution sets of $(GMVE)^i, (GVVI)^i$, and $(VVI)^i$ will be denoted by S_{GMVE}^i, S_{GVVI}^i , and S_{VVI}^i , respectively.

In the paper, we intend to investigate gap functions and error bounds of (GMVE), (GVVI), and (VVI). We shall recall some notations and definitions, which will be used in the sequel.

Definition 2.1 A real-valued function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex (resp. concave) over \mathbb{R}^n , if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

$$(\text{resp. } f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)),$$

for every $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$.

Definition 2.2 A vector-valued function $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is strongly monotone over \mathbb{R}^n with modulus $\kappa > 0$, if for any $x, y \in \mathbb{R}^n$,

$$\langle h(y) - h(x), y - x \rangle \geq \kappa \|y - x\|^2.$$

Definition 2.3 A real-valued function $\vartheta : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a scalar-valued gap function of (GMVE) (resp. (GVVI) and (VVI)), if it satisfies the following conditions:

- (i) $\vartheta(x) \geq 0$, for any $x \in K$,
- (ii) $\vartheta(x_0) = 0$ if and only if $x_0 \in K$ is a solution of (GMVE) (resp. (GVVI) and (VVI)).

3 Main results

Notice that Sun and Chai [14] introduced a new scalar-valued gap function of (GVVI) without any scalarization approach; the gap function discussed in [14] is simpler from the computational view. In terms of an approach due to Sun and Chai [14] and Yamashita *et al.* [18], we construct the function $\vartheta_\alpha : K \rightarrow \mathbb{R}$ for $\alpha > 0$,

$$\vartheta_\alpha(x) := \sup_{y \in K} \left\{ \min_{1 \leq i \leq m} \left\{ -F_i(x, y) + \langle A_i(x), x - y \rangle + g_i(x) - g_i(y) \right\} - \alpha \varphi(x, y) \right\}, \tag{7}$$

$$\psi_\alpha(x) := \sup_{y \in K} \left\{ \min_{1 \leq i \leq m} \left\{ \langle A_i(x), x - y \rangle + g_i(x) - g_i(y) \right\} - \alpha \varphi(x, y) \right\}, \tag{8}$$

and

$$\phi_\alpha(x) := \sup_{y \in K} \min_{1 \leq i \leq m} \left\{ \langle A_i(x), x - y \rangle - \alpha \varphi(x, y) \right\}, \tag{9}$$

respectively, where $F_i, \varphi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, 2, \dots, m$ are real-valued functions. In the following, let φ be a continuously differentiable function, which has the following property with the associated constants $\gamma, \beta > 0$.

(P) For all $x, y \in \mathbb{R}^n$,

$$\beta \|x - y\|^2 \leq \varphi(x, y) \leq (\gamma - \beta) \|x - y\|^2, \quad \gamma \geq 2\beta > 0. \tag{10}$$

For example, let $\kappa > 0$ and let $\varphi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by

$$\varphi(x, y) = \kappa \|x - y\|^2, \quad x, y \in \mathbb{R}^n.$$

Then φ satisfies condition (P), with $\gamma = 2\kappa$ and $\beta = \kappa$.

For any $i = 1, 2, \dots, m$, we suppose that F_i satisfies the following conditions:

- (A1) F_i is a convex function about the second variable on $\mathbb{R}^n \times \mathbb{R}^n$.
- (A2) $F_i(x, y) = 0, \forall x, y \in \mathbb{R}^n$, if and only if $x = y$.
- (A3) For any $x, y, z \in K$, $F_i(x, y) + F_i(y, z) \leq F_i(x, z)$.

