Panyanak *Fixed Point Theory and Applications* (2015) 2015:186
DOI 10.1186/s13663-015-0438-7

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

On an open problem of Kyung Soo Kim

Bancha Panyanak*

*Correspondence:
bancha.p@cmu.ac.th
Department of Mathematics,
Faculty of Science, Chiang Mai
University, Chiang Mai, 50200,
Thailand

Abstract

We prove a convergence theorem of the Mann iteration scheme for a uniformly L-Lipschitzian asymptotically demicontractive mapping in a CAT(κ) space with $\kappa > 0$. We also obtain a convergence theorem of the Ishikawa iteration scheme for a uniformly L-Lipschitzian asymptotically hemicontractive mapping. Our results provide a complete solution to an open problem raised by Kim (Abstr. Appl. Anal. 2013:381715, 2013).

MSC: 47H09; 49J53

Keywords: Mann iteration; Ishikawa iteration; strong convergence; fixed point; CAT(κ) space

1 Introduction

Roughly speaking, $\mathrm{CAT}(\kappa)$ spaces are geodesic spaces of bounded curvature and generalizations of Riemannian manifolds of sectional curvature bounded above. The precise definition is given below. The letters C, A, and T stand for Cartan, Alexandrov, and Toponogov, who have made important contributions to the understanding of curvature via inequalities for the distance function, and κ is a real number that we impose it as the curvature bound of the space.

Fixed point theory in $\mathrm{CAT}(\kappa)$ spaces was first studied by Kirk [1, 2]. His work was followed by a series of new works by many authors, mainly focusing on $\mathrm{CAT}(0)$ spaces (see *e.g.*, [3–25]). Since any $\mathrm{CAT}(\kappa)$ space is a $\mathrm{CAT}(\kappa')$ space for $\kappa' \geq \kappa$, all results for $\mathrm{CAT}(0)$ spaces immediately apply to any $\mathrm{CAT}(\kappa)$ space with $\kappa \leq 0$. However, there are only a few articles that contain fixed point results in the setting of $\mathrm{CAT}(\kappa)$ spaces with $\kappa > 0$, because in this case the proof seems to be more complicated.

The notion of uniformly L-Lipschitzian mappings, which is more general than the notion of asymptotically nonexpansive mappings, was introduced by Goebel and Kirk [26]. In 1991, Schu [27] proved the strong convergence of Mann iteration for asymptotically nonexpansive mappings in Hilbert spaces. Qihou [28] extended Schu's result to the general setting of asymptotically demicontractive mappings and also obtained the strong convergence of Ishikawa iteration for asymptotically hemicontractive mappings. Recently, Kim [29] proved the analogous results of Qihou in the framework of the so-called $\mathrm{CAT}(0)$ spaces. Precisely, Kim obtained the following theorems.

Theorem A *Let (X, ρ) be a complete* $\mathrm{CAT}(0)$ *space, C be a nonempty bounded closed convex subset of X, and $T : C \to C$ be a completely continuous and uniformly L-Lipschitzian* *asymptotically demicontractive mapping with constant $k \in [0,1)$ and sequence $\{a_n\}$ in $[1,\infty)$ such that $\sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in $[\varepsilon, 1-k-\varepsilon]$ for some $\varepsilon > 0$. Given $x_1 \in C$, define the iteration scheme $\{x_n\}$ by*

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n, \quad n \geq 1.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Theorem B *Let (X,ρ) be a complete* CAT(0) *space, let C be a nonempty bounded closed convex subset of X, and let $T : C \to C$ be a completely continuous and uniformly L-Lipschitzian asymptotically hemicontractive mapping with sequence $\{a_n\}$ in $[1,\infty)$ such that $\sum_{n=1}^{\infty}(a_n - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ be such that $\varepsilon \leq \alpha_n \leq \beta_n \leq b$ for some $\varepsilon > 0$ and $b \in (0, \frac{\sqrt{1+L^2}-1}{L^2})$. Given $x_1 \in C$, define the iteration scheme $\{x_n\}$ by*

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n y_n,$$
$$y_n = (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \geq 1.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

In [29], the author raised the following problem.

Problem Can we extend Theorems A and B to the general setting of CAT(κ) spaces with $\kappa > 0$?

The purpose of the paper is to solve this problem. Our main discoveries are Theorems 3.2 and 3.6.

