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# Projective contractions, generalized metrics, and fixed points

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**Abstract**

In 1981, Borsík and Doboš studied the aggregation problem for metric spaces. Thus, they characterized those functions that allow one to merge a collection of metrics providing a single metric as a result (Borsík and Doboš in Math. Slovaca 31:193-205, 1981).

Later on, in 1994, the notion of partial metric space was introduced by Matthews with the aim of providing an appropriate mathematical tool for program verification (Matthews in Ann. N.Y. Acad. Sci. 728:183-197, 1994). In the aforesaid reference, an extension of the well-known Banach fixed point theorem to the partial metric framework was given and, in addition, an application of such a result to denotational semantics and program verification was provided.

Inspired by the applicability of partial metric spaces to computer science and by the fact that there are partial metrics useful in such a field which can be induced through aggregation, in 2012 Massanet and Valero analyzed the aggregation problem in the partial metric framework (Massanet and Valero in Proc. of the 17th Spanish Conference on Fuzzy Technology and Fuzzy Logic (Estylf 2012), pp.558-563, 2012).

In this paper, motivated by the fact that fixed point techniques are essential in order to apply partial metric spaces to computer science and that, as we have pointed out above, some of such partial metrics can be induced by aggregation, we introduce a new notion of contraction between partial metric spaces which involves aggregation functions. Besides, since fixed point theory in partial metric spaces from an aggregation viewpoint still is without exploring, we provide a fixed point theorem in the spirit of Matthews for the new type of contractions and, in addition, we give examples which illustrate that the assumptions in such a result cannot be weakened. Furthermore, we provide conditions that vouch the existence and uniqueness of fixed point for this new class of contractions. Finally, we discuss the well-posedness for this kind of fixed point problem and the limit shadowing property for the new sort of contractions.

**Keywords:** partial metric; fixed point; aggregation function; homogeneous function; projective contraction

## 1 Introduction

In many practical problems appears the need to process simultaneously a few numerical values which are provided by some sources (possibly of different natures) with the aim of making a decision in order to elaborate a plan of action or to solve a problem. A natural way to achieve this goal consists of merging the aforementioned values into a simple one by means of an aggregation function in such a way that the obtained numerical value is used to take the desired decision under an appropriate criterion for the problem under consideration (see, for instance, [1]). In many situations the numerical values that are merged represent distances between different objects. Motivated, in part, by the aforesaid facts Borsík and Doboš began in 1981 a research line on the aggregation of metrics [2]. Concretely, they studied the properties that a function must satisfy in order to merge a collection of metrics into a single one. Next we recall a few facts about the aforementioned functions.

From now on, the letters $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}$ will denote the set of real numbers, the set of nonnegative real numbers, and the set of positive integer numbers, respectively.

Following [2], a function $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ ($n \in \mathbb{N}$) is a metric aggregation function (metric preserving in [2]) provided that the function $M_\Phi : X \times X \to \mathbb{R}_+$ is a metric for every collection of metric spaces $\{(X_i, d_i)\}_{i=1}^n$, where

$$M_\Phi(x,y) = \Phi\bigl(d_1(x_1,y_1),\dots,d_n(x_n,y_n)\bigr)$$

for all $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n) \in X$ with $X = \prod_{i=1}^n X_i$.

According to [2], we will denote by $\mathcal{O}$ the set of all functions $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ satisfying $\Phi(x) = 0 \Leftrightarrow x = \bar{0}$, where $\bar{0} \in \mathbb{R}_+^n$ with $\bar{0} = (0,\dots,0)$. We will consider the pointwise order relation $\preceq$ on $\mathbb{R}_+^n$, *i.e.*, $x \preceq y \Leftrightarrow x_i \le y_i$ for all $i = 1,\dots,n$. Of course, a function $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ is said to be:

(i) monotone provided that $\Phi(x) \le \Phi(y)$ for all $x, y \in \mathbb{R}_+^n$ with $x \preceq y$,

(ii) subadditive provided that $\Phi(x+y) \le \Phi(x) + \Phi(y)$ for all for all $x, y \in \mathbb{R}_+^n$.

On account of [3], a function $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ is called homogeneous provided that $\Phi(\alpha x) = \alpha\Phi(x)$ for all $x \in \mathbb{R}_+^n$ and $\alpha \in \mathbb{R}_+$.

The next result yields the first interesting property about metric aggregation functions that Borsík and Doboš proved.

**Proposition 1** *Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$. If $\Phi$ is monotone and subadditive, and $\Phi \in \mathcal{O}$, then $\Phi$ is a metric aggregation function.*

It is clear that Proposition 1 does not characterize metric aggregation functions because there are metric aggregation functions that are not monotone (see Example 8 in [4]). Inspired by this fact, Borsík and Doboš gave a characterization of metric aggregation functions by means of triplets (for a detailed discussion we refer the reader to [2]). Let us recall that the triplet of nonnegative real numbers $(a, b, c)$ forms a triangle triplet whenever $a \le b + c$, $b \le a + c$, and $c \le b + a$. The result that characterizes metric aggregation functions states the following (Theorem 2.6, [2]).

**Theorem 2** *A function $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ is a metric aggregation function if and only if the following properties hold*:

(1) $\Phi \in \mathcal{O}$.

(2) *Let $x, y, z \in \mathbb{R}_+^n$. If $(a_i, b_i, c_i)$ is a triangle triplet for all $i = 1,\dots,n$, then so is* $(\Phi(x), \Phi(y), \Phi(z))$.

More recently, inspired by the original work of Borsík and Doboš, Castiñeira *et al.* studied in depth the aggregation problem when different classes of generalized metrics, as pseudometrics, are considered in [5–7] and, Petruşel *et al.* discussed when the contractive character of a mapping is preserved by transformations of metrics obtained by means of metric aggregation functions in [8].

In the last years the interest in partial metric spaces, a generalization of the notion of metric space, has grown because they are efficient tools in modeling some processes that arise in a natural way in the few fields of computer science. For instance, applications of partial metric spaces to denotational semantics and program verification can be found in [9–11], applications to logic programming have been given in [12, 13] and applications to asymptotic complexity analysis of algorithms can be looked up in [14, 15].

Let us introduce the basics of partial metric spaces in order to achieve our main target in our subsequent study.