Theorem 3.1 *If $F_i : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is convex about the second variable and g_i is convex over \mathbb{R}^n for any $i = 1, 2, \dots, m$, then the function ϑ_α , with $\alpha > 0$, defined by (7) is a gap function for (GMVE).*

Proof (i) It is clear that for any $x \in K$,

$$\vartheta_\alpha(x) := \sup_{y \in K} \left\{ \min_{1 \leq i \leq m} \left\{ -F_i(x, y) + \langle A_i(x), x - y \rangle + g_i(x) - g_i(y) \right\} - \alpha \varphi(x, y) \right\} \geq 0$$

follows simply by setting $x = y$ in the right hand side of the expression for $\vartheta_\alpha(x)$.

(ii) If there exists $x_0 \in K$ such that $\vartheta_\alpha(x_0) = 0$, then

$$\min_{1 \leq i \leq m} \left\{ -F_i(x_0, y) + \langle A_i(x_0), x_0 - y \rangle + g_i(x_0) - g_i(y) \right\} \leq \alpha \varphi(x_0, y), \quad \forall y \in K.$$

For arbitrary $x \in K$ and $\kappa \in (0, 1)$, let $y = x + \kappa(x_0 - x)$. Since K is convex, we get $y \in K$ and

$$\begin{aligned} & \min_{1 \leq i \leq m} \left\{ -F_i(x_0, x + \kappa(x_0 - x)) + \langle A_i(x_0), x_0 - (x + \kappa(x_0 - x)) \rangle + g_i(x_0) - g_i(x + \kappa(x_0 - x)) \right\} \\ & \leq \alpha \varphi(x_0, x + \kappa(x_0 - x)). \end{aligned}$$

Since F_i is convex over $\mathbb{R}^n \times \mathbb{R}^n$ about the second variable and g_i is convex over \mathbb{R}^n for any $i = 1, 2, \dots, m$, we have

$$\begin{aligned} & \min_{1 \leq i \leq m} \left\{ -\kappa F_i(x_0, x_0) - (1 - \kappa) F_i(x_0, x) + \langle A_i(x_0), (1 - \kappa)(x_0 - x) \rangle \right. \\ & \quad \left. + g_i(x_0) - \kappa g_i(x_0) - (1 - \kappa) g_i(x) \right\} \\ & \leq \min_{1 \leq i \leq m} \left\{ -F_i(x_0, x + \kappa(x_0 - x)) + \langle A_i(x_0), (1 - \kappa)(x_0 - x) \rangle + g_i(x_0) - g_i(x + \kappa(x_0 - x)) \right\} \\ & \leq \alpha \varphi(x_0, x + \kappa(x_0 - x)). \end{aligned} \tag{11}$$

For the function $\varphi(x, y)$, by using (10), we have

$$\varphi(x_0, x + \kappa(x_0 - x)) \leq (1 - \kappa)^2 (\gamma - \beta) \|x_0 - x\|^2. \tag{12}$$

By the property (A2) of the function F_i , we obtain $F_i(x_0, x_0) = 0$.

Hence, from (11) and (12), we get

$$\begin{aligned} & (1 - \kappa) \min_{1 \leq i \leq m} \left\{ -F_i(x_0, x) + \langle A_i(x_0), x_0 - x \rangle + g_i(x_0) - g_i(x) \right\} \\ & \leq \alpha (\gamma - \beta) (1 - \kappa)^2 \|x - x_0\|^2. \end{aligned}$$

So,

$$\min_{1 \leq i \leq m} \left\{ -F_i(x_0, x) + \langle A_i(x_0), x_0 - x \rangle + g_i(x_0) - g_i(x) \right\} \leq \alpha (\gamma - \beta) (1 - \kappa) \|x - x_0\|^2.$$

Taking the limit as $\kappa \rightarrow 1$, we obtain

$$\min_{1 \leq i \leq m} \left\{ -F_i(x_0, x) + \langle A_i(x_0), x_0 - x \rangle + g_i(x_0) - g_i(x) \right\} \leq 0.$$

Then, for any $x \in K$, there exists $1 \leq i_0 \leq m$ such that

$$-F_{i_0}(x_0, x) + \langle A_{i_0}(x_0), x_0 - x \rangle + g_{i_0}(x_0) - g_{i_0}(x) \leq 0.$$

This means that

$$F(x_0, x) + \langle A_i(x_0), x - x_0 \rangle + g(x) - g(x_0) \notin -\text{int } \mathbb{R}_+^m, \quad \forall x \in K.$$

Thus, $x_0 \in S_{GMVE}$.

