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Nielsen type numbers and homotopy minimal periods for maps on solvmanifolds with Sol_1^4-geometry

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Abstract

For all maps f on the special solvmanifolds with Sol_1^4-geometry, we give explicit formulas for a complete computation of the Nielsen type numbers $\mathrm{NP}_n(f)$ and $\mathrm{N\Phi}_n(f)$. We also give a complete description of the sets of homotopy minimal periods of all such maps.

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1 Introduction

For a self-map $f : X \to X$, the Lefschetz number $L(f)$ and the Nielsen number $N(f)$ give information concerning the fixed points of f. It is known that the Nielsen number gives more precise information about the existence of fixed points than the Lefschetz number, but its computation is in general very difficult. For the periodic points, two Nielsen type numbers $\mathrm{NP}_n(f)$ and $\mathrm{N\Phi}_n(f)$ were introduced by Jiang [1], which are lower bounds for the number of periodic points of least period exactly n and the set of periodic points of period n, respectively.

It is obvious that these Nielsen numbers are much more powerful than the Lefschetz numbers in describing the periodic point sets of self-maps. Using fiber techniques on nilmanifolds and some solvmanifolds, Heath and Keppelmann [2] (see also [3]) succeeded in showing that the Nielsen numbers and the two Nielsen type numbers are related to each other under certain conditions. However, the computation of the Nielsen type numbers even on low dimensional infra-homogeneous spaces is a hard problem. See [4] for the Klein bottle and [5] for a three-dimensional flat Riemannian manifold.

One of the natural problems in dynamical systems is the study of the existence of periodic points of least period exactly n. Homotopically, a new concept, namely homotopy minimal periods,

$$\mathrm{HPer}(f) = \bigcap_{g\simeq f} \{ n \in \mathbb{N} \mid P_n(g) \neq \emptyset \},$$

where $P_n(g) = \mathrm{Fix}(g^n) - \bigcup_{k<n} \mathrm{Fix}(g^k)$, was introduced by Alsedà *et al.* [6]. Since the homotopy minimal period is preserved under a small perturbation of a self-map f on a manifold X, we can say that the set $\mathrm{HPer}(f)$ of homotopy minimal periods of f describes the rigid part of dynamics of f. A complete description of the set of homotopy minimal periods of all self-maps was obtained on the nilmanifolds with Nil^3-geometry [7, 8] and on the special solvmanifolds with Sol^3-geometry [9, 10].

There are four-dimensional geometries which were classified by Filipkiewicz [11], see also [12]. One of their model spaces is a simply connected four-dimensional unimodular solvable Lie group Sol_1^4. This group contains Nil^3 as a nil-radical and the quotient by its center is Sol^3. Recall that Nil^3 and Sol^3 are model spaces for three-dimensional geometries.

In this paper, we are concerned with the special solvmanifolds with Sol_1^4-geometry, *i.e.*, the closed manifolds $\Gamma\backslash \mathrm{Sol}_1^4$ which are quotient spaces of Sol_1^4 by its lattices Γ. For all continuous maps f on any special solvmanifolds with Sol_1^4-geometry, we will give a complete description of the Nielsen type numbers $\mathrm{NP}_n(f)$ and $\mathrm{N\Phi}_n(f)$, and the homotopy minimal periods $\mathrm{HPer}(f)$.

2 The Lie group Sol_1^4 and its Lie algebra

Consider the connected and simply connected four-dimensional matrix Lie group

$$\mathrm{Sol}_1^4 = \left\{ \begin{bmatrix} 1 & y & x \\ 0 & e^\theta & z \\ 0 & 0 & 1 \end{bmatrix} : x, y, z, \theta \in \mathbb{R} \right\}.$$

This Lie group is one of the four-dimensional geometries which were classified by Filipkiewicz [11], see also [12]. The Lie algebra of Sol_1^4 is

$$\mathfrak{sol}_1^4 = \left\{ \begin{bmatrix} 0 & b & a \\ 0 & \theta & c \\ 0 & 0 & 0 \end{bmatrix} : a, b, c, \theta \in \mathbb{R} \right\}.$$

Denote

$$\mathbf{e}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{e}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{e}_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Its nontrivial brackets are

$$[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1, \qquad [\mathbf{e}_2, \mathbf{e}_4] = \mathbf{e}_2, \qquad [\mathbf{e}_3, \mathbf{e}_4] = -\mathbf{e}_3,$$

and hence ${\mathfrak{sol}_1^4}^{(1)} := [\mathfrak{sol}_1^4, \mathfrak{sol}_1^4] = \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle$ and ${\mathfrak{sol}_1^4}^{(2)} = \mathcal{Z}(\mathfrak{sol}_1^4) = \langle \mathbf{e}_1 \rangle$.

Let $\varphi : \mathfrak{sol}_1^4 \to \mathfrak{sol}_1^4$ be a Lie algebra endomorphism. Since φ preserves ${\mathfrak{sol}_1^4}^{(1)}$ and ${\mathfrak{sol}_1^4}^{(2)}$, we have

$$\varphi(\mathbf{e}_1) = p_{11}\mathbf{e}_1,$$

$$\varphi(\mathbf{e}_2) = p_{12}\mathbf{e}_1 + p_{22}\mathbf{e}_2 + p_{32}\mathbf{e}_3,$$
$$\varphi(\mathbf{e}_3) = p_{13}\mathbf{e}_1 + p_{23}\mathbf{e}_2 + p_{33}\mathbf{e}_3,$$
$$\varphi(\mathbf{e}_4) = p_{14}\mathbf{e}_1 + p_{24}\mathbf{e}_2 + p_{34}\mathbf{e}_3 + p_{44}\mathbf{e}_4,$$

for some constants p_{ij}. Because φ preserves the Lie brackets, it follows that φ can be expressed as a matrix of one of the following three forms:

$$\begin{bmatrix} xy & wx & vy & u \\ 0 & x & 0 & v \\ 0 & 0 & y & w \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} xy & wx & vy & u \\ 0 & x & 0 & v \\ 0 & 0 & y & w \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & u \\ 0 & 0 & 0 & v \\ 0 & 0 & 0 & w \\ 0 & 0 & 0 & p \end{bmatrix}$$
$$(p \neq \pm 1).$$

In particular, we have

$$\operatorname{Aut}_0\left(\mathfrak{sol}_1^4\right) = \left\{ \begin{bmatrix} xy & wx & vy & u \\ 0 & x & 0 & v \\ 0 & 0 & y & w \\ 0 & 0 & 0 & 1 \end{bmatrix} \,\middle|\, x, y, u, v, w \in \mathbb{R}, xy \neq 0 \right\}.$$

Let

$$\tau = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The subgroup of $\operatorname{Aut}_0(\mathfrak{sol}_1^4)$ with $x, y = \pm 1$ and $u = v = w = 0$ is isomorphic to $(\mathbb{Z}_2)^2$, which is a maximal compact subgroup. Hence $\operatorname{Aut}(\mathfrak{sol}_1^4)$ has a subgroup generated by $(\mathbb{Z}_2)^2$ and τ as a maximal compact subgroup. This group is isomorphic to the dihedral group $D(4)$ of order 8.