According to [9], a partial metric on a (nonempty) set $X$ is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(i) $p(x,x) = p(x,y) = p(y,y) \Leftrightarrow x = y$.
(ii) $p(x,x) \leq p(x,y)$.
(iii) $p(x,y) = p(y,x)$.
(iv) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

Of course, a partial metric space is a pair $(X,p)$ such that $X$ is a (nonempty) set and $p$ is a partial metric on $X$. Clearly, the metric notion can be retrieved as a particular case of the partial metric one. In particular, a metric on a set $X$ is a partial metric $p$ on $X$ such that $p(x,x) = 0$ for all $x \in X$. Moreover, every partial metric $p$ on $X$ generates a $T_0$ topology $\mathcal{T}(p)$ on $X$ which has as a base the family of open $p$-balls $\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$. One deduces immediately that a sequence $(x_n)_{n\in\mathbb{N}}$ in a partial metric space $(X,p)$ converges to a point $x \in X$ with respect to $\mathcal{T}(p) \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

On account of [9], a sequence $(x_n)_{n\in\mathbb{N}}$ in a partial metric space $(X,p)$ is said to be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists in $\mathbb{R}_+$. Moreover, a partial metric space $(X,p)$ is called complete provided that every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ converges, with respect to $\mathcal{T}(p)$, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

The aforementioned applications of partial metric spaces are obtained by means of fixed point techniques which are based on the so-called Matthews fixed point theorem. In order to recall such a fixed point theorem, let us introduce the notion of a contraction in the partial metric framework (see [9, 16]):

A mapping from a partial metric space $(X,p)$ into itself is said to be a contraction if there exists $c \in [0,1[$ such that

$$p\bigl(f(x),f(y)\bigr) \leq cp(x,y) \tag{1}$$

for all $x, y \in X$. The preceding constant $c$ is said to be the contractive constant of the contraction $f$. In the light of the above notion, the Matthews fixed point theorem can be stated as follows.

**Theorem 3** *Let $(X,p)$ be a complete partial metric space and let $F : X \to X$. If $F$ is a contraction from $(X,p)$ into itself, then $F$ has a unique fixed point $x_0$. Moreover, $p(x_0,x_0) = 0$ and for each $x \in X$, $\lim_{n\to+\infty} F^n(x) = x_0$ in $(X,p)$.*

Motivated by the above-mentioned utility of partial metric spaces and the fact that there are partial metrics used in such applications (see for instance [14, 15]) which are obtained by means of aggregation, Massanet and Valero introduced the notion of a partial metric aggregation function in [17].

Let us recall that a function $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ is a partial metric aggregation function provided that the function $P_\Phi : X \times X \to \mathbb{R}_+$ is a partial metric for every arbitrary collection of partial metric spaces $\{(X_i, p_i)\}_{i=1}^n$, where

$$P_\Phi(x, y) = \Phi\bigl(p_1(x_1, y_1), \ldots, p_n(x_n, y_n)\bigr)$$

for all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in X$ with $X = \prod_{i=1}^n X_i$.

Examples of partial metric aggregation functions are given; see Examples 13, 14, and 15 in Section 3.

The characterization and a few properties, in the spirit of Theorem 2 and Proposition 1, of partial metric aggregation functions were stated without proof in [17] and they will be detailed in Section 2.

As we have pointed out before, there are partial metrics that are used in the applications to computer science that can be obtained by means of suitable aggregation functions and, besides, fixed point theory is crucial in such applications. So it seems natural and interesting to ask whether Theorem 3 can be extended to the aggregation framework, *i.e.*, for contractions defined between partial metric spaces such that the partial metric is induced by a partial metric aggregation function. Thus the main proposal of this paper is twofold. On the one hand, we introduce a new notion of contraction in the partial metric context which involves aggregation functions in such a way that the contractive notion of Matthews can be retrieved as a particular case of our new one. On the other hand, we provide conditions that ensure the existence and uniqueness of fixed point for this new kind of contractions. In particular we show that homogeneity and a sort of contractivity are required for the involved aggregation function. Moreover, completeness is also required for the partial metric induced by such an aggregation function. Furthermore, examples which illustrate that the assumptions in such a fixed point result cannot be weakened are given. Finally, we discuss the well-posedness for this kind of fixed point problem and the limit shadowing property for the new sort of contractions.

## 2 The aggregation of partial metrics

This section is devoted to the presentation of the basics about the partial metric aggregation problem. Concretely, we introduce those properties of partial aggregation functions that will be required to accomplish our target, *i.e.*, to develop a fixed point theory in partial metric spaces from an aggregation perspective. Thus we provide the characterization, in the spirit of Theorem 2, of partial metric aggregation functions and we show, in addition, that, contrary to the metric aggregation functions, they are monotone functions. It is worth to mention that the aforesaid results were published for the first time in [17]. Nonetheless, they were presented in the aforesaid reference without proof. We have included their detailed proofs, on the one hand, because their proof cannot be found in the literature and, on the other hand, because they are crucial in our subsequent study and the inclusion of the proof will help the reader to delve into the techniques that will be made use of in the next section.

First of all we show that, contrary to the metric case (see Section 1), every partial aggregation function is always monotone. We will take advantage of this fact in the characterization of partial metric aggregation functions and in the next section.

**Proposition 4** *Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ be a partial metric aggregation function. Then $\Phi$ is monotone.*

*Proof* Define $p_{\max} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by $p_{\max}(a,b) = \max\{a,b\}$ for all $a,b \in \mathbb{R}_+$. It is clear that $p_{\max}$ is a partial metric on $\mathbb{R}_+$. Since $\Phi$ is a partial metric aggregation function we see that the mapping $P_\phi : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+$, defined by

$$P_\Phi(x,y) = \Phi\bigl(p_{\max}(x_1,y_1),\dots,p_{\max}(x_n,y_n)\bigr)$$

for all $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n) \in \mathbb{R}_+^n$, is a partial metric.

Consider $x,y \in \mathbb{R}_+^n$ with $x \preceq y$. Then

$$\begin{aligned}\Phi(x) &= \Phi\bigl(p_{\max}(x_1,x_1),\dots,p_{\max}(x_n,x_n)\bigr) = P_\Phi(x,x)\\ &\leq P_\Phi(x,y) = \Phi\bigl(p_{\max}(x_1,y_1),\dots,p_{\max}(x_n,y_n)\bigr) = \Phi(y).\end{aligned}$$ □

The following lemmas are technical results that prove to be extremely useful in order to provide the announced characterization.