Conversely, if $x_0 \in S_{GMVE}$, then there exists $1 \leq i_0 \leq m$ such that

$$F_{i_0}(x_0, y) + \langle A_{i_0}(x_0), y - x_0 \rangle + g_{i_0}(y) - g_{i_0}(x_0) \geq 0 \quad \text{for any } y \in K.$$

This means that

$$\min_{1 \leq i \leq m} \{-F_i(x_0, y) + \langle A_i(x_0), x_0 - y \rangle + g_i(x_0) - g_i(y)\} \leq 0 \quad \text{for any } y \in K.$$

So,

$$\vartheta_\alpha(x_0) \leq 0.$$

Since $\vartheta_\alpha(x_0) \geq 0$ for any $x \in K$,

$$\vartheta_\alpha(x_0) = 0.$$

This completes the proof. □

By a similar method, we conclude the following results for (GVVI) and (VVI), respectively.

Corollary 3.1 *The function ψ_α , with $\alpha > 0$, defined by (8) is a gap function for (GVVI).*

Corollary 3.2 *The function ϕ_α , with $\alpha > 0$, defined by (9) is a gap function for (VVI).*

Now, by using the gap function $\vartheta_\alpha(x)$, we obtain an error bound result for (GMVE).

Theorem 3.2 *Assume that each A_i are strongly monotone over K with the modulus $\kappa_i > 0$. Assume F_i is convex about the second variable over $\mathbb{R}^n \times \mathbb{R}^n$ and g_i is convex over \mathbb{R}^n for any $i = 1, 2, \dots, m$. Further assume that $\bigcap_{i=1}^m S_{GMVE}^i \neq \emptyset$. Moreover, let $\kappa = \min_{1 \leq i \leq m} \kappa_i$ and $\alpha > 0$ be chosen such that $\kappa > \alpha(\gamma - \beta)$, where $\gamma \geq 2\beta > 0$ are constants associated with the function φ . Then for any $x \in K$,*

$$d(x, S_{GMVE}) \leq \frac{1}{\sqrt{\kappa - \alpha(\gamma - \beta)}} \sqrt{\vartheta_\alpha(x)},$$

where $d(x, S_{GMVE})$ denotes the distance from the point x to the solution set S_{GMVE} .

Proof By (7), we get

$$\vartheta_\alpha(x) \geq \min_{1 \leq i \leq m} \{-F_i(x, y) + \langle A_i(x), x - y \rangle + g_i(x) - g_i(y)\} - \alpha\varphi(x, y) \quad \text{for any } y \in K.$$

Since $\bigcap_{i=1}^m S_{GMVE}^i \neq \emptyset$, all $(GMVE)^i$ have the same solution. Without loss of generality, we can assume that the same solution is $x_0 \in K$. Obviously, $x_0 \in S_{GMVE}$ and

$$\vartheta_\alpha(x) \geq \min_{1 \leq i \leq m} \{-F_i(x, x_0) + \langle A_i(x), x - x_0 \rangle + g_i(x) - g_i(x_0)\} - \alpha\varphi(x, x_0).$$

Without loss of generality, we assume that

$$\begin{aligned}
 & -F_1(x, x_0) + \langle A_1(x), x - x_0 \rangle + g_1(x) - g_1(x_0) \\
 & = \min_{1 \leq i \leq m} \{ -F_i(x, x_0) + \langle A_i(x), x - x_0 \rangle + g_i(x) - g_i(x_0) \}.
 \end{aligned}$$