A connected solvable Lie group S is called of type (R) (or completely solvable) if $\operatorname{ad}(X) : \mathfrak{S} \to \mathfrak{S}$ has only real eigenvalues for each $X \in \mathfrak{S}$. It is known that any Lie group S of type (R) is of type (E), *i.e.*, $\exp : \mathfrak{S} \to S$ is surjective. Remark that $\mathfrak{sol}_1^4$ is a 3-step unimodular and completely solvable (or of type (R)) Lie algebra. Hence the exponential map $\exp : \mathfrak{sol}_1^4 \to \mathrm{Sol}_1^4$ is a diffeomorphism with inverse log, and they are given explicitly as follows:

$$\exp : \begin{bmatrix} 0 & b & a \\ 0 & \theta & c \\ 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & \frac{e^\theta - 1}{\theta} b & a + \frac{e^\theta - 1 - \theta}{\theta^2} bc \\ 0 & \theta & \frac{e^\theta - 1}{\theta} c \\ 0 & 0 & 1 \end{bmatrix},$$
$$\log : \begin{bmatrix} 1 & y & x \\ 0 & e^\theta & z \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & \frac{\theta}{e^\theta - 1} y & x - \frac{e^\theta - 1 - \theta}{(e^\theta - 1)^2} yz \\ 0 & \theta & \frac{\theta}{e^\theta - 1} z \\ 0 & 0 & 0 \end{bmatrix}.$$

Because Sol_1^4 is simply connected, every Lie group endomorphism is understood as the composition $\exp \circ \varphi \circ \log$.

The group Sol_1^4 has the three-dimensional Heisenberg group Nil^3 ($\theta = 0$) as its nilradical. Indeed, the derived group of Sol_1^4 is Nil^3. On the other hand, Sol_1^4 has the center $\mathcal{Z}(\mathrm{Sol}_1^4) = \mathbb{R}$ ($y = z = \theta = 0$), and the quotient turns out to be isomorphic to a three-dimensional solvable Lie group Sol^3, see Remark 2.1. That is,

$$\begin{array}{ccccccccc} & & & & & & \mathrm{Sol}^3 & & \\ & & & & & & \uparrow\cong & & \\ 1 & \longrightarrow & \mathcal{Z}(\mathrm{Sol}_1^4) & \longrightarrow & \mathrm{Sol}_1^4 & \longrightarrow & \mathrm{Sol}_1^4/\mathcal{Z}(\mathrm{Sol}_1^4) & \longrightarrow & 1 \end{array}$$

where the isomorphism is given by

$$\begin{bmatrix} 1 & y & * \\ 0 & e^{\theta} & z \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \left(\begin{bmatrix} e^{-\theta} y \\ z \end{bmatrix}, -\theta \right).$$

Consequently, Sol_1^4 fits in the following commutative diagram between short exact sequences (*cf.* [13]):

$$\begin{array}{ccccccccc} & & & & 1 & & 1 & & \\ & & & & \uparrow & & \uparrow & & \\ & & & & \mathbb{R}^+ & \xrightarrow{\log} & \mathbb{R} & & \\ & & & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathrm{Sol}_1^4 & \longrightarrow & \mathrm{Sol}^3 & \longrightarrow & 1 \\ & & \uparrow = & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathrm{Nil}^3 & \longrightarrow & \mathbb{R}^2 & \longrightarrow & 1 \\ & & & & \uparrow & & \uparrow & & \\ & & & & 1 & & 1 & & \end{array} \qquad \text{(S)}$$

Remark 2.1 Recall that Sol^3 is one of the eight geometries in dimension 3. We denote by $\mathbb{R}^{1,1}$ the vector space $\mathbb{R}^2$ with the bilinear form

$$\mathbf{b}(\mathbf{x}, \mathbf{y}) = -x_1 y_1 + x_2 y_2.$$

Let $\mathbb{E}(1,1)$ denote the group of all isometries of $\mathbb{R}^{1,1}$. Then $\mathbb{E}(1,1) = \mathbb{R}^2 \rtimes \mathrm{O}(1,1)$, where $\mathrm{O}(1,1)$ denotes the orthogonal group of $\mathbf{b}$:

$$\mathrm{O}(1,1) = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \,\middle|\, \alpha \neq 0 \right\}.$$

Hence

$$\mathrm{S}\mathbb{E}(1,1) = \mathbb{R}^2 \rtimes \mathrm{SO}(1,1) = \mathbb{R}^2 \rtimes \left\{ \begin{bmatrix} e^{\theta} & 0 \\ 0 & e^{-\theta} \end{bmatrix} \,\middle|\, \theta \in \mathbb{R} \right\}.$$

This shows that $\mathrm{S}\mathbb{E}(1,1) \cong \mathbb{R}^2 \rtimes_\sigma \mathbb{R}$, where

$$\sigma : \mathbb{R} \to \mathrm{GL}(2,\mathbb{R}), \qquad \theta \mapsto \begin{bmatrix} e^\theta & 0 \\ 0 & e^{-\theta} \end{bmatrix}.$$

Recall that Sol^3 is the group $\mathbb{R}^2 \rtimes_\sigma \mathbb{R}$. We also remark that the map $\mathrm{Sol}_1^4 \to \mathrm{GL}(3,\mathbb{R})$,

$$\begin{bmatrix} 1 & y & * \\ 0 & e^\theta & z \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ e^{-\theta}y + z & \cosh(-\theta) & \sinh(-\theta) \\ e^{-\theta}y - z & \sinh(-\theta) & \cosh(-\theta) \end{bmatrix},$$

is a Lie group homomorphism with kernel $\mathcal{Z}(\mathrm{Sol}_1^4) = \mathbb{R}$. Consequently, this induces that Sol^3 can be embedded into $\mathrm{GL}(3,\mathbb{R})$ as

$$\left(\begin{bmatrix} y \\ z \end{bmatrix}, \theta\right) \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ y + z & \cosh\theta & \sinh\theta \\ y - z & \sinh\theta & \cosh\theta \end{bmatrix}.$$

As observed above, Sol^3 can be embedded naturally into the affine group $\mathrm{Aff}(\mathbb{R}^2)$ as

$$\left(\begin{bmatrix} y \\ z \end{bmatrix}, \theta\right) \longmapsto \begin{bmatrix} e^\theta & 0 & y \\ 0 & e^{-\theta} & z \\ 0 & 0 & 1 \end{bmatrix}.$$

3 The lattices of Sol_1^4

In this section we study lattices Γ of Sol_1^4 with [12] as our basic reference, and then we study continuous maps on the solvmanifold $\Gamma\backslash \mathrm{Sol}_1^4$ up to homotopy.

Theorem 3.1 *Every lattice* Γ *of* Sol_1^4 *can be generated by* $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ *with relations*

$$[\gamma_1, \gamma_2] = \gamma_3^k, \qquad [\gamma_1, \gamma_3] = [\gamma_2, \gamma_3] = 1,$$
$$\gamma_0\gamma_1\gamma_0^{-1} = \gamma_1^{n_{11}}\gamma_2^{n_{21}}\gamma_3^{p_1}, \qquad \gamma_0\gamma_2\gamma_0^{-1} = \gamma_1^{n_{12}}\gamma_2^{n_{22}}\gamma_3^{p_2}, \qquad \gamma_0\gamma_3\gamma_0^{-1} = \gamma_3$$

for some integers k, p_1, p_2 *with* $k \neq 0$ *and* $N = [n_{ij}] \in \mathrm{SL}(2,\mathbb{Z})$ *with trace* > 2.