2 Preliminaries

Let (X,ρ) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $\rho(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, c is an isometry and $\rho(x,y) = l$. The image $c([0,l])$ of c is called a *geodesic segment* joining x and y. When it is unique this geodesic segment is denoted by $[x,y]$. This means that $z \in [x,y]$ if and only if there exists $\alpha \in [0,1]$ such that

$$\rho(x,z) = (1-\alpha)\rho(x,y) \quad \text{and} \quad \rho(y,z) = \alpha\rho(x,y).$$

In this case, we write $z = \alpha x \oplus (1-\alpha)y$. The space (X,ρ) is said to be a *geodesic space* (*D-geodesic space*) if every two points of X (every two points of distance smaller than D) are joined by a geodesic, and X is said to be *uniquely geodesic* (*D-uniquely geodesic*) if there is exactly one geodesic joining x and y for each $x, y \in X$ (for $x, y \in X$ with $\rho(x,y) < D$). A subset C of X is said to be *convex* if C includes every geodesic segment joining any two of its points. The set C is said to be *bounded* if

$$\operatorname{diam}(C) := \sup\{\rho(x,y) : x, y \in C\} < \infty.$$

Now we introduce the model spaces M_κ^n, for more details on these spaces the reader is referred to [30, 31]. Let $n \in \mathbb{N}$. We denote by $\mathbb{E}^n$ the metric space $\mathbb{R}^n$ endowed with the usual Euclidean distance. We denote by $(\cdot|\cdot)$ the Euclidean scalar product in $\mathbb{R}^n$, that is,

$$(x|y) = x_1y_1 + \cdots + x_ny_n, \quad \text{where } x = (x_1, \dots, x_n), y = (y_1, \dots, y_n).$$

Let $\mathbb{S}^n$ denote the *n-dimensional sphere* defined by

$$\mathbb{S}^n = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : (x|x) = 1\},$$

with metric $d_{\mathbb{S}^n}(x,y) = \arccos(x|y)$, $x, y \in \mathbb{S}^n$.

Let $\mathbb{E}^{n,1}$ denote the vector space $\mathbb{R}^{n+1}$ endowed with the symmetric bilinear form which associates to vectors $u = (u_1, \dots, u_{n+1})$ and $v = (v_1, \dots, v_{n+1})$ the real number $\langle u|v\rangle$ defined by

$$\langle u|v\rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_iv_i.$$

Let $\mathbb{H}^n$ denote the *hyperbolic n-space* defined by

$$\mathbb{H}^n = \{u = (u_1, \dots, u_{n+1}) \in \mathbb{E}^{n,1} : \langle u|u\rangle = -1, u_{n+1} > 0\},$$

with metric $d_{\mathbb{H}^n}$ such that

$$\cosh d_{\mathbb{H}^n}(x,y) = -\langle x|y\rangle, \quad x, y \in \mathbb{H}^n.$$

Definition 2.1 Given $\kappa \in \mathbb{R}$, we denote by M_κ^n the following metric spaces:

(i) if $\kappa = 0$ then M_0^n is the Euclidean space $\mathbb{E}^n$;

(ii) if $\kappa > 0$ then M_κ^n is obtained from the spherical space $\mathbb{S}^n$ by multiplying the distance function by the constant $1/\sqrt{\kappa}$;

(iii) if $\kappa < 0$ then M_κ^n is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by the constant $1/\sqrt{-\kappa}$.

A *geodesic triangle* $\triangle(x,y,z)$ in a geodesic space (X,ρ) consists of three points x, y, z in X (the *vertices* of $\triangle$) and three geodesic segments between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for a geodesic triangle $\triangle(x,y,z)$ in (X,ρ) is a triangle $\overline{\triangle}(\bar{x},\bar{y},\bar{z})$ in M_κ^2 such that

$$\rho(x,y) = d_{M_\kappa^2}(\bar{x},\bar{y}), \qquad \rho(y,z) = d_{M_\kappa^2}(\bar{y},\bar{z}) \quad \text{and} \quad \rho(z,x) = d_{M_\kappa^2}(\bar{z},\bar{x}).$$

If $\kappa \le 0$ then such a comparison triangle always exists in M_κ^2. If $\kappa > 0$ then such a triangle exists whenever $\rho(x,y) + \rho(y,z) + \rho(z,x) < 2D_\kappa$, where $D_\kappa = \pi/\sqrt{\kappa}$. A point $\bar{p} \in [\bar{x},\bar{y}]$ is called a *comparison point* for $p \in [x,y]$ if $\rho(x,p) = d_{M_\kappa^2}(\bar{x},\bar{p})$.