**Lemma 5** *For every $a,b,c,d \in \mathbb{R}^+$ such that $a \leq c + d - b$ with $b \leq c$ and $b \leq d$, there exist $x,y,z \in \mathbb{R}_+^2$ with $p_2(x,y) = c + d - b$, $p_2(x,z) = c$, $p_2(z,y) = d$, and $p_2(z,z) = b$, where $p_2 : \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}_+$ is the partial metric defined by $p_2(x,y) = \max\{x_1,y_1\} + \max\{x_2,y_2\}$ for all $x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}_+^2$.*

*Proof* It is not hard to see that $p_2$ is a partial metric on $\mathbb{R}_+^2$. Moreover, a straightforward computation shows that the following points of $\mathbb{R}_+^2$ hold the required conditions:

$$x = \left(c - \frac{b}{2}, \frac{b}{2}\right), \qquad y = \left(\frac{b}{2}, d - \frac{b}{2}\right), \qquad z = \left(\frac{b}{2}, \frac{b}{2}\right).$$ □

**Lemma 6** *For every $a,b,c \in \mathbb{R}_+$ such that $a \geq b$ and $a \geq c$, there exist $x,y \in \mathcal{I}(\mathbb{R})$ with $p_{\mathcal{I}(\mathbb{R})}(x,y) = a$, $p_{\mathcal{I}(\mathbb{R})}(x,x) = b$, and $p_{\mathcal{I}(\mathbb{R})}(y,y) = c$, where $p_{\mathcal{I}(\mathbb{R})} : \mathcal{I}(\mathbb{R}) \times \mathcal{I}(\mathbb{R}) \to \mathbb{R}_+$ is the partial metric defined by $p_{\mathcal{I}(\mathbb{R})}([x_1,x_2],[y_1,y_2]) = \max\{x_2,y_2\} - \min\{x_1,y_1\}$ for all $[x_1,x_2],[y_1,y_2] \in \mathcal{I}(\mathbb{R})$.*

*Proof* It is not hard to see that $p_{\mathcal{I}(\mathbb{R})}$ is a partial metric on $\mathcal{I}(\mathbb{R})$. Moreover, an easy computation shows that the following elements of $\mathcal{I}(\mathbb{R})$ satisfy the required conditions:

$$x = [-b,0], \qquad y = [-a,-a+c].$$ □

The next result describes those functions that able one to merge a collection of partial metrics into a single one.

**Theorem 7** *Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$. Then $\Phi$ is a partial metric aggregation function if and only if it satisfies the following two properties for all $x,y,z,w \in \mathbb{R}_+^n$:*

(1) *$\Phi(x) + \Phi(y) \leq \Phi(z) + \Phi(w)$ whenever $x + y \preceq z + w$, $y \preceq z$, and $y \preceq w$.*
(2) *If $y \preceq x$, $z \preceq x$, and $\Phi(x) = \Phi(y) = \Phi(z)$, then $x = y = z$.*

*Proof* Assume that $\Phi$ is a partial metric aggregation function. Next we prove assertions (1) and (2). Let $x,y,z,w \in \mathbb{R}_+^n$ such that $x + y \preceq z + w$, $y \preceq z$, and $y \preceq w$. Lemma 5 guarantees the existence of $\bar{x}_i, \bar{y}_i, \bar{z}_i \in \mathbb{R}^2_+$ such that $p_2(\bar{x}_i, \bar{y}_i) = z_i + w_i - y_i$, $p_2(\bar{x}_i, \bar{z}_i) = z_i$, $p_2(\bar{z}_i, \bar{y}_i) = w_i$, and $p_2(\bar{z}_i, \bar{z}_i) = y_i$ for all $i = 1, \ldots, n$. Set $X = \prod_{i=1}^n \mathbb{R}^2_+$. Then $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n), \bar{y} = (\bar{y}_1, \ldots, \bar{y}_n), \bar{z} = (\bar{z}_1, \ldots, \bar{z}_n) \in X$. Consider the partial metric $P_\Phi$ on $X$ given by

$$P_\Phi(\bar{x}, \bar{y}) = \Phi\bigl(p_2(\bar{x}_1, \bar{y}_1), \ldots, p_2(\bar{x}_n, \bar{y}_n)\bigr)$$

for all $\bar{x}, \bar{y} \in X$. Clearly we have

$$\Phi(z + w - y) = P_\Phi(\bar{x}, \bar{y}) \leq P_\Phi(\bar{x}, \bar{z}) + P_\Phi(\bar{z}, \bar{y}) - P_\Phi(\bar{z}, \bar{z}) = \Phi(z) + \Phi(w) - \Phi(y).$$

Moreover, $\Phi(x) \leq \Phi(z + w - y) \leq \Phi(z) + \Phi(w) - \Phi(y)$, since $x \preceq z + w - y$, and, by Proposition 4, the partial metric aggregation function is monotone. It follows that

$$\Phi(x) + \Phi(y) \leq \Phi(z) + \Phi(w)$$

whenever $x + y \preceq z + w$, $y \preceq z$, and $y \preceq w$. So we have proved that assertion (1) holds.

Next, let $x, y, z \in \mathbb{R}^n_+$ with $y \preceq x$, $z \preceq x$, and $\Phi(x) = \Phi(y) = \Phi(z)$. By Lemma 6 there exist $\bar{x}_i, \bar{y}_i \in \mathcal{I}(\mathbb{R})$ such that $p_{\mathcal{I}(\mathbb{R})}(\bar{x}_i, \bar{y}_i) = x_i$, $p_{\mathcal{I}(\mathbb{R})}(\bar{x}_i, \bar{x}_i) = y_i$, and $p_{\mathcal{I}(\mathbb{R})}(\bar{y}_i, \bar{y}_i) = z_i$ for all $i = 1, \ldots, n$. Set $X = \prod_{i=1}^n \mathcal{I}(\mathbb{R})$. Then $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n), \bar{y} = (\bar{y}_1, \ldots, \bar{y}_n), \bar{z} = (\bar{z}_1, \ldots, \bar{z}_n) \in X$. Consider the partial metric $P_\Phi$ on $X$ induced by

$$P_\Phi(\bar{x}, \bar{y}) = \Phi\bigl(p_2(\bar{x}_1, \bar{y}_1), \ldots, p_2(\bar{x}_n, \bar{y}_n)\bigr)$$

for all $\bar{x}, \bar{y} \in X$. Clearly we have $P_\Phi(\bar{x}, \bar{y}) = P_\phi(\bar{x}, \bar{x}) = P_\Phi(\bar{y}, \bar{y})$, since $\Phi(x) = \Phi(y) = \Phi(z)$. Consequently, $\bar{x} = \bar{y}$ and, thus, we conclude that $x = y = z$ because $\bar{x} = [-y, 0]$ and $\bar{y} = [-x, -x + z]$. So we have proved that assertion (2) holds.