Notation We denote such a lattice of Sol_1^4 by $\Gamma_{k,N,\mathbf{p}}$.

Proof Consider the derived series of Sol_1^4: $\mathrm{Sol}_1^4 \supset \mathrm{Nil}^3 \supset \mathcal{Z}(\mathrm{Nil}^3)$. Taking intersections with Γ, we obtain

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2,$$

where Γ_1 is a lattice of Nil^3. From the commutative diagram (S), we obtain a commutative diagram between lattices

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \Gamma_0/\Gamma_1 & \xrightarrow{=} & \Gamma_0/\Gamma_1 & & \\
 & & & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Gamma_2 & \longrightarrow & \Gamma_0 & \longrightarrow & \Gamma_0/\Gamma_2 & \longrightarrow & 1\\
 & & \uparrow{\scriptstyle =} & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Gamma_2 & \longrightarrow & \Gamma_1 & \longrightarrow & \Gamma_1/\Gamma_2 & \longrightarrow & 1\\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 1 & & 1 & &
\end{array} \tag{L}
$$

Remark that the bottom exact sequence comes from the short exact sequence $1 \to \mathbb{R} \to \mathrm{Nil}^3 \to \mathbb{R}^2 \to 1$. Then it is well known, for example, in [14] that such Γ_1 is generated by γ_1, γ_2, γ_3 satisfying the relations

$$[\gamma_1, \gamma_3] = [\gamma_2, \gamma_3] = 1, \qquad [\gamma_1, \gamma_2] = \gamma_3^k$$

for some nonzero integer k. In particular, γ_3 is a generator of $\Gamma_2 \cong \mathbb{Z}$ and $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ generate $\Gamma_1/\Gamma_2 \cong \mathbb{Z}^2$. Now, from the middle vertical, we can choose $\gamma_0 \in \Gamma_0$ so that $\{\gamma_0, \dots, \gamma_3\}$ generates Γ_0. We denote by $\tilde{\gamma}_0$ and $\tilde{\tilde{\gamma}}_0$ the images of γ_0 under the projections $\Gamma_0 \to \Gamma_0/\Gamma_2$ and $\Gamma_0 \to \Gamma_0/\Gamma_1$, respectively. Remark also that $\{\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_2\}$ is a generator set of Γ_0/Γ_2, which is a lattice of Sol^3. Because $\{\tilde{\gamma}_1, \tilde{\gamma}_2\}$ generates $\Gamma_1/\Gamma_2 \cong \mathbb{Z}^2$ and $\tilde{\tilde{\gamma}}_0$ generates $\Gamma_0/\Gamma_1 \cong \mathbb{Z}$, we must have

$$[\tilde{\gamma}_1, \tilde{\gamma}_2] = 1, \qquad \tilde{\gamma}_0 \tilde{\gamma}_j \tilde{\gamma}_0^{-1} = \tilde{\gamma}_1^{n_{1j}} \tilde{\gamma}_2^{n_{2j}}$$

for some integers n_{ij}. Let $N = [n_{ij}]$. Then it can be seen that $N \in \mathrm{SL}(2, \mathbb{Z})$ with trace > 2. For details about lattices of Sol^3, we refer to [10, 15]. On the other hand, the conjugation by γ_0 induces an automorphism on Γ_1. Because this automorphism must preserve the relation $[\gamma_1, \gamma_2] = \gamma_3^k$, it follows that $\gamma_0 \gamma_3 \gamma_0^{-1} = \gamma_3^{\det(N)} = \gamma_3$. Consequently, the theorem is proved. □

Now we will study an embedding of an abstract group $\Gamma_{k,N,\mathbf{p}}$ into Sol_1^4 as a lattice. Let N be a 2×2 hyperbolic integer matrix with trace > 2. Then N has two distinct irrational eigenvalues e^{θ_0} and $e^{-\theta_0}$ with corresponding eigenvectors (y_1, z_1) and (y_2, z_2). This means that $NQ = Q\Theta$, where Q is the matrix with columns $(y_1, z_1)^t$ and $(y_2, z_2)^t$ and Θ is the diagonal matrix with entries e^{θ_0} and $e^{-\theta_0}$. From the identity $NQ = Q\Theta$, we can now check that the assignment

$$\tilde{\gamma}_0 \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-\theta_0} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \tilde{\gamma}_j \mapsto \begin{bmatrix} 1 & y_j & 0 \\ 0 & 1 & z_j \\ 0 & 0 & 1 \end{bmatrix}$$

realizes Γ_0/Γ_2 as a lattice of $\mathrm{Sol}^3 = \mathrm{Sol}_1^4/\mathcal{Z}(\mathrm{Sol}_1^4)$. Thus, by lifting $\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_2 \in \Gamma_0/\Gamma_2$ in Γ_0, we have

$$\gamma_0 \mapsto \begin{bmatrix} 1 & 0 & x_0 \\ 0 & e^{-\theta_0} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \gamma_j \mapsto \begin{bmatrix} 1 & y_j & x_j \\ 0 & 1 & z_j \\ 0 & 0 & 1 \end{bmatrix}, \qquad \gamma_3 \mapsto \begin{bmatrix} 1 & 0 & x_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for some $x_0, x_1, x_2, x_3 \in \mathbb{R}$. The relation $[\gamma_1, \gamma_2] = \gamma_3^k$ yields $x_3 = (y_1 z_2 - y_2 z_1)/k$, and then the relation $\gamma_0 \gamma_j \gamma_0^{-1} = \gamma_1^{n_{1j}} \gamma_2^{n_{2j}} \gamma_3^{p_j}$ yields that $x_j = (y_1 z_2 - y_2 z_1) p_j / k$. We can choose x_0 arbitrarily. Therefore, this gives an embedding Γ into Sol_1^4 as a lattice.

In the theorem below, we study the homomorphisms on any lattice of Sol_1^4.

Theorem 3.2 *Let*

$$\Gamma_{k,N,\mathbf{p}} = \left\langle \gamma_0, \gamma_1, \gamma_2, \gamma_3 \;\middle|\; \begin{array}{c} [\gamma_1, \gamma_2] = \gamma_3^k, [\gamma_0, \gamma_3] = [\gamma_1, \gamma_3] = [\gamma_2, \gamma_3] = 1, \\ \gamma_0 \gamma_1 \gamma_0^{-1} = \gamma_1^{n_{11}} \gamma_2^{n_{21}} \gamma_3^{p_1}, \gamma_0 \gamma_2 \gamma_0^{-1} = \gamma_1^{n_{12}} \gamma_2^{n_{22}} \gamma_3^{p_2} \end{array} \right\rangle$$

be a lattice of Sol_1^4. *Then any homomorphism* ϕ *on* $\Gamma_{k,N,\mathbf{p}}$ *is either one of the following forms*:

Type (I) $\phi(\gamma_0) = \gamma_0 \gamma_1^{r_1} \gamma_2^{r_2} \gamma_3^{q_0}$, $\phi(\gamma_1) = \gamma_1^{\mu} \gamma_2^{\frac{n_{21}}{n_{12}}\nu} \gamma_3^{q_1}$, $\phi(\gamma_2) = \gamma_1^{\nu} \gamma_2^{\mu + \frac{n_{22}-n_{11}}{n_{12}}\nu} \gamma_3^{q_2}$,
$\phi(\gamma_3) = \gamma_3^{\mu(\mu + \frac{n_{22}-n_{11}}{n_{12}}\nu) - \frac{n_{21}}{n_{12}}\nu^2}$;

Type (II) $\phi(\gamma_0) = \gamma_0^{-1} \gamma_1^{r_1} \gamma_2^{r_2} \gamma_3^{q_0}$, $\phi(\gamma_1) = \gamma_1^{-\mu} \gamma_2^{\nu} \gamma_3^{q_1}$, $\phi(\gamma_2) = \gamma_1^{\frac{n_{11}-n_{22}}{n_{21}}\mu - \frac{n_{12}}{n_{21}}\nu} \gamma_2^{\mu} \gamma_3^{q_2}$,
$\phi(\gamma_3) = \gamma_3^{-\mu^2 - (\frac{n_{11}-n_{22}}{n_{21}}\mu - \frac{n_{12}}{n_{21}}\nu)\nu}$;

Type (III) $\phi(\gamma_0) = \gamma_0^{m} \gamma_1^{r_1} \gamma_2^{r_2} \gamma_3^{q_0}$, $\phi(\gamma_1) = \gamma_3^{q_1}$, $\phi(\gamma_2) = \gamma_3^{q_2}$, $\phi(\gamma_3) = 1$ *with* $m \neq \pm 1$.

Proof For simplicity, we write $\Gamma = \Gamma_{k,N,\mathbf{p}}$. Let $\phi : \Gamma \to \Gamma$ be any homomorphism. Consider the derived series of Sol_1^4: $\mathrm{Sol}_1^4 \supset \mathrm{Nil}^3 \supset \mathcal{Z}(\mathrm{Nil}^3)$, and its associated sequence $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2$ obtained by taking intersections with Γ. Note that $\Gamma_1 = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$ and $\Gamma_2 = \langle \gamma_3 \rangle$. Because Sol_1^4 is of type (R), the homomorphism ϕ on Γ extends uniquely to a Lie group homomorphism Φ on Sol_1^4, see, for example, [16], Corollary 8.3, or [17, 18]. Now, Φ preserves the derived series of Sol_1^4, and so ϕ preserves the associated sequence. In particular, $\phi|_{\Gamma_1}$ is a homomorphism on the lattice Γ_1 of Nil^3. Since $\Gamma_1 = \langle \gamma_1, \gamma_2, \gamma_2 \mid [\gamma_1, \gamma_2] = \gamma_3^k \rangle$, it follows that

$$\begin{aligned} \phi(\gamma_0) &= \gamma_0^{m} \gamma_1^{r_1} \gamma_2^{r_2} \gamma_3^{q_0}, \\ \phi(\gamma_1) &= \gamma_1^{m_{11}} \gamma_2^{m_{21}} \gamma_3^{q_1}, \\ \phi(\gamma_2) &= \gamma_1^{m_{12}} \gamma_2^{m_{22}} \gamma_3^{q_2}, \\ \phi(\gamma_3) &= \gamma_3^{m_{11}m_{22} - m_{12}m_{21}}, \end{aligned}$$

where $m_{ij}, r_i, q_j \in \mathbb{Z}$, see [8], Lemma 2.1, for $\phi|_{\Gamma_1}$. Further, ϕ induces a homomorphism $\tilde{\phi}$ on the lattice Γ_0/Γ_2 of Sol^3 so that

$$\tilde{\phi}(\tilde{\gamma}_0) = \tilde{\gamma}_0^{m} \tilde{\gamma}_1^{r_1} \tilde{\gamma}_2^{r_2}, \qquad \tilde{\phi}(\tilde{\gamma}_1) = \tilde{\gamma}_1^{m_{11}} \tilde{\gamma}_2^{m_{21}}, \qquad \tilde{\phi}(\tilde{\gamma}_2) = \tilde{\gamma}_1^{m_{12}} \tilde{\gamma}_2^{m_{22}}.$$

Note that if we put $M = [m_{ij}]$, then the relations in the lattice

$$\Gamma_0/\Gamma_2 = \langle \tilde{\gamma}_0 \cdot \tilde{\gamma}_1, \tilde{\gamma}_2 \mid [\tilde{\gamma}_1, \tilde{\gamma}_2] = 1, \tilde{\gamma}_0\tilde{\gamma}_1\tilde{\gamma}_0^{-1} = \tilde{\gamma}_1^{n_{11}}\tilde{\gamma}_2^{n_{21}}, \tilde{\gamma}_0\tilde{\gamma}_2\tilde{\gamma}_0^{-1} = \tilde{\gamma}_1^{n_{12}}\tilde{\gamma}_2^{n_{22}} \rangle$$

induce the identity $N^m M = MN$. Recall from the observation earlier that there exists a matrix Q such that $Q^{-1}NQ = \Theta$. Hence $\Theta^m(Q^{-1}MQ) = (Q^{-1}MQ)\Theta$. Because Θ is a diagonal matrix of determinant 1, it follows that when $m = 1$, $Q^{-1}MQ$ is a diagonal matrix; when $m = -1$, $Q^{-1}MQ$ is an off-diagonal matrix; when $m \neq \pm 1$, $Q^{-1}MQ$ and hence M is a zero matrix. These yield the three possibilities for $\tilde{\phi}$. For details, we refer to [10], Theorem 2.4. All these observations deduce the proof. □

Remark 3.3 (Homomorphisms on Γ up to conjugacy) Let ϕ be any homomorphism on $\Gamma = \Gamma_{k,N,\mathbf{p}}$ given as in Theorem 3.2. We will observe the effect of ϕ under the conjugation $\mu(\gamma)$ by an element $\gamma \in \Gamma$. Assume

$$\phi(\gamma_0) = \gamma_0^m \gamma_1^{r_1}\gamma_2^{r_2}\gamma_3^{q_0}, \qquad \phi(\gamma_1) = \gamma_1^{m_{11}}\gamma_2^{m_{21}}\gamma_3^{q_1},$$
$$\phi(\gamma_2) = \gamma_1^{m_{12}}\gamma_2^{m_{22}}\gamma_3^{q_2}, \qquad \phi(\gamma_3) = \gamma_3^{m_{11}m_{22}-m_{12}m_{21}}.$$

For any element $\gamma = \gamma_0^\ell \gamma_1^s \gamma_2^t \gamma_3^q$ in Γ, we have

$$\mu(\gamma)\circ\phi(\gamma_0) = \gamma_0^{m'}\gamma_1^{r_1'}\gamma_2^{r_2'}\gamma_3^{q_0'}, \qquad \mu(\gamma)\circ\phi(\gamma_1) = \gamma_1^{m_{11}'}\gamma_2^{m_{21}'}\gamma_3^{q_1'},$$
$$\mu(\gamma)\circ\phi(\gamma_2) = \gamma_1^{m_{12}'}\gamma_2^{m_{22}'}\gamma_3^{q_2'}, \qquad \mu(\gamma)\circ\phi(\gamma_3) = \gamma_3^{m_{11}m_{22}-m_{12}m_{21}}.$$