Since A is strongly monotone, we obtain

$$\vartheta_\alpha(x) \geq -F_1(x, x_0) + \langle A_1(x_0), x - x_0 \rangle + g_1(x) - g_1(x_0) + \kappa \|x - x_0\|^2 - \alpha\varphi(x, x_0).$$

It follows from the property (P) of the function φ that

$$\begin{aligned}
 \vartheta_\alpha(x) & \geq -F_1(x, x_0) + \langle A_1(x_0), x - x_0 \rangle + g_1(x) - g_1(x_0) \\
 & \quad + \kappa \|x - x_0\|^2 - \alpha(\gamma - \beta)\|x - x_0\|^2.
 \end{aligned} \tag{13}$$

By $x_0 \in S_{GMVE}^1$, we get

$$F_1(x_0, x) + \langle A_1(x_0), x - x_0 \rangle + g_1(x) - g_1(x_0) \geq 0. \tag{14}$$

For the function F_1 , by using (A3), we get from (13) and (14)

$$\vartheta_\alpha(x) = \vartheta_\alpha(x) + F_1(x, x) \geq \vartheta_\alpha(x) + F_1(x, x_0) + F(x_0, x) \geq [\kappa - \alpha(\gamma - \beta)]\|x - x_0\|^2.$$

Namely,

$$\vartheta_\alpha(x) \geq [\kappa - \alpha(\gamma - \beta)]\|x - x_0\|^2.$$

Then

$$\|x - x_0\| \leq \frac{1}{\sqrt{\kappa - \alpha(\gamma - \beta)}} \sqrt{\vartheta_\alpha(x)},$$

which means that

$$d(x, S_{GMVE}) \leq \frac{1}{\sqrt{\kappa - \alpha(\gamma - \beta)}} \sqrt{\vartheta_\alpha(x)}.$$

This complete the proof. □

The following example shows that, in general, the conditions of Theorem 3.2 can be achieved.

Example 3.1 Let $n = 1, m = 2, K \subseteq R$, and $K = [-1, 1]$. Define $A_1, A_2, g_1, g_2 : R \rightarrow R, F_1 : R \times R \rightarrow R, F_2 : R \times R \rightarrow R$ by

$$\begin{aligned}
 A_1(x) &= x, & A_2(x) &= x^3 + 4x, & g_1(x) &= 2x^2, & g_2(x) &= 3x^4, \\
 F_1(x, y) &= -x^2 + y^2, & F_2(x, y) &= -3x^4 + 3y^4.
 \end{aligned}$$

Then

$$F(x, y) = (-x^2 + y^2, -3x^4 + 3y^4), \quad A(x) = (x, x^3 + 2x), \quad g(x) = (2x^2, 3x^4).$$

Obviously, $F_1(x, y)$ and $F_2(x, y)$ are convex about the second variable, respectively. A_1 and A_2 are strongly monotone over K with the modulus $\kappa_1 = 1$ and $\kappa_2 = 2$, respectively. Moreover, g_1 and g_2 are convex over R . On the other hand, by direct calculations, we have

$$\bigcap_{i=1}^2 S_{GMVE}^i = \{0\}.$$

Thus, the conditions of Theorem 3.2 are satisfied.

Similarly, by using gap functions ψ_α and ϕ_α , we can also obtain error bound results for (GVVI) and (VVI), respectively.

Corollary 3.3 *Assume that each A_i are strongly monotone over K with the modulus $\kappa_i > 0$. If g_i is convex over \mathbb{R}^n for any $i = 1, 2, \dots, m$. Further assume that $\bigcap_{i=1}^m S_{GVVI}^i \neq \emptyset$. Moreover, let $\kappa = \min_{1 \leq i \leq m} \kappa_i$ and $\alpha > 0$ be chosen such that $\kappa > \alpha(\gamma - \beta)$, where $\gamma \geq 2\beta > 0$ are constants associated with the function φ . Then, for any $x \in K$,*

$$d(x, S_{GVVI}) \leq \frac{1}{\sqrt{\kappa - \alpha(\gamma - \beta)}} \sqrt{\psi_\alpha(x)},$$

where $d(x, S_{GVVI})$ denotes the distance from the point x to the solution set S_{GVVI} .