A geodesic triangle $\triangle(x,y,z)$ in X is said to satisfy the CAT(κ) *inequality* if for any $p, q \in \triangle(x,y,z)$ and for their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x},\bar{y},\bar{z})$, one has

$$\rho(p,q) \le d_{M_\kappa^2}(\bar{p},\bar{q}).$$

Definition 2.2 If $\kappa \le 0$, then X is called a CAT(κ) *space* if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(κ) inequality.

If $\kappa > 0$, then X is called a CAT(κ) *space* if X is D_κ-geodesic and any geodesic triangle $\triangle(x,y,z)$ in X with $\rho(x,y)+\rho(y,z)+\rho(z,x) < 2D_\kappa$ satisfies the CAT(κ) inequality.

Notice that in a CAT(0) space (X,ρ), if $x,y,z \in X$ then the CAT(0) inequality implies

$$\text{(CN)}\quad \rho^2\left(x, \frac{1}{2}y \oplus \frac{1}{2}z\right) \le \frac{1}{2}\rho^2(x,y) + \frac{1}{2}\rho^2(x,z) - \frac{1}{4}\rho^2(y,z).$$

This is the (CN) *inequality* of Bruhat and Tits [32]. This inequality is extended by Dhompongsa and Panyanak [9] as

$$\text{(CN}^*\text{)}\quad \rho^2\big(x,(1-\alpha)y \oplus \alpha z\big) \le (1-\alpha)\rho^2(x,y) + \alpha\rho^2(x,z) - \alpha(1-\alpha)\rho^2(y,z)$$

for all $\alpha \in [0,1]$ and $x,y,z \in X$. In fact, if X is a geodesic space then the following statements are equivalent:

(i) X is a CAT(0) space;
(ii) X satisfies (CN);
(iii) X satisfies (CN*).

Let $R \in (0,2]$. Recall that a geodesic space (X,ρ) is said to be *R-convex* for R [33] if for any three points $x,y,z \in X$, we have

$$\rho^2\big(x,(1-\alpha)y \oplus \alpha z\big) \le (1-\alpha)\rho^2(x,y) + \alpha\rho^2(x,z) - \frac{R}{2}\alpha(1-\alpha)\rho^2(y,z). \tag{1}$$

It follows from (CN*) that a geodesic space (X,ρ) is a CAT(0) space if and only if (X,ρ) is R-convex for $R=2$. The following lemma generalizes Proposition 3.1 of Ohta [33].

Lemma 2.3 *Let κ be an arbitrary positive real number and (X,ρ) be a* CAT(κ) *space with* $\operatorname{diam}(X) \le \frac{\pi/2-\eta}{\sqrt{\kappa}}$ *for some $\eta \in (0,\pi/2)$. Then (X,ρ) is R-convex for $R = (\pi - 2\eta)\tan(\eta)$.*

Proof Let $x,y,z \in X$. Since $\operatorname{diam}(X) < \frac{\pi}{2\sqrt{\kappa}}$, $\rho(x,y)+\rho(x,z)+\rho(y,z) < 2D_\kappa$ where $D_\kappa = \frac{\pi}{\sqrt{\kappa}}$. Let $\triangle(x,y,z)$ be the geodesic triangle constructed from x, y, z and $\overline{\triangle}(\bar{x},\bar{y},\bar{z})$ its comparison triangle. Then

$$\rho(x,y) = d_{M_\kappa^2}(\bar{x},\bar{y}), \qquad \rho(x,z) = d_{M_\kappa^2}(\bar{x},\bar{z}) \quad \text{and} \quad \rho(y,z) = d_{M_\kappa^2}(\bar{y},\bar{z}). \tag{2}$$

It is sufficient to prove (1) only the case of $\alpha = 1/2$. Let $a = d_{\mathbb{S}^2}(\bar{x},\bar{y})$, $b = d_{\mathbb{S}^2}(\bar{x},\bar{z})$, $c = d_{\mathbb{S}^2}(\bar{y},\bar{z})/2$, and $d = d_{\mathbb{S}^2}(\bar{x},\frac{1}{2}\bar{y} \oplus \frac{1}{2}\bar{z})$ and define

$$f(a,b,c) := \frac{2}{c^2}\left(\frac{1}{2}a^2 + \frac{1}{2}b^2 - d^2\right).$$

By using the same argument in the proof of Proposition 3.1 in [33], we obtain

$$d_{\mathbb{S}^2}^2\left(\bar{x}, \frac{1}{2}\bar{y} \oplus \frac{1}{2}\bar{z}\right) \le \frac{1}{2}d_{\mathbb{S}^2}^2(\bar{x},\bar{y}) + \frac{1}{2}d_{\mathbb{S}^2}^2(\bar{x},\bar{z}) - \left(\frac{R}{2}\right)\left(\frac{1}{4}\right)d_{\mathbb{S}^2}^2(\bar{y},\bar{z}),$$