Conversely we suppose that assertions (1) and (2) hold. In order to show that $\Phi$ is a partial aggregation function let $\{(X_i, p_i)\}_{i=1}^n$ be an arbitrary family of partial metric spaces. Consider $x, y \in X = \prod_{i \in I} X_i$ such that $P_\Phi(x, y) = P_\Phi(x, x) = P_\Phi(y, y)$. Then assertion (2) guarantees that $p_i(x_i, y_i) = p_i(x_i, x_i) = p_i(y_i, y_i)$ for all $i = 1, \ldots, n$, since $p_i(x_i, x_i) \leq p_i(x_i, y_i)$ and $p_i(y_i, y_i) \leq p_i(x_i, y_i)$ for all $i = 1, \ldots, n$ and $\Phi(p_1(x_1, y_1), \ldots, p_n(x_n, y_n)) = \Phi(p_1(x_1, x_1), \ldots, p_n(x_n, x_n)) = \Phi(p_1(y_1, y_1), \ldots, p_n(y_n, y_n))$. It follows that $x_i = y_i$ for all $i = 1, \ldots, n$ and, hence, that $x = y$.

Now, consider $x, y \in X$. Taking $y = w = \bar{0}$ in assertion (1) we see that $\Phi$ is monotone. Since $p_i(x_i, x_i) \leq p_i(x_i, y_i)$ for all $i = 1, \ldots, n$ we obtain

$$P_\Phi(x, x) = \Phi\bigl(p_1(x_1, x_1), \ldots, p_n(x_n, x_n)\bigr) \leq \Phi\bigl(p_1(x_1, y_1), \ldots, p_n(x_n, y_n)\bigr) = P_\Phi(x, y),$$

where $P_\Phi$ is the partial metric induced by aggregation of the family of partial metric spaces $\{(X_i, p_i)\}_{i=1}^n$ through $\Phi$.

Moreover, we have

$$P_\Phi(x, y) = \Phi\bigl(p_1(x_1, y_1), \ldots, p_n(x_n, y_n)\bigr) = \Phi\bigl(p_1(y_1, x_1), \ldots, p_n(y_n, x_n)\bigr) = P_\Phi(y, x)$$

for all $x, y \in X$.

Finally, consider $x, y, z \in X$. Assertion (1) provides

$$\begin{aligned} P_\Phi(x,y) &= \Phi\big(p_1(x_1,y_1),\ldots,p_n(x_n,y_n)\big) \\ &\leq \Phi\big(p_1(x_1,z_1),\ldots,p_n(x_n,z_n)\big) \\ &\quad + \Phi\big(p_1(z_1,y_1),\ldots,p_n(z_n,y_n)\big) \\ &\quad - \Phi\big(p_1(z_1,z_1),\ldots,p_n(z_n,z_n)\big) \\ &= P_\Phi(x,z) + P_\Phi(z,y) - P_\Phi(z,z), \end{aligned}$$

since

$$p_i(x_i,y_i) \leq p_i(x_i,z_i) + p_i(z_i,y_i) - p_i(z_i,z_i),$$
$$p_i(z_i,z_i) \leq p_i(x_i,z_i) \quad \text{and} \quad p_i(z_i,z_i) \leq p_i(z_i,y_i)$$

for all $i = 1,\ldots,n$. □

Notice that from the preceding result we deduce that, in general, if $\Phi$ is a partial metric aggregation then $\Phi \notin \mathcal{O}$ (see also Proposition 9). An instance of a partial metric aggregation function which does not belong to $\mathcal{O}$ will be provided in Example 13 later on. Observe that the preceding fact shows a distinguishing feature of partial metric aggregation functions with respect to the metric ones.

## 3 Fixed point theory in partial metric spaces and aggregation functions

In this section we present the promised fixed point theorem for partial metric spaces from an aggregation perspective. In order to state a fixed point theorem in the spirit of Matthews (Theorem 3) in the aggregation framework one needs to have two main elements, namely, appropriate notions of contraction and completeness. In order to introduce both type of notions in our new context, let us recall that if $\{X_i\}_{i=1}^n$ is a family of nonempty sets, $X = \prod_{i=1}^n X_i$ and $F$ is a mapping from $X$ into $X$, then the coordinate functions of $F$ are the functions $F_i : X \to X_i$, $i = 1,\ldots,n$, such that

$$F(x) = \big(F_1(x),\ldots,F_n(x)\big)$$

for all $x \in X$.

Next we introduce the new kind of contraction concept in those partial metric spaces whose partial metrics have been obtained via the aggregation of a family of partial metric spaces.

**Definition 8** Let $\{(X_i,p_i)\}_{i=1}^n$ be a family of arbitrary partial metric spaces, $X = \prod_{i=1}^n X_i$ and $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ a partial metric aggregation function. A mapping $F : X \to X$ is said to be a projective $\Phi$-contraction from $(X, P_\Phi)$ into itself, provided there exist $n$ constants $c_1,\ldots,c_n \in [0,1[$ such that

$$p_i\big(F_i(x),F_i(y)\big) \leq c_i\Phi\big(p_1(x_1,y_1),\ldots,p_n(x_n,y_n)\big)$$

for all $x, y \in X$ and for all $i = 1,\ldots,n$, where $P_\Phi$ is the partial metric induced by aggregation of the family of partial metric spaces $\{(X_i,p_i)\}_{i=1}^n$ through $\Phi$.

Notice that if $n = 1$ and $\Phi$ is the identity function in Definition 8, then one obtains as a particular case of the projective $\Phi$-contraction notion the classical one, due to Matthews (see inequality (1) in Section 1).