We will first have a look at $\mu(\gamma)\circ\phi \bmod \Gamma_2$, *i.e.*, $\mu(\tilde{\gamma})\circ\tilde{\phi}$ on the lattice Γ_0/Γ_2. Since in $\bmod \Gamma_2$, $[\gamma_1,\gamma_2] \equiv 1$ and the conjugation by γ_0 is the multiplication by N on $\mathbb{Z}^2 \equiv \langle \gamma_1, \gamma_2\rangle$, we can show easily that

$$m' = m, \qquad \begin{bmatrix} m_{1i}' \\ m_{2i}' \end{bmatrix} = N^\ell \begin{bmatrix} m_{1i} \\ m_{2i} \end{bmatrix}, \qquad \begin{bmatrix} r_1' \\ r_2' \end{bmatrix} = N^\ell \left(\left(N^{-m} - I\right) \begin{bmatrix} s \\ t \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \right).$$

4 Continuous maps on solvmanifolds $\Gamma\backslash \mathrm{Sol}_1^4$

Let $f : \Gamma\backslash \mathrm{Sol}_1^4 \to \Gamma\backslash \mathrm{Sol}_1^4$ be a continuous map. Fixing a lift $\tilde{f}$ of f on the universal cover Sol_1^4, f defines a homomorphism φ on the group of covering transformations Γ, namely,

$$\varphi(\gamma)\circ\tilde{f} = \tilde{f}\circ\gamma, \quad \forall \gamma \in \Gamma.$$

Thus φ is a homomorphism in Theorem 3.2. This homomorphism extends uniquely to a Lie group homomorphism Φ on Sol_1^4. This implies that Φ restricts to a map Φ_Γ on the quotient space $\Gamma\backslash \mathrm{Sol}_1^4$ which induces the same homomorphism φ on Γ. Further, since $\Gamma\backslash \mathrm{Sol}_1^4$ is aspherical, Φ_Γ is homotopic to f. Remark also that another choice of a lift of f results in a new homomorphism on Γ and on Sol_1^4 which differ by the conjugation by an element of Γ.

Since the invariants that we are going to deal with are all homotopy invariants, we will assume in what follows that *every continuous map on* $\Gamma\backslash \mathrm{Sol}_1^4$ *is induced by a Lie group homomorphism* Φ *on* Sol_1^4 *preserving* Γ.

Let Φ be a Lie group homomorphism on Sol_1^4. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Sol}_1^4 & \xrightarrow{\Phi} & \mathrm{Sol}_1^4 \\ \downarrow{\scriptstyle\log} & & \downarrow{\scriptstyle\log} \\ \mathrm{Sol}_1^4 & \xrightarrow{\Phi_*} & \mathrm{Sol}_1^4 \end{array}$$

where Φ_* is the differential of Φ. Assume that Φ restricts to a homomorphism φ on Γ. Recall that Φ preserves the derived series of Sol_1^4, $\mathrm{Sol}_1^4 \supset \mathrm{Nil}^3 \supset \mathcal{Z}(\mathrm{Sol}_1^4)$, and its associated sequence $\Gamma \supset \Gamma_1 \supset \Gamma_2$. Then we could choose a set of generators $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ so that $\gamma_1, \gamma_2 \in \mathrm{Nil}^3$ and $\gamma_3 \in \mathcal{Z}(\mathrm{Sol}_1^4)$. Furthermore, the image of γ_0 under the projection $\Gamma \to \Gamma/\Gamma_1$ generates $\Gamma/\Gamma_1 \cong \mathbb{Z}$, the images of γ_1, γ_2 under the projection $\Gamma_1 \to \Gamma_1/\Gamma_2$ generate $\Gamma_1/\Gamma_2 \cong \mathbb{Z}^2$, and γ_3 generates $\Gamma_3 \cong \mathbb{Z}$. Therefore, $\{\log\gamma_0, \log\gamma_1, \log\gamma_2, \log\gamma_3\}$ forms a linear ordered basis of Sol_1^4 with respect to which Φ_* can be expressed as a matrix of the form

$$\Phi_* = \left[\begin{array}{c:c:c} \varphi_0 & 0 & 0 \\ \hdashline * & \varphi_1 & 0 \\ \hdashline * & * & \varphi_2 \end{array}\right],$$

where φ_0, φ_2 are integers and φ_1 is a 2×2 integer matrix. When f is a map on $\Gamma\backslash\mathrm{Sol}_1^4$, which is homotopic to a map induced by Φ, we say that Φ_* is a *linearization* of f. Because in this paper we are only concerned with the eigenvalues of Φ_*, we shall denote by Φ_* the diagonal block integer matrix $\mathrm{diag}\{\varphi_0, \varphi_1, \varphi_2\}$.

The commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Nil}^3 & \longrightarrow & \mathrm{Sol}_1^4 & \longrightarrow & \mathbb{R} & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \Gamma_1 & \longrightarrow & \Gamma & \longrightarrow & \Gamma/\Gamma_1 & \longrightarrow & 1 \end{array}$$

gives rise to a fibration structure on $\Gamma\backslash\mathrm{Sol}_1^4$, called the *Mostow fibration*,

$$\Gamma_1\backslash\mathrm{Nil}^3 \longrightarrow \Gamma\backslash\mathrm{Sol}_1^4 \longrightarrow \mathbb{Z}\backslash\mathbb{R}$$

over the circle base with nilmanifold fiber. We remark that any map f on $\Gamma\backslash\mathrm{Sol}_1^4$ induced by a Lie group homomorphism of Sol_1^4 is a fibration map with respect to the above bundle structure. Indeed, such a Lie group homomorphism Φ on Sol_1^4 induces maps f, f' and f_0 so that the diagram is commutative

$$\begin{array}{ccccc} \Gamma_1\backslash\mathrm{Nil}^3 & \longrightarrow & \Gamma\backslash\mathrm{Sol}_1^4 & \longrightarrow & (\Gamma/\Gamma_1)\backslash\mathbb{R} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} \\ \Gamma_1\backslash\mathrm{Nil}^3 & \longrightarrow & \Gamma\backslash\mathrm{Sol}_1^4 & \longrightarrow & (\Gamma/\Gamma_1)\backslash\mathbb{R} \end{array}$$

Thus, φ_0 is the degree of the induced map f_0 on the base space $\mathbb{Z}\backslash\mathbb{R}$, and

$$\Phi'_* = \left[\begin{array}{c:c} \varphi_1 & 0 \\ \hdashline * & \varphi_2 \end{array}\right]$$

is a linearization of f', see [8] for details. In particular, $\varphi_2 = \det \varphi_1$ which was observed already in Theorem 3.2. Furthermore, we also have the following commutative diagram induced by Φ:

$$\begin{array}{ccccc}
\Gamma_2\backslash\mathcal{Z}(\mathrm{Sol}_1^4) & \longrightarrow & \Gamma\backslash\mathrm{Sol}_1^4 & \longrightarrow & (\Gamma/\Gamma_2)\backslash\mathrm{Sol}^3 \\
\big\downarrow{\scriptstyle f_2} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \hat{f}} \\
\Gamma_2\backslash\mathcal{Z}(\mathrm{Sol}_1^4) & \longrightarrow & \Gamma\backslash\mathrm{Sol}_1^4 & \longrightarrow & (\Gamma/\Gamma_2)\backslash\mathrm{Sol}^3
\end{array}$$