where $R = (\pi - 2\eta)\tan(\eta)$. This implies that

$$d_{M_\kappa^2}^2\left(\bar{x}, \frac{1}{2}\bar{y} \oplus \frac{1}{2}\bar{z}\right) \le \frac{1}{2}d_{M_\kappa^2}^2(\bar{x},\bar{y}) + \frac{1}{2}d_{M_\kappa^2}^2(\bar{x},\bar{z}) - \left(\frac{R}{2}\right)\left(\frac{1}{4}\right)d_{M_\kappa^2}^2(\bar{y},\bar{z}). \tag{3}$$

By (2) and (3), we get

$$\rho^2\left(x, \frac{1}{2}y \oplus \frac{1}{2}z\right) \leq \frac{1}{2}\rho^2(x,y) + \frac{1}{2}\rho^2(x,z) - \left(\frac{R}{2}\right)\left(\frac{1}{4}\right)\rho^2(y,z).$$

This completes the proof. □

The following lemma is also needed.

Lemma 2.4 *Let $\{s_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers satisfying*

$$s_{n+1} \leq s_n + t_n \quad \text{for all } n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} t_n < \infty$ and $\{s_n\}$ has a subsequence converging to 0, then $\lim_{n\to\infty} s_n = 0$.

Definition 2.5 Let C be a nonempty subset of a CAT(κ) space (X,ρ) and $T : C \to C$ be a mapping. We denote by $F(T)$ the set of all fixed points of T, *i.e.*, $F(T) = \{x \in C : x = Tx\}$. Then T is said to

(i) be *completely continuous* if T is continuous and for any bounded sequence $\{x_n\}$ in C, $\{Tx_n\}$ has a convergent subsequence in C;

(ii) be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\rho\left(T^n x, T^n y\right) \leq L\rho(x,y) \quad \text{for all } x,y \in C \text{ and all } n \in \mathbb{N};$$

(iii) be *asymptotically demicontractive* if $F(T) \neq \emptyset$ and there exist $k \in [0,1)$ and a sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = 1$ such that

$$\rho^2\left(T^n x, p\right) \leq a_n^2 \rho^2(x,p) + k\rho^2\left(x, T^n x\right) \quad \text{for all } x \in C, p \in F(T) \text{ and } n \in \mathbb{N};$$

(iv) be *asymptotically hemicontractive* if $F(T) \neq \emptyset$ and there exists a sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = 1$ such that

$$\rho^2\left(T^n x, p\right) \leq a_n \rho^2(x,p) + \rho^2\left(x, T^n x\right) \quad \text{for all } x \in C, p \in F(T) \text{ and } n \in \mathbb{N}.$$

It follows from the definition that every asymptotically demicontractive mapping is asymptotically hemicontractive. For more details as regards these classes of mappings the reader is referred to [27, 28].

Let C be a nonempty convex subset of a CAT(κ) space (X,ρ) and $T : C \to C$ be a mapping. Given $x_1 \in C$.

Algorithm 1 The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \geq 1, \end{aligned}$$

is called an *Ishikawa iterative sequence* (see [34]).

If $\beta_n = 0$ for all $n \in \mathbb{N}$, then Algorithm 1 reduces to the following.

Algorithm 2 The sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n, \quad n \geq 1,$$

is called a *Mann iterative sequence* (see [35]).

3 Main results

We first discuss the strong convergence of Mann iteration for uniformly L-Lipschitzian asymptotically demicontractive mappings. The following lemma follows immediately from Lemma 6 of [29] and [30], p.176.

Lemma 3.1 *Let $\kappa > 0$ and (X,ρ) be a* $\mathrm{CAT}(\kappa)$ *space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\eta}{\sqrt{\kappa}}$ *for some $\eta \in (0,\pi/2)$. Let C be a nonempty convex subset of X, $T : C \to C$ be a uniformly L-Lipschitzian mapping, and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0,1]$. Given $x_1 \in C$, define the iteration scheme $\{x_n\}$ by*

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n y_n,$$
$$y_n = (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \geq 1.$$

Then

$$\rho(x_n, Tx_n) \leq \rho\bigl(x_n, T^n x_n\bigr) + L\bigl(1+2L+L^2\bigr)\rho\bigl(x_{n-1}, T^{n-1}x_{n-1}\bigr)$$

for all $n \geq 1$.