The result below yields a property of partial metric aggregation functions that will be crucial in order to guarantee the completeness of the partial metric obtained via aggregation later on.

**Proposition 9** *Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ be a partial metric aggregation function. If there exists $x \in \mathbb{R}_+^n$ such that $\Phi(x) = 0$, then $x = \bar{0}$.*

*Proof* Consider the partial metric $P_\phi : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+$ introduced in the proof of Proposition 4. Assume that $\Phi(x) = 0$. Then, by Proposition 4, we see that $\Phi$ is monotone and, thus, that $\Phi(\frac{x}{2}) \leq \Phi(x)$. Therefore $\Phi(\frac{x}{2}) = 0$, since $\Phi(x) = 0$. It follows that

$$P_\Phi\left(\frac{x}{2}, x\right) = \Phi\left(p_{\max}\left(\frac{x_1}{2}, x_1\right), \ldots, p_{\max}\left(\frac{x_n}{2}, x_n\right)\right) = \Phi(x) = 0,$$

$$P_\Phi(x, x) = \Phi\big(p_{\max}(x_1, x_1), \ldots, p_{\max}(x_n, x_n)\big) = \Phi(x) = 0,$$

$$P_\Phi\left(\frac{x}{2}, \frac{x}{2}\right) = \Phi\left(p_{\max}\left(\frac{x_1}{2}, \frac{x_1}{2}\right), \ldots, p_{\max}\left(\frac{x_n}{2}, \frac{x_n}{2}\right)\right) = \Phi\left(\frac{x}{2}\right) = 0.$$

Hence we obtain $\frac{x}{2} = x$; and therefore, $x = \bar{0}$. □

Note that, similar to Proposition 4 and Theorem 7, the preceding result gives a property of partial aggregation functions that permits one to discriminate between metric and partial aggregation functions (compare Theorem 2).

The next result affords the second requirement in order to state the fixed point theorem. Thus it guarantees the completeness of the partial metric obtained by means of aggregation.

From now on, we will set $1_i = (0, \ldots, 0, \overbrace{1}^{i}, 0, \ldots, 0)$ for all $i = 1, \ldots, n$.

**Lemma 10** *Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ be an homogeneous partial metric aggregation function such that $\Phi(1, \ldots, 1) = 1 = \Phi(1_i)$ for all $i = 1, \ldots, n$. Let $\{(X_i, p_i)\}_{i=1}^n$ be a family of arbitrary partial metric spaces and $X = \prod_{i=1}^n X_i$. Assume that, for each $i = 1, \ldots, n$, the partial metric space $(X_i, p_i)$ is complete. Then the partial metric space $(X, P_\Phi)$ is complete, where $P_\Phi$ is the partial metric induced by aggregation of the family of partial metric spaces $\{(X_i, p_i)\}_{i=1}^n$ through $\Phi$.*

*Proof* Let $(x^k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $(X, P_\Phi)$. Then we see that there exists $r \in \mathbb{R}^+$ such that $\lim_{k,j\to\infty} P_\Phi(x^k, x^j) = r$. Then, given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $P_\Phi(x^k, x^j) < \epsilon + r$ for all $k, j \geq k_0$. It follows that

$$\Phi\big(p_1\big(x_1^k, x_1^j\big), \ldots, p_n\big(x_n^k, x_n^j\big)\big) < \epsilon + r$$

for all $k, j \geq k_0$. By Proposition 4, $\Phi$ is monotone, and the monotonicity of $\Phi$ shows that

$$\Phi\big(p_i\big(x_i^k, x_i^j\big) \cdot 1_i\big) \leq \Phi\big(p_1\big(x_1^k, x_1^j\big), \ldots, p_n\big(x_n^k, x_n^j\big)\big) < \epsilon + r$$

for all $i = 1, \ldots, n$. The homogeneity of $\Phi$ yields $p_i(x_i^k, x_i^j) = p_i(x_i^k, x_i^j)\Phi(1_i) = \Phi(p_i(x_i^k, x_i^j) \cdot 1_i)$ for all $i = 1, \ldots, n$. It follows that $p_i(x_i^k, x_i^j) < \epsilon + r$ for all $k, j \geq k_0$ and for all $i = 1, \ldots, n$. Thus we deduce that there exist $x_i \in X_i$ such that $\lim_{k\to\infty} x_i^k = x_i$ in $(X_i, p_i)$ and $p_i(x_i, x_i) = r = \lim_{k,j\to\infty} p_i(x_i^k, x_i^j)$ for all $i = 1, \ldots, n$, since $(X_i, p_i)$ is a complete partial metric space for all $i = 1, \ldots, n$. Then, given $\epsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that $p_i(x_i, x_i^j) - p(x_i, x_i) < \frac{\epsilon}{2}$ for all $j \geq j_0$ for all $i = 1, \ldots, n$.

Next we prove that $\lim_{k\to\infty} x^k = x$ in $(X, P_\Phi)$ with $x = (x_1, \ldots, x_n)$. Indeed, by assertion (1) in the statement of Theorem 7, we have

$$\begin{aligned} P_\Phi\bigl(x, x^j\bigr) - P_\Phi(x, x) &= \Phi\bigl(p_1\bigl(x_1, x_1^j\bigr), \ldots, p_n\bigl(x_n, x_n^j\bigr)\bigr) - \Phi\bigl(p_1(x_1, x_1), \ldots, p_n(x_n, x_n)\bigr) \\ &\leq \Phi\left(\frac{\epsilon}{2}, \ldots, \frac{\epsilon}{2}\right). \end{aligned}$$

Since $\Phi$ is homogeneous we obtain $\Phi(\frac{\epsilon}{2}, \ldots, \frac{\epsilon}{2}) = \frac{\epsilon}{2}\Phi(1, \ldots, 1) < \epsilon$. So

$$P_\Phi\bigl(x, x^j\bigr) - P_\Phi(x, x) < \epsilon$$

for all $j \geq j_0$. Hence we deduce that $P_\Phi(x, x^j) - P_\Phi(x, x) < \epsilon$ eventually and, thus, that $\lim_{j\to\infty} P_\Phi(x, x^j) = P_\Phi(x, x)$. Moreover, $P_\Phi(x, x) = \Phi(p_1(x_1, x_1), \ldots, p_n(x_n, x_n)) = \Phi(r, \ldots, r) = r\Phi(1, \ldots, 1) = r$. Consequently, we see that the partial metric space $(X, P_\Phi)$ is complete. □

In the next result we prove that every projective $\Phi$-contraction is a contraction from the partial metric space obtained through aggregation into itself.