Thus,

$$\hat{\Phi}_* = \left[\begin{array}{c:c} \varphi_0 & 0 \\ \hdashline * & \varphi_1 \end{array}\right]$$

is a linearization of $\hat{f}$, see [10] for details. In all, we have

$$\Phi_* = \left[\begin{array}{c:c:c} \varphi_0 & 0 & 0 \\ \hdashline * & \varphi_1 & 0 \\ \hdashline * & * & \varphi_2 \end{array}\right] = \left[\begin{array}{c:c} \varphi_0 & 0 \\ \hdashline * & \Phi'_* \end{array}\right] = \left[\begin{array}{c:c} \hat{\Phi}_* & 0 \\ \hdashline * & \varphi_2 \end{array}\right].$$

From Theorem 3.2, we immediately obtain the following.

Proposition 4.1 *Let* $f : \Gamma\backslash\mathrm{Sol}_1^4 \to \Gamma\backslash\mathrm{Sol}_1^4$ *be a continuous map where* $\Gamma = \Gamma_{k,N,\mathbf{p}}$. *Then a linearization of* f *is an integer matrix of one of the following*:

$$\left[\begin{array}{c:cc:c} 1 & & & \\ \hdashline & \mu & \nu & \\ & \frac{n_{21}}{n_{12}}\nu & \mu + \frac{n_{22}-n_{11}}{n_{12}}\nu & \\ \hdashline & & & \varphi_2 \end{array}\right], \quad \left[\begin{array}{c:cc:c} -1 & & & \\ \hdashline & -\mu & \frac{n_{11}-n_{22}}{n_{21}}\mu - \frac{n_{12}}{n_{21}}\nu & \\ & \nu & \mu & \\ \hdashline & & & \varphi_2 \end{array}\right],$$

$$\left[\begin{array}{c:cc:c} m(\neq \pm 1) & & & \\ \hdashline & 0 & 0 & \\ & 0 & 0 & \\ \hdashline & & & 0 \end{array}\right].$$

Each linearization is respectively conjugate to

$$\left[\begin{array}{c:cc:c} 1 & & & \\ \hdashline & \mu - \frac{n_{11}-n_{22}-\sqrt{(n_{11}+n_{22})^2-4}}{2n_{12}}\nu & 0 & \\ & 0 & \mu - \frac{n_{11}-n_{22}+\sqrt{(n_{11}+n_{22})^2-4}}{2n_{12}}\nu & \\ \hdashline & & & \varphi_2 \end{array}\right],$$

$$\left[\begin{array}{c:cc:c} -1 & & & \\ \hdashline & 0 & \mu + \frac{n_{11}-n_{22}-\sqrt{(n_{11}+n_{22})^2-4}}{2n_{21}}\nu & \\ & \mu + \frac{n_{11}-n_{22}+\sqrt{(n_{11}+n_{22})^2-4}}{2n_{21}}\nu & 0 & \\ \hdashline & & & \varphi_2 \end{array}\right],$$

$$\left[\begin{array}{c:cc:c} m(\neq \pm 1) & & & \\ \hdashline & 0 & 0 & \\ & 0 & 0 & \\ \hdashline & & & 0 \end{array}\right].$$

Proof The first part follows from Theorem 3.2. Taking conjugation $Q^{-1}(\cdot)Q$ by the matrix

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{n_{11}-n_{22}+\sqrt{(n_{11}+n_{22})^2-4}}{2n_{21}} & \frac{n_{11}-n_{22}-\sqrt{(n_{11}+n_{22})^2-4}}{2n_{21}} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

we obtain the second part of our assertion. Notice that the middle block of Q consists of eigenvectors of N. □

Note that homomorphisms of distinct types are not conjugate to each other. We shall say a map on $\Gamma\backslash\mathrm{Sol}_1^4$ is of type (I), (II) or (III) according to its homomorphism on Γ.

Corollary 4.2 *Each linearization of a continuous map on* $\Gamma\backslash\mathrm{Sol}_1^4$ *is conjugate to one of the following*:

$$\text{(I)}\ \begin{bmatrix} 1 & & & \\ & \alpha & 0 & \\ & 0 & \beta & \\ & & & \alpha\beta \end{bmatrix}, \qquad \text{(II)}\ \begin{bmatrix} -1 & & & \\ & 0 & \gamma & \\ & \delta & 0 & \\ & & & -\gamma\delta \end{bmatrix}, \qquad \text{(III)}\ \begin{bmatrix} \xi & & & \\ & 0 & 0 & \\ & 0 & 0 & \\ & & & 0 \end{bmatrix}, \qquad (*)$$

where $\alpha+\beta, \alpha\beta, \gamma\delta, \xi \in \mathbb{Z}$ *with* $\xi \neq \pm 1$. *Moreover*, $\alpha = 0$ *if and only if* $\beta = 0$, *and* $\gamma = 0$ *if and only if* $\delta = 0$.

Proof Since $\operatorname{tr} N = n_{11} + n_{12} > 2$, the number $\sqrt{(n_{11}+n_{22})^2 - 4}$ must be irrational. If $\alpha = \mu - \frac{n_{11}-n_{22}-\sqrt{(n_{11}+n_{22})^2-4}}{2n_{12}}\nu = 0$, then ν must be zero and hence $\mu = 0$ because μ, ν, n_{11}, n_{12}, n_{22} are all integers. It follows that $\beta = \alpha = 0$. The converse is the same. Similarly, we can show that $\gamma = 0$ if and only if $\delta = 0$. □

Remark 4.3 If f is of type (II), then f^2 is of type (I).

Proposition 4.4 *Let* f *and* f' *be two continuous maps on the solvmanifold* $\Gamma_{k,N,\mathbf{p}}\backslash\mathrm{Sol}_1^4$ *with linearizations* $\Phi_* = \operatorname{diag}\{\varphi_0, \varphi_1, \varphi_2\}$ *and* $\Phi'_* = \operatorname{diag}\{\varphi'_0, \varphi'_1, \varphi'_2\}$, *respectively. If* f *and* f' *are homotopic, then*

$$\varphi'_0 = \varphi_0, \qquad \varphi'_1 = N^\ell \varphi_1, \qquad \varphi'_2 = \varphi_2$$

for some ℓ.

Proof As it was mentioned earlier, we may assume that f, f' are induced respectively by Lie group homomorphisms $\Phi, \Phi' : \mathrm{Sol}_1^4 \to \mathrm{Sol}_1^4$, both of which restrict to homomorphisms $\varphi, \varphi' : \Gamma_{k,N,\mathbf{p}} \to \Gamma_{k,N,\mathbf{p}}$. Because $f \simeq f'$, φ and φ' differ by the conjugation by an element of $\Gamma_{k,N,\mathbf{p}}$. Now our assertion follows from Remark 3.3. □

According to this result, $\det \Phi_*$ and φ_0, φ_2 are all homotopy invariants. A map becomes of type (I), (II) or (III) according to what the value of φ_0 is 1, –1 or the others. It should be noticed that the φ_1 is not a homotopy invariant.