The following theorem is one of our main results.

Theorem 3.2 *Let $\kappa > 0$ and (X,ρ) be a* $\mathrm{CAT}(\kappa)$ *space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\eta}{\sqrt{\kappa}}$ *for some $\eta \in (0,\pi/2)$. Let C be a nonempty closed convex subset of X, and $T : C \to C$ be a completely continuous and uniformly L-Lipschitzian asymptotically demicontractive mapping with constant $k \in [0,1)$ and sequence $\{a_n\}$ in $[1,\infty)$ such that $\sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in $[\varepsilon, R/2 - k - \varepsilon]$ for some $\varepsilon > 0$ where $R = (\pi - 2\eta)\tan(\eta)$. Given $x_1 \in C$, define the iteration scheme $\{x_n\}$ by*

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n, \quad n \geq 1.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof Let $p \in F(T)$. By (1), we have

$$\rho^2(x_{n+1},p) \leq (1-\alpha_n)\rho^2(x_n,p) + \alpha_n\rho^2\bigl(T^n x_n,p\bigr) - \frac{R}{2}\alpha_n(1-\alpha_n)\rho^2\bigl(x_n, T^n x_n\bigr).$$

It follows from the asymptotically demicontractiveness of T that

$$\begin{aligned}\rho^2(x_{n+1},p) \leq{}& (1-\alpha_n)\rho^2(x_n,p) + \alpha_n\bigl[a_n^2\rho^2(x_n,p) + k\rho^2\bigl(x_n, T^n x_n\bigr)\bigr] \\ &- \frac{R}{2}\alpha_n(1-\alpha_n)\rho^2\bigl(x_n, T^n x_n\bigr)\end{aligned}$$

$$= \rho^2(x_n,p) + \alpha_n\big(a_n^2 - 1\big)\rho^2(x_n,p) - \alpha_n\left(\frac{R}{2} - \frac{R}{2}\alpha_n - k\right)\rho^2\big(x_n, T^n x_n\big)$$

$$\le \rho^2(x_n,p) + \alpha_n\big(a_n^2 - 1\big)\rho^2(x_n,p) - \alpha_n\left(\frac{R}{2} - \alpha_n - k\right)\rho^2\big(x_n, T^n x_n\big). \tag{4}$$

Since $\varepsilon \le \alpha_n \le R/2 - k - \varepsilon$, we have $\varepsilon \le R/2 - \alpha_n - k$. Thus,

$$\varepsilon^2 \le \alpha_n(R/2 - \alpha_n - k). \tag{5}$$

By (4) and (5), we have

$$\rho^2(x_{n+1},p) \le \rho^2(x_n,p) + \alpha_n\big(a_n^2 - 1\big)\rho^2(x_n,p) - \varepsilon^2\rho^2\big(x_n, T^n x_n\big)$$

$$\le \rho^2(x_n,p) + \frac{\pi^2(a_n^2 - 1)}{4\kappa} - \varepsilon^2\rho^2\big(x_n, T^n x_n\big). \tag{6}$$

Therefore,

$$\varepsilon^2\rho^2\big(x_n, T^n x_n\big) \le \rho^2(x_n,p) - \rho^2(x_{n+1},p) + \frac{\pi^2(a_n^2 - 1)}{4\kappa}.$$

Since $\sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$, $\sum_{n=1}^{\infty}\rho^2(x_n, T^n x_n) < \infty$, which implies that $\lim_{n\to\infty}\rho(x_n, T^n x_n) = 0$. By Lemma 3.1, we have

$$\lim_{n\to\infty}\rho(x_n, Tx_n) = 0. \tag{7}$$

Since T is completely continuous, $\{Tx_n\}$ has a convergent subsequence in C. By (7), $\{x_n\}$ has a convergent subsequence, say $x_{n_k} \to q \in C$. Moreover,

$$\rho(q, Tq) \le \rho(q, x_{n_k}) + \rho(x_{n_k}, Tx_{n_k}) + \rho(Tx_{n_k}, Tq) \to 0 \quad \text{as } k \to \infty.$$

That is $q \in F(T)$. It follows from (6) that

$$\rho^2(x_{n+1},p) \le \rho^2(x_n,p) + \frac{\pi^2(a_n^2 - 1)}{4\kappa}.$$

Since $\sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$, by Lemma 2.4 we have $x_n \to q$. This completes the proof. □