**Theorem 11** *Let $\{(X_i, p_i)\}_{i=1}^n$ be a family of arbitrary partial metric spaces and $X = \prod_{i=1}^n X_i$. If $\Phi$ is an homogeneous partial metric aggregation function such that $\Phi(1, \ldots, 1) \leq 1$ and $F$ is a $\Phi$-projective contraction, then $F$ is a contraction from the partial metric space $(X, P_\Phi)$ into itself, where $P_\Phi$ is the partial metric induced by aggregation of the family of partial metric spaces $\{(X_i, p_i)\}_{i=1}^n$ through $\Phi$.*

*Proof* Proposition 4 guarantees that $\Phi$ is a monotone mapping. Let $x, y \in X$. Then the monotonicity of $\Phi$ and the fact that $F$ is a projective $\Phi$-contraction yields

$$\begin{aligned} &P_\Phi\bigl(F(x), F(y)\bigr) \\ &\quad = \Phi\bigl(p_1\bigl(F_1(x), F_1(y)\bigr), \ldots, p_n\bigl(F_n(x), F_n(y)\bigr)\bigr) \\ &\quad \leq \Phi\bigl(c_1\Phi\bigl(p_1(x_1, y_1), \ldots, p_n(x_n, y_n)\bigr), \ldots, c_n\Phi\bigl(p_1(x_1, y_1), \ldots, p_n(x_n, y_n)\bigr)\bigr) \\ &\quad \leq \Phi\bigl(c\Phi\bigl(p_1(x_1, y_1), \ldots, p_n(x_n, y_n)\bigr), \ldots, c\Phi\bigl(p_1(x_1, y_1), \ldots, p_n(x_n, y_n)\bigr)\bigr), \end{aligned}$$

where $c = \max\{c_1, \ldots, c_n\}$. From the fact that $\Phi$ is homogeneous we deduce that

$$\begin{aligned} &\Phi\bigl(c\Phi\bigl(p_1(x_1, y_1), \ldots, p_n(x_n, y_n)\bigr), \ldots, c\Phi\bigl(p_1(x_1, y_1), \ldots, p_n(x_n, y_n)\bigr)\bigr) \\ &\quad = c\Phi(1, \ldots, 1)\Phi\bigl(p_1(x_1, y_1), \ldots, p_n(x_n, y_n)\bigr) \end{aligned}$$

and, hence,

$$P_\Phi\bigl(F(x), F(y)\bigr) \leq c\Phi(1, \ldots, 1)P_\Phi(x, y) \leq cP_\Phi(x, y).$$

Therefore, the mapping $F : X \to X$ is a contraction from the partial metric space $(X, P_\Phi)$ into itself. □

The existence and uniqueness of fixed point for Φ-projective contractions are warranted by the result below.

**Corollary 12** *Let* $\{(X_i, p_i)\}_{i=1}^n$ *be a family of arbitrary complete partial metric spaces and* $X = \prod_{i=1}^n X_i$. *If* Φ *is an homogeneous partial metric aggregation function such that* $\Phi(1, \ldots, 1) = 1 = \Phi(1_i)$ *for all* $i = 1, \ldots, n$ *and* $F$ *is a* Φ*-projective contraction, then* $F$ *has a unique fixed point* $x_0$. *Moreover,* $P_\Phi(x_0, x_0) = 0$ *and for each* $x \in X$, $\lim_{n\to+\infty} F^n(x) = x_0$ *in* $(X, P_\Phi)$, *where* $P_\Phi$ *is the partial metric induced by aggregation of the family of partial metric spaces* $\{(X_i, p_i)\}_{i=1}^n$ *through* Φ.

*Proof* By Theorem 11 we see that $F$ is a contraction from the partial metric space $(X, P_\Phi)$ into itself. Lemma 10 guarantees that the partial metric space $(X, P_\Phi)$ is complete. By Theorem 3 we deduce that $F$ has a unique fixed point $x_0 \in X$ such that, for each $x \in X$, $\lim_{n\to\infty} F^n(x) = x_0$ in $(X, P_\Phi)$ and $P_\Phi(x_0, x_0) = 0$. □

Observe that whenever $n = 1$ and Φ is the identity function in the statements of Theorem 11 and Corollary 12 we retrieve as a particular case Theorem 3. Thus the former results can be understood as an extension of the latter one.

The following example shows that the assumption 'Φ is homogeneous' cannot be omitted in the statement of Theorem 11 in order to guarantee that a projective Φ-contraction is also a contraction from $(X, P_\Phi)$ into itself.

**Example 13** Let $([0,1], p_{\max})$ be the complete partial metric space such that $p_{\max}$ denotes the restriction of the partial metric introduced in Proposition 4 to $[0,1]$. Consider the family of complete partial metric spaces $\{([0,1], p_i)\}_{i=1,2}$ such that $p_1 = p_2 = p_{\max}$. Define the function $\Phi_2 : \mathbb{R}_+^2 \to \mathbb{R}_+$ by $\Phi_2(x) = \frac{x_1+x_2}{4} + \frac{1}{2}$ for all $x \in \mathbb{R}_+^2$. It is not hard to see that for the function $\Phi_2$ assertions (1) and (2) hold in the statement of Theorem 7 and, thus, it is a partial metric aggregation function. Moreover, it is clear that $\Phi_2(1,1) \leq 1$. However, $\Phi_2$ is not homogeneous. Indeed, $\frac{3}{2} = \Phi_2(2,2) \neq 2 \cdot \Phi_2(1,1) = 2$.

Next, consider the mapping $F : [0,1]^2 \to [0,1]^2$ defined by $F(x) = (0,0)$ for all $x \in [0,1]^2$. It is clear that $F$ is a projective Φ-contraction. Nevertheless $F$ is not a contraction from $([0,1]^2, P_{\Phi_2})$ into itself, where $P_{\Phi_2}$ is the partial metric induced by aggregation of the family of partial metric spaces $\{([0,1], p_i)\}_{i=1,2}$ through $\Phi_2$. Indeed,

$$P_{\Phi_2}\big(F(0,0), F(0,0)\big) = P_{\Phi_2}\big((0,0),(0,0)\big) = \Phi_2(0,0) = \frac{1}{2}.$$

Therefore, there does not exist $c \in [0,1[$ such that

$$P_{\Phi_2}\big(F(0,0), F(0,0)\big) \leq c P_{\Phi_2}\big((0,0),(0,0)\big).$$

In the next example we show that the assumption '$\Phi(1, \ldots, 1) \leq 1$' cannot also be omitted in the statement of Theorem 11 in order to guarantee that a projective Φ-contraction is also a contraction from $(X, P_\Phi)$ into itself.