Corollary 3.4 *Assume that each A_i are strongly monotone over K with the modulus $\kappa_i > 0$. Further assume that $\bigcap_{i=1}^m S_{VVI}^i \neq \emptyset$. Moreover, let $\kappa = \min_{1 \leq i \leq m} \kappa_i$ and $\alpha > 0$ be chosen such that $\kappa > \alpha(\gamma - \beta)$, where $\gamma \geq 2\beta > 0$ are constants associated with the function φ . Then, for any $x \in K$,*

$$d(x, S_{VVI}) \leq \frac{1}{\sqrt{\kappa - \alpha(\gamma - \beta)}} \sqrt{\phi_\alpha(x)},$$

where $d(x, S_{VVI})$ denotes the distance from the point x to the solution set S_{VVI} .

Remark 3.1 (i) In [14], there exist some mistakes in the proof of Theorem 3.2, which lead to the requirement of Lipschitz properties of $g_i, i = 1, 2, \dots, n$. Hence, we give the modified error bound for (GVVI) in Corollary 3.3 without Lipschitz assumption.

(ii) In [12], Charitha and Dutta established error bounds for (VVI) by using the projection operator method and strongly monotone assumptions, whereas it seems that our method is more simple from the computational view since there are not any scalarization parameters.

(iii) Under the conditions of Theorem 3.2, the strong assumption that $\bigcap_{i=1}^m S_{GMVE}^i \neq \emptyset$ shows that S_{GMVE} is a singleton set but not a general set. In fact, by $\bigcap_{i=1}^m S_{GMVE}^i \neq \emptyset$, there exists $x \in K$ such that $x \in \bigcap_{i=1}^m S_{GMVE}^i$, namely, for every $i = 1, 2, \dots, m$,

$$F_i(x, y) + \langle A_i(x), y - x \rangle + g_i(y) - g_i(x) \geq 0, \quad \forall y \in K. \tag{15}$$

It is clear that $x \in S_{GMVE}$. If S_{GMVE} is not a singleton set, there exists $x' \in S_{GMVE}$ with $x' \neq x$. Therefore, there exists $j \in \{1, 2, \dots, m\}$ such that

$$F_j(x', y) + \langle A_j(x'), y - x' \rangle + g_j(y) - g_j(x') \geq 0, \quad \forall y \in K. \quad (16)$$

Thus, from (15), we have

$$F_j(x, x') + \langle A_j(x), x' - x \rangle + g_j(x') - g_j(x) \geq 0. \quad (17)$$

From (16), we have

$$F_j(x', x) + \langle A_j(x'), x - x' \rangle + g_j(x) - g_j(x') \geq 0. \quad (18)$$

According to (17) and (18), we get

$$F_j(x, x') + F_j(x', x) + \langle A_j(x), x' - x \rangle + \langle A_j(x'), x - x' \rangle \geq 0. \quad (19)$$

However, by the properties (A2) and (A3) of the function F_j ,

$$F_j(x, x') + F_j(x', x) \leq 0. \quad (20)$$

As A_j is strongly monotone, we have

$$\langle A_j(x) - A_j(x'), x - x' \rangle \geq \kappa \|x - x'\|^2 > 0. \quad (21)$$

By combining (20) and (21), we have

$$F_j(x, x') + F_j(x', x) + \langle A_j(x) - A_j(x'), x' - x \rangle < 0. \quad (22)$$

This, however, contradicts (18).

Now we ask: How do we establish error bounds for S_{GMVE} in terms of the gap function ϑ_α , under mild assumptions, such that S_{GMVE} need not be a singleton set in general? This problem may be interesting and valuable in vector optimization.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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