Corollary 3.3 (Theorem 7 of [29]) *Let (X,ρ) be a* CAT(0) *space, C be a nonempty bounded closed convex subset of X, and $T : C \to C$ be a completely continuous and uniformly L-Lipschitzian asymptotically demicontractive mapping with constant $k \in [0,1)$ and sequence $\{a_n\}$ in $[1,\infty)$ such that $\sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in $[\varepsilon, 1 - k - \varepsilon]$ for some $\varepsilon > 0$. Given $x_1 \in C$, define the iteration scheme $\{x_n\}$ by*

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n T^n x_n, \quad n \ge 1.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof It is well known that every convex subset of a CAT(0) space, equipped with the induced metric, is a CAT(0) space. Then (C,ρ) is a CAT(0) space and hence it is a CAT(κ)

space for all $\kappa > 0$. Notice also that C is R-convex for $R = 2$. Since C is bounded, we can choose $\eta \in (0, \pi/2)$ and $\kappa > 0$ so that $\operatorname{diam}(C) \leq \frac{\pi/2-\eta}{\sqrt{\kappa}}$. The conclusion follows from Theorem 3.2. □

Next, we prove the strong convergence of Ishikawa iteration for uniformly L-Lipschitzian asymptotically hemicontractive mappings. The following lemmas are also needed.

Lemma 3.4 *Let $\kappa > 0$ and (X, ρ) be a* $\mathrm{CAT}(\kappa)$ *space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\eta}{\sqrt{\kappa}}$ *for some $\eta \in (0, \pi/2)$. Let $R = (\pi - 2\eta)\tan(\eta)$, C be a nonempty convex subset of X, and $T : C \to C$ be a uniformly L-Lipschitzian and asymptotically hemicontractive mapping with sequence $\{a_n\}$ in $[1, \infty)$. Given $x_1 \in C$, define the iteration scheme $\{x_n\}$ by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \geq 1, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. Then the following inequality holds:

$$\begin{aligned} \rho^2(x_{n+1}, p) \leq{}& \bigl[1 + \alpha_n(a_n - 1)(1 + a_n\beta_n)\bigr]\rho^2(x_n, p) \\ &- \alpha_n\beta_n\biggl[\frac{R}{2}(1-\beta_n)(1+a_n) - \bigl(a_n + L^2\beta_n^2\bigr)\biggr]\rho^2\bigl(x_n, T^n x_n\bigr) \\ &- \alpha_n\biggl[\frac{R}{2}(1-\alpha_n) - (1-\beta_n)\biggr]\rho^2\bigl(x_n, T^n y_n\bigr) \end{aligned}$$

for all $p \in F(T)$.

Proof Let $p \in F(T)$. By (1), we have

$$\rho^2(x_{n+1}, p) \leq (1-\alpha_n)\rho^2(x_n, p) + \alpha_n\rho^2\bigl(T^n y_n, p\bigr) - \frac{R}{2}\alpha_n(1-\alpha_n)\rho^2\bigl(x_n, T^n y_n\bigr) \tag{8}$$

and

$$\rho^2(y_n, p) \leq (1-\beta_n)\rho^2(x_n, p) + \beta_n\rho^2\bigl(T^n x_n, p\bigr) - \frac{R}{2}\beta_n(1-\beta_n)\rho^2\bigl(x_n, T^n x_n\bigr). \tag{9}$$

Since T is asymptotically hemicontractive,

$$\rho^2\bigl(T^n y_n, p\bigr) \leq a_n\rho^2(y_n, p) + \rho^2\bigl(y_n, T^n y_n\bigr) \tag{10}$$

and

$$\rho^2\bigl(T^n x_n, p\bigr) \leq a_n\rho^2(x_n, p) + \rho^2\bigl(x_n, T^n x_n\bigr). \tag{11}$$

It follows from (9) and (11) that

$$\begin{aligned} \rho^2(y_n, p) \leq{}& (1-\beta_n)\rho^2(x_n, p) + \beta_n\bigl[a_n\rho^2(x_n, p) + \rho^2\bigl(x_n, T^n x_n\bigr)\bigr] \\ &- \frac{R}{2}\beta_n(1-\beta_n)\rho^2\bigl(x_n, T^n x_n\bigr) \\ ={}& \bigl(1 + \beta_n(a_n - 1)\bigr)\rho^2(x_n, p) + \beta_n\biggl(1 - \frac{R}{2}(1-\beta_n)\biggr)\rho^2\bigl(x_n, T^n x_n\bigr). \end{aligned} \tag{12}$$