**Example 14** Let $([0,1],p_i)_{i=1,2}$ be the family of complete partial metric spaces such that $p_1 = p_2 = p_{\max}$. Define the function $\Phi_+ : \mathbb{R}^2_+ \to \mathbb{R}_+$ by $\Phi_+(x) = x_1 + x_2$ for all $x \in \mathbb{R}^2_+$. It is obvious that the function $\Phi_+$ is a homogeneous partial metric aggregation function. Nevertheless $\Phi_+(1,1) = 2$. Next consider the mapping $F : [0,1]^2 \to [0,1]^2$ defined by $F(x) = (\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2})$ for all $x \in [0,1]^2$. Then we have

$$\begin{aligned} p_{\max}\bigl(F_i(x),F_i(y)\bigr) &= p_{\max}\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right) \\ &= \frac{1}{2}p_{\max}(x_1+x_2,y_1+y_2) \\ &\leq \frac{1}{2}\max\{x_1,y_1\} + \frac{1}{2}\max\{x_2,y_2\} \\ &= \frac{1}{2}\Phi_+\bigl(p_{\max}(x_1,y_1),p_{\max}(x_2,y_2)\bigr) \end{aligned}$$

for all $x,y \in [0,1]^2$ and for $i = 1,2$. So, $F$ is a projective $\Phi_+$-contraction. However, $F$ is not a contraction from the quasi-metric space $([0,1]^2,P_{\Phi_+})$ into itself, where $P_{\Phi_+}$ is the partial metric induced by aggregation of the family of partial metric spaces $\{([0,1],p_i)\}_{i=1,2}$ through $\Phi_+$. Indeed, take $x,y \in [0,1]^2$ given by $x = (0,0)$ and $y = (1,1)$. Then there does not exist $c \in [0,1[$ such that

$$P_{\Phi_+}\bigl(F(0,0),F(1,1)\bigr) \leq cP_{\Phi_+}\bigl((0,0),(1,1)\bigr),$$

since $P_{\Phi_+}(F(0,0),F(1,1)) = P_{\Phi_+}((0,0),(1,1)) = 2$.

Now, a natural question is whether any contraction from the partial metric space $(X,P_\Phi)$ into itself is a projective $\Phi$-contraction provided the partial metric aggregation function $\Phi$ satisfies $\Phi(1,\dots,1) \leq 1$ and is homogeneous. In the next example we show that the answer to such an inquiry is negative and, thus, Theorem 11 is not a trivial consequence of Theorem 3.

**Example 15** Let $([0,1],p_i)_{i=1,2}$ be the family of complete partial metric spaces such that $p_1 = p_2 = p_{\max}$. Define the function $\Phi_{\frac{1}{2}} : \mathbb{R}^2_+ \to \mathbb{R}_+$ by $\Phi_{\frac{1}{2}}(x) = \frac{x_1+x_2}{2}$ for all $x \in \mathbb{R}^2_+$. Clearly $\Phi_{\frac{1}{2}}$ is an homogeneous partial metric aggregation function for which $\Phi_{\frac{1}{2}}(1,1) \leq 1$. Consider the mapping $F : [0,1]^2 \to [0,1]^2$ defined by $F(x) = (\frac{x_1+x_2}{2},0)$ for all $x \in [0,1]^2$. Then we have

$$\begin{aligned} P_{\Phi_{\frac{1}{2}}}\bigl(F(x),F(y)\bigr) &= \max\left\{\frac{x_1+x_2}{4},\frac{y_1+y_2}{4}\right\} \\ &\leq \frac{1}{2}\max\left\{\frac{x_1}{2},\frac{y_1}{2}\right\} + \frac{1}{2}\max\left\{\frac{x_2}{2},\frac{y_2}{2}\right\} \\ &= \frac{1}{2}P_{\Phi_{\frac{1}{2}}}(x,y) \end{aligned}$$

for all $x,y \in [0,1]^2$, where $P_{\Phi_{\frac{1}{2}}}$ is the partial metric induced by aggregation of the family of partial metric spaces $\{([0,1],p_i)\}_{i=1,2}$ through $\Phi_{\frac{1}{2}}$.

It follows that $F$ is a contraction from the partial metric space $([0,1]^2,P_{\Phi_{\frac{1}{2}}})$ into itself. Next we show that $F$ is not a projective $\Phi_{\frac{1}{2}}$-contraction. To this end, consider $x,y \in [0,1]^2$

with $x = (0,0)$ and $y = (1,1)$. Thus $p_{\max}(F_1(x),F_1(y)) = p_{\max}(0,1) = 1$ and $p_{\max}(x_1,y_1) = p_{\max}(x_2,y_2) = p_{\max}(1,0) = 1$. Consequently there does not exist $c \in [0,1[$ such that

$$p_{\max}\bigl(F_1(x),F_1(y)\bigr) \le c\Phi_{\frac{1}{2}}\bigl(p_{\max}(x_1,y_1),p_{\max}(x_2,y_2)\bigr),$$

since $\Phi_{\frac{1}{2}}(p_{\max}(x_1,y_1),p_{\max}(x_2,y_2)) = \Phi_{\frac{1}{2}}(1,1) = 1$.

According to [18], given a partial metric space $(X,p)$ and a contraction $f : X \to X$ with fixed point $x^* \in X$, we will say that:

(i) The fixed point problem for such a mapping is well-posed with respect to $p$ whenever the following property is satisfied:
If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $X$ such that $\lim_{n\to\infty} p(x_n,f(x_n)) = 0$, then $\{x_n\}_{n\in\mathbb{N}}$ converges to $x^*$ with respect to $\tau(p)$.