5 Lefschetz numbers and Nielsen type numbers

Recall that Sol_1^4 is of type (R). Let f be a map on the special solvmanifold $\Gamma\backslash\mathrm{Sol}_1^4$ of type (R) with $\Gamma = \Gamma_{k,N,\mathbf{p}}$. Then we may assume that f is induced by a Lie group homomorphism Φ. Consider a linearization $\Phi_* = \mathrm{diag}\{\varphi_0, \varphi_1, \varphi_2\}$ of f. Then the main result, Theorem 3.1, of [19] implies that $L(f) = \det(I - \Phi_*)$. By [20], Theorem 2.1, we also have $N(f) = |L(f)|$. See also the main result, Theorem 3.1, of [3].

Next we consider the iterations f^n of f. Since Φ induces a map homotopic to f, Φ^n induces a map homotopic to f^n. Therefore, $\Phi_*^n = \mathrm{diag}\{\varphi_0^n, \varphi_1^n, \varphi_2^n\}$ is a linearization of f^n, and so we have

$$L(f^n) = \det(I - \Phi_*^n) = (1 - \varphi_0^n)\det(I - \varphi_1^n)(1 - \varphi_2^n).$$

Proposition 5.1 *Let $f : \Gamma\backslash\mathrm{Sol}_1^4 \to \Gamma\backslash\mathrm{Sol}_1^4$ be any continuous map on the solvmanifold $\Gamma\backslash\mathrm{Sol}_1^4$ with linearization $\Phi_* = \mathrm{diag}\{\varphi_0, \varphi_1, \varphi_2\}$. Then, for all positive integers n,*

$$L(f^n) = \begin{cases} 0 & \text{when } f \text{ is of type (I)}, \\ (1 - (-1)^n)(1 - \varphi_2^{2n}) & \text{when } f \text{ is of type (II)}, \\ 1 - \varphi_0^n & \text{when } f \text{ is of type (III)}, \end{cases}$$

and $N(f^n) = |L(f^n)|$.

Proof If f is of type (I), then $\varphi_0 = \varphi_0^n = 1$ and hence $L(f^n) = 0$. If f is of type (III), then $\varphi_1 = \varphi_1^n = 0$ and $\varphi_2 = \varphi_2^n = 0$, and hence $L(f^n) = 1 - \varphi_0^n$.

Assume that f is of type (II). By Corollary 4.2, we may assume that

$$\Phi_* = \left[\begin{array}{c|cc|c} -1 & & & \\ \hline & 0 & \gamma & \\ & \delta & 0 & \\ \hline & & & -\gamma\delta \end{array}\right], \qquad \Phi_*^{2k+1} = \left[\begin{array}{c|cc|c} -1 & & & \\ \hline & 0 & \gamma^{k+1}\delta^k & \\ & \gamma^k\delta^{k+1} & 0 & \\ \hline & & & -(\gamma\delta)^{2k+1} \end{array}\right].$$

Hence

$$\begin{aligned} L(f^{2k+1}) &= \det(I - \Phi_*^{2k+1}) = 2(1 + \det\varphi_1^{2k+1})(1 - \varphi_2^{2k+1}) \\ &= 2(1 - (\gamma\delta)^{2k+1})(1 + (\gamma\delta)^{2k+1}) \\ &= 2(1 - \varphi_2^{2(2k+1)}). \end{aligned}$$

Since $\varphi_0 = -1$, $L(f^{2k}) = 0$, and hence the theorem is proved. □

A connected solvable Lie group S is called of type (NR) (for 'no roots') if the eigenvalues of $\mathrm{Ad}(x) : \mathfrak{S} \to \mathfrak{S}$ are always either equal to 1 or else they are not roots of unity. Solvable Lie groups of type (NR) were considered first in Keppelmann and McCord [3]. Since our solvmanifold $\Gamma\backslash\mathrm{Sol}_1^4$ is of type (NR), we have the following.

Theorem 5.2 *Let $f : \Gamma\backslash\mathrm{Sol}_1^4 \to \Gamma\backslash\mathrm{Sol}_1^4$ be a continuous map with linearization $\Phi_* = \mathrm{diag}\{\varphi_0, \varphi_1, \varphi_2\}$. Then we have*

(1) *If f is of type* (I), *then*

$$\mathrm{NP}_n(f) = \mathrm{N}\Phi_n(f) = 0.$$

(2) *Suppose f is of type* (II). *Then we have*:

(a) *If n is odd and $\varphi_2 = \pm 1$, then* $\mathrm{NP}_n(f) = \mathrm{N}\Phi_n(f) = 0$.

(b) *If n is odd and $\varphi_2 \neq \pm 1$, then*

$$\mathrm{NP}_n(f) = 2\sum_{m|n} \mu(n/m)\left|1-\varphi_2^{2m}\right|, \qquad \mathrm{N}\Phi_n(f) = 2\left|1-\varphi_2^{2n}\right|.$$

(c) *If $n = 2^r n_0$ with n_0 odd, then*

$$\mathrm{NP}_n(f) = 0, \qquad \mathrm{N}\Phi_n(f) = \begin{cases} 0 & \textit{when } \varphi_2 = \pm 1, \\ 2|1-\varphi_2^{2n_0}| & \textit{when } \varphi_2 \neq \pm 1. \end{cases}$$

(3) *If f is of type* (III), *then*

$$\mathrm{NP}_n(f) = \sum_{m|n} \mu(n/m)\left|1-\varphi_0^{2m}\right|, \qquad \mathrm{N}\Phi_n(f) = \left|1-\varphi_0^{2n}\right|.$$

Proof In case (1), the Nielsen number $N(f^m) = 0$ for all positive integers m. The map f has no essential periodic orbit classes of any period. It follows that $\mathrm{NP}_n(f) = \mathrm{N}\Phi_n(f) = 0$.

Since $\Gamma\backslash\mathrm{Sol}_1^4$ is a solvmanifold of type (NR), by [2], Theorem 1.2, we have $\mathrm{N}\Phi_m(f) = N(f^m)$ and $\mathrm{NP}_m(f) = \sum_{q|m}\mu(q)N(f^{\frac{m}{q}})$ for all $m|n$ provided $N(f^n) \neq 0$. This proves our case (2)(b) and case (3) because of the following reason: When f is of type (III), $N(f^n) \neq 0$; when f is of type (II), $N(f^n) = (1-(-1)^n)|1-\varphi_2^{2n}|$, and $N(f^n) \neq 0$ if and only if n is odd and $\varphi_2 \neq \pm 1$. In case (2) when n is odd and $\varphi_2 = \pm 1$, *i.e.*, in case (2)(a), we note that $N(f^m) = 0$ for all positive integers m. Hence $\mathrm{NP}_n(f) = \mathrm{N}\Phi_n(f) = 0$.