Substituting (12) into (10) and using (1), we get

$$\begin{aligned}
\rho^2\big(T^n y_n, p\big) &\leq a_n\big(1+\beta_n(a_n-1)\big)\rho^2(x_n,p) \\
&\quad + a_n\beta_n\left(1-\frac{R}{2}(1-\beta_n)\right)\rho^2\big(x_n, T^n x_n\big) + \rho^2\big(y_n, T^n y_n\big) \\
&\leq a_n\big(1+\beta_n(a_n-1)\big)\rho^2(x_n,p) + a_n\beta_n\left(1-\frac{R}{2}(1-\beta_n)\right)\rho^2\big(x_n, T^n x_n\big) \\
&\quad + (1-\beta_n)\rho^2\big(x_n, T^n y_n\big) + \beta_n\rho^2\big(T^n x_n, T^n y_n\big) \\
&\quad - \frac{R}{2}\beta_n(1-\beta_n)\rho^2\big(x_n, T^n x_n\big) \\
&\leq a_n\big(1+\beta_n(a_n-1)\big)\rho^2(x_n,p) \\
&\quad + \left[a_n\beta_n - a_n\beta_n\frac{R}{2}(1-\beta_n) - \beta_n\frac{R}{2}(1-\beta_n)\right]\rho^2\big(x_n, T^n x_n\big) \\
&\quad + (1-\beta_n)\rho^2\big(x_n, T^n y_n\big) + \beta_n L^2\rho^2(x_n, y_n) \\
&\leq a_n\big(1+\beta_n(a_n-1)\big)\rho^2(x_n,p) \\
&\quad + \left[a_n\beta_n - a_n\beta_n\frac{R}{2}(1-\beta_n) - \beta_n\frac{R}{2}(1-\beta_n) + \beta^3 L^2\right]\rho^2\big(x_n, T^n x_n\big) \\
&\quad + (1-\beta_n)\rho^2\big(x_n, T^n y_n\big).
\end{aligned} \tag{13}$$

Substituting (13) into (8), we obtain

$$\begin{aligned}
\rho^2(x_{n+1},p) &\leq (1-\alpha_n)\rho^2(x_n,p) + \alpha_n a_n\big(1+\beta_n(a_n-1)\big)\rho^2(x_n,p) \\
&\quad + \alpha_n\left[a_n\beta_n - a_n\beta_n\frac{R}{2}(1-\beta_n) - \beta_n\frac{R}{2}(1-\beta_n) + \beta^3 L^2\right]\rho^2\big(x_n, T^n x_n\big) \\
&\quad + \alpha_n(1-\beta_n)\rho^2\big(x_n, T^n y_n\big) - \frac{R}{2}\alpha_n(1-\alpha_n)\rho^2\big(x_n, T^n y_n\big) \\
&= \big[1+\alpha_n(a_n-1)(1+a_n\beta_n)\big]\rho^2(x_n,p) \\
&\quad - \alpha_n\beta_n\left[\frac{R}{2}(1-\beta_n)(1+a_n) - \big(a_n + L^2\beta_n^2\big)\right]\rho^2\big(x_n, T^n x_n\big) \\
&\quad - \alpha_n\left[\frac{R}{2}(1-\alpha_n) - (1-\beta_n)\right]\rho^2\big(x_n, T^n y_n\big).
\end{aligned}$$

This completes the proof. □

Lemma 3.5 *Let $\kappa > 0$ and (X,ρ) be a $\mathrm{CAT}(\kappa)$ space with $\mathrm{diam}(X) \leq \frac{\pi/2-\eta}{\sqrt{\kappa}}$ for some $\eta \in (0,\pi/2)$. Let C be a nonempty convex subset of X, and $T : C \to C$ be a uniformly L-Lipschitzian and asymptotically hemicontractive mapping with sequence $\{a_n\}$ in $[1,\infty)$ such that $\sum_{n=1}^{\infty}(a_n-1) < \infty$. Let $\{\alpha_n\},\{\beta_n\} \subset [0,1]$ be such that $\frac{1-\beta_n}{1-\alpha_n} \leq \frac{R}{2}$ where $R = (\pi-2\eta)\tan(\eta)$ and $\alpha_n,\beta_n \in [\varepsilon,b]$ for some $\varepsilon > 0$ and $b \in (0, \frac{\sqrt{R^2+4RL^2-4L^2}-R}{2L^2})$. Given $x_1 \in C$, define the iteration scheme $\{x_n\}$ by*

$$\begin{aligned}
x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n T^n y_n, \\
y_n &= (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \geq 1.
\end{aligned}$$