(ii) The contraction $f$ has the limit shadowing property with respect to $p$ provided that the following property is satisfied:
If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $X$ such that $\lim_{n\to\infty} p(x_{n+1},f(x_n)) = 0$, then $\lim_{n\to\infty} p(x_n,f^n(x)) = 0$ for all $x \in X$.

In Theorem 3.1 of [18], it was proved that the fixed point problem for contractions in the partial metric context is well-posed and, in addition, that every contraction has the limit shadowing property provided that the partial metric space is complete. Taking into account the preceding facts we discuss the same properties in the case of projective $\Phi$-contractions in the result below.

**Corollary 16** *Let $\{(X_i,p_i)\}_{i=1}^n$ be a family of arbitrary complete partial metric spaces and $X = \prod_{i=1}^n X_i$. Assume that $\Phi$ is an homogeneous partial metric aggregation function such that $\Phi(1,\dots,1) = 1 = \Phi(1_i)$ for all $i = 1,\dots,n$ and that $F$ is a projective $\Phi$-contraction, then*

(1) *the fixed point problem for $F$ is well-posed with respect to $P_\Phi$,*

(2) *$F$ has the shadowing property with respect to $P_\Phi$,*

*where $P_\Phi$ is the partial metric induced by aggregation of the family of partial metric spaces $\{(X_i,p_i)\}_{i=1}^n$ through $\Phi$.*

*Proof* By Theorem 11 we see that a projective $\Phi$-contraction is a contraction from $(X,P_\Phi)$ into itself. Moreover, by Lemma 10, we see that the partial metric space $(X,P_\Phi)$ is complete. Thus we deduce, by Theorem 3.1 in [18], that the fixed point problem for $F$ is well-posed with respect to $P_\Phi$ and, besides, $F$ has the shadowing property with respect to $P_\Phi$. □

In Theorem 3.1 of [18], it was also proved that if $f$ is a contraction from a complete partial metric space $(X,p)$ into itself, $x^*$ is the fixed point of $f$ and $g$ is a mapping from $X$ into itself with fixed point $y^*$, then the following property holds:

If there exists $\eta > 0$ such that $p(f(x),g(x)) < \eta$ for all $x \in X$ and $c$ is the contractive constant of $f$, then $p(x^*,y^*) \le \frac{\eta}{1-c}$.

Hereafter we will say that a contraction $f$ from a partial metric space $(X,p)$ into itself with a unique fixed point $x^*$ and contractive constant $c$ has the bounding fixed point property with respect to $p$ whenever, given a mapping $g$ from $X$ into itself with fixed point $y^*$, the existence of $\eta > 0$ such that $p(f(x),g(x)) < \eta$ for all $x \in X$ implies that $p(x^*,y^*) \le \frac{\eta}{1-c}$.

The next result provides conditions under which projective $\Phi$-contractions enjoy the bounding fixed point property.

**Corollary 17** *Let $\{(X_i, p_i)\}_{i=1}^n$ be a family of arbitrary complete partial metric spaces and $X = \prod_{i=1}^n X_i$. Assume that $\Phi$ is an homogeneous partial metric aggregation function such that $\Phi(1, \ldots, 1) = 1 = \Phi(1_i)$ for all $i = 1, \ldots, n$ and that $F$ is a projective $\Phi$-contraction. Then $F$ has the bounding fixed point property with respect to $P_\Phi$, where $P_\Phi$ is the partial metric induced by aggregation of the family of partial metric spaces $\{(X_i, p_i)\}_{i=1}^n$ through $\Phi$.*

*Proof* The same arguments as those in the proof of Corollary 16 give the thesis. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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**References**
1. Grabisch, M, Marichal, JL, Mesiar, R, Pap, E: Aggregation Functions. Cambridge University Press, New York (2009)
2. Borsík, J, Doboš, J: On a product of metric spaces. Math. Slovaca **31**, 193-205 (1981)
3. Herburt, I, Moszyńska, M: On metric products. Colloq. Math. **62**, 121-133 (1995)
4. Mayor, G, Valero, O: Aggregation of asymmetric distances in computer science. Inf. Sci. **180**, 803-812 (2010)
5. Castiñeira, E, Pradera, A, Trillas, E: On the aggregation of some classes of fuzzy relations. In: Technologies for Constructing Intelligent Systems, pp. 125-147. Springer, Berlin (2002)
6. Castiñeira, E, Pradera, A, Trillas, E: On distances aggregation. In: Proc. of the Information Processing and Management of Uncertainty in Knowledge-Based Systems International Conference (IPMU 2000), pp. 693-700 (2000)
7. Pradera, A, Trillas, E: A note on pseudometrics aggregation. Int. J. Gen. Syst. **31**, 41-51 (2002)
8. Petruşel, A, Rus, IA, Şerban, MA: The role of equivalent metrics in fixed point theory. Topol. Methods Nonlinear Anal. **41**, 85-112 (2013)
9. Matthews, SG: Partial metric topology. Ann. N.Y. Acad. Sci. **728**, 183-197 (1994)
10. Matthews, SG: An extensional treatment of lazy data flow deadlock. Theor. Comput. Sci. **151**, 195-205 (1995)
11. Shahzad, N, Valero, O: On 0-complete partial metric spaces and quantitative fixed point techniques in denotational semantics. Abstr. Appl. Anal. **2013**, Article ID 985095 (2013)
12. Hitzler, P, Seda, AK: Generalized distance functions in the theory of computation. Comput. J. **53**, 443-464 (2010)
13. Hitzler, P, Seda, AK: Mathematical Aspects of Logic Programming Semantics. CRC Press, Boca Raton (2011)
14. Alghamdi, MA, Shahzad, N, Valero, O: Fixed point theorems in generalized metric spaces with applications to computer science. Fixed Point Theory Appl. **2013**, 118 (2013)
15. Romaguera, S, Schellekens, MP: Partial metric monoids and semivaluation spaces. Topol. Appl. **153**, 948-962 (2005)
16. Kirk, WA, Shahzad, N: Fixed Point Theory in Distance Spaces. Springer, Cham (2014)
17. Massanet, S, Valero, O: New results on metric aggregation. In: Proceedings of the 17th Spanish Conference on Fuzzy Technology and Fuzzy Logic (Estylf 2012), pp. 558-563 (2012)
18. Rus, IA: Fixed point theory in partial metric spaces. An. Univ. Vest. Timiş., Ser. Mat.-Inform. **XLVI**, 149-160 (2008)