Consider the case (2) with n even. Since $N(f^n) = 0$, it follows that $\mathrm{NP}_n(f) = 0$. Let $n = 2^r n_0$ for odd n_0. By Proposition 5.1, f^q has no essential fixed point class for every even factor q of n. Thus, the set of essential fixed point classes of f^q with $q \mid n$ is the same as that of f^q with $q \mid n_0$. Thus, $\mathrm{N}\Phi_n(f) = \mathrm{N}\Phi_{n_0}(f)$, which is just $N(f^{n_0}) = 2|1-\varphi_2^{2n_0}|$ if $\varphi_2 \neq \pm 1$ and 0 if $\varphi_2 = \pm 1$. □

6 Homotopy minimal periods

In this section, we shall present the homotopy minimal periods for all maps on four-dimensional solvmanifolds $\Gamma\backslash\mathrm{Sol}_1^4$, which is of type (NR). Our main tool is

$$\mathrm{HPer}(f) = \left\{n \mid N\left(f^n\right) \neq 0, N\left(f^n\right) \neq N\left(f^{\frac{n}{q}}\right) \text{ for all prime } q \mid n\right\}. \tag{H}$$

This formula can be obtained immediately from the following theorem.

Theorem 6.1 ([21], Theorem 6.1) *Let $f: M \to M$ be a self-map on a compact PL-manifold M of dimension ≥ 3. Then f is homotopic to a map g with $P_n(g) = \emptyset$ if and only if* $\mathrm{NP}_n(f) = 0$.

Table 1 Homotopy minimal periods

HPer(*f*)	Linearization matrix *F* is conjugate to
∅	$\left[\begin{array}{c\|cc\|c} 1 & & & \\ \hline & \alpha & 0 & \\ & 0 & \beta & \\ \hline & & & \alpha\beta \end{array}\right]$ or $\left[\begin{array}{c\|cc\|c} -1 & & & \\ \hline & 0 & \gamma & \\ & \delta & 0 & \\ \hline & & & -\gamma\delta \end{array}\right]$ $(\gamma\delta = \pm 1)$
{1}	$\left[\begin{array}{c\|cc\|c} -1 & & & \\ \hline & 0 & \gamma & \\ & \delta & 0 & \\ \hline & & & -\gamma\delta \end{array}\right]$ $(\gamma\delta = 0)$ or $\left[\begin{array}{c\|cc\|c} 0 & & & \\ \hline & 0 & 0 & \\ & 0 & 0 & \\ \hline & & & 0 \end{array}\right]$
$\mathbb{N} - \{2\}$	$\left[\begin{array}{c\|cc\|c} -2 & & & \\ \hline & 0 & 0 & \\ & 0 & 0 & \\ \hline & & & 0 \end{array}\right]$
$\mathbb{N}$	$\left[\begin{array}{c\|cc\|c} \zeta & & & \\ \hline & 0 & 0 & \\ & 0 & 0 & \\ \hline & & & 0 \end{array}\right]$ $(\zeta \neq 0, \pm 1, -2)$
$\mathbb{N} - 2\mathbb{N}$	$\left[\begin{array}{c\|cc\|c} -1 & & & \\ \hline & 0 & \gamma & \\ & \delta & 0 & \\ \hline & & & -\gamma\delta \end{array}\right]$ $(\gamma\delta \neq 0, \pm 1)$

Proposition 6.2 ([9], Proposition 3.2) *Let $f : M \to M$ be a self-map of a compact solvmanifold M of type* (NR). *Then $\mathrm{NP}_n(f) = 0$ if and only if either $N(f^n) = 0$ or $N(f^n) = N(f^{n/q})$ for some prime factor $q \mid n$.*

Now the following is one of our main results.

Theorem 6.3 *Let $f : \Gamma\backslash \mathrm{Sol}_1^4 \to \Gamma\backslash \mathrm{Sol}_1^4$ be a continuous map with linearization $\Phi_* = \mathrm{diag}\{\varphi_0, \varphi_1, \varphi_2\}$. Then we have*

$$\mathrm{HPer}(f) = \begin{cases} \{1\} & \textit{when } \varphi_0 = 0; \\ \emptyset & \textit{when } \varphi_0 = 1; \\ \emptyset & \textit{when } \varphi_0 = -1 \textit{ and } \varphi_2 = \pm 1; \\ \{1\} & \textit{when } \varphi_0 = -1 \textit{ and } \varphi_2 = 0; \\ \mathbb{N} - 2\mathbb{N} & \textit{when } \varphi_0 = -1 \textit{ and } \varphi_2 \neq 0, \pm 1; \\ \mathbb{N} - \{2\} & \textit{when } \varphi_0 = -2; \\ \mathbb{N} & \textit{otherwise, i.e.}, \varphi_0 \neq 0, \pm 1, -2. \end{cases}$$

Proof (1) If $\varphi_0 = 0$, then f is of type (III) and $N(f^n) = 1$ for all n by Proposition 5.1. Hence $\mathrm{HPer}(f) = \{1\}$.

(2) If $\varphi_0 = 1$, then f is of type (I) and so $N(f^n) = 0$ for all n. This implies that $\mathrm{HPer}(f) = \emptyset$.

(3) If $\varphi_0 = -1$, then f is of type (II). By Proposition 5.1, $N(f^n) = 0$ for all even n. It follows that $\mathrm{HPer}(f)$ does not contain any even number, *i.e.*, $\mathrm{HPer}(f) \subset \mathbb{N} - 2\mathbb{N}$. Let us consider its subcases:

(3-1) If $\varphi_2 = \det\varphi_1 = \pm 1$, by Proposition 5.1, $N(f^n) = 0$ for all n. Thus $\mathrm{HPer}(f) = \emptyset$.

(3-2) If $\varphi_2 = \det\varphi_1 = 0$, by Proposition 5.1, $N(f^n) = 2$ for all odd n. Since $N(f) \neq 0$, we have $1 \in \mathrm{HPer}(f)$. By (H), we have $n \notin \mathrm{HPer}(f)$ for all odd n with $n > 1$. Thus $\mathrm{HPer}(f) = \{1\}$.

(3-3) If $\varphi_2 = \det\varphi_1 \neq 0, \pm 1$, by Proposition 5.1, $N(f^n)$'s are all distinct for all odd n. Hence $\mathrm{HPer}(f) = \mathbb{N} - 2\mathbb{N}$.

(4) If $\varphi_0 = -2$, by Proposition 5.1, we have $N(f^n) = |1 - (-2)^n|$. Especially, $N(f) = N(f^2) = 3$. By (H), $1 \in \mathrm{HPer}(f)$ but $2 \notin \mathrm{HPer}(f)$. Since $N(f^{n+1}) > N(f^n)$ for all $n > 2$, (H) induces that $\mathrm{HPer}(f) = \mathbb{N} - \{2\}$.

(5) Consider finally the case where $\varphi_0 \neq -2, -1, 0, 1$. By Proposition 5.1 again, we still have $N(f^n) = |1 - \varphi_0^n|$. In this case, we have that $N(f^{n+1}) > N(f^n)$ for all n. (H) induces that $\mathrm{HPer}(f) = \mathbb{N}$. □

We tabulate this result according to linearizations in Table 1. In fact, for each subset $S \subset \mathbb{N}$ appearing as $\mathrm{HPer}(f)$ and each form of linearization Φ_* listed above, there exists a self-map $f : \Gamma\backslash \mathrm{Sol}_1^4 \to \Gamma\backslash \mathrm{Sol}_1^4$ such that $\mathrm{HPer}(f) = S$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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