Then

$$\lim_{n\to\infty} \rho(x_n, Tx_n) = 0. \tag{14}$$

Proof First, we prove that $\lim_{n\to\infty} \rho(x_n, T^n x_n) = 0$. Since $\frac{1-\beta_n}{1-\alpha_n} \leq \frac{R}{2}$, by Lemma 3.4 we have

$$\begin{aligned}\rho^2(x_{n+1},p) - \rho^2(x_n,p) &\leq \alpha_n(a_n-1)(1+a_n\beta_n)\rho^2(x_n,p)\\ &\quad - \alpha_n\beta_n\left[\frac{R}{2}(1-\beta_n)(1+a_n) - \left(a_n + L^2\beta_n^2\right)\right]\rho^2\left(x_n, T^n x_n\right).\end{aligned}$$

Since $\{\alpha_n(1+a_n\beta_n)\rho^2(x_n,p)\}_{n=1}^{\infty}$ is a bounded sequence, there exists $M > 0$ such that

$$\begin{aligned}\rho^2(x_{n+1},p) - \rho^2(x_n,p) &\leq (a_n-1)M\\ &\quad - \alpha_n\beta_n\left[\frac{R}{2}(1-\beta_n)(1+a_n) - \left(a_n + L^2\beta_n^2\right)\right]\rho^2\left(x_n, T^n x_n\right).\end{aligned} \tag{15}$$

Let $D = R(1-b) - (1+L^2b^2) > 0$. Since $\lim_{n\to\infty} a_n = 1$, there exists a natural number N such that

$$\frac{R}{2}(1-\beta_n)(1+a_n) - \left(a_n + L^2\beta_n^2\right) \geq \frac{R}{2}(1-b)(1+a_n) - \left(a_n + L^2b^2\right) \geq \frac{D}{2} > 0 \tag{16}$$

for all $n \geq N$. Suppose that $\lim_{n\to\infty} \rho(x_n, T^n x_n) \neq 0$. Then there exist $\varepsilon_0 > 0$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\rho^2\left(x_{n_i}, T^{n_i}x_{n_i}\right) \geq \varepsilon_0. \tag{17}$$

Without loss of generality, we let $n_1 \geq N$. From (15), we have

$$\alpha_n\beta_n\left[\frac{R}{2}(1-\beta_n)(1+a_n) - \left(a_n + L^2\beta_n^2\right)\right]\rho^2\left(x_n, T^n x_n\right) \leq (a_n-1)M + \rho^2(x_n,p) - \rho^2(x_{n+1},p).$$

Then

$$\begin{aligned}&\sum_{l=1}^{i} \alpha_{n_l}\beta_{n_l}\left[\frac{R}{2}(1-\beta_{n_l})(1+a_{n_l}) - \left(a_{n_l} + L^2\beta_{n_l}^2\right)\right]\rho^2\left(x_{n_l}, T^{n_l}x_{n_l}\right)\\ &\quad = \sum_{m=n_1}^{n_i} \alpha_m\beta_m\left[\frac{R}{2}(1-\beta_m)(1+a_m) - \left(a_m + L^2\beta_m^2\right)\right]\rho^2\left(x_m, T^m x_m\right)\\ &\quad \leq \sum_{m=n_1}^{n_i} (a_m-1)M + \rho^2(x_{n_1},p) - \rho^2(x_{n_i+1},p).\end{aligned}$$

From this, together with (16), (17) and the fact that $\varepsilon \leq \alpha_n \leq \beta_n$, we obtain

$$i \cdot \varepsilon^2 \cdot \frac{D}{2} \cdot \varepsilon_0 \leq \sum_{m=n_1}^{n_i} (a_m-1)M + \rho^2(x_{n_1},p) - \rho^2(x_{n_i+1},p). \tag{18}$$

If we take $i \to \infty$, the right side of (18) is bounded while the left side is unbounded. This is a contradiction. Therefore $\lim_{n\to\infty} \rho(x_n, T^n x_n) = 0$, and hence $\lim_{n\to\infty} \rho(x_n, Tx_n) = 0$ by Lemma 3.1. □

Theorem 3.6 *Let $\kappa > 0$ and (X,ρ) be a* $\mathrm{CAT}(\kappa)$ *space with* $\operatorname{diam}(X) \le \frac{\pi/2-\eta}{\sqrt{\kappa}}$ *for some $\eta \in (0,\pi/2)$. Let C be a nonempty closed convex subset of X, and $T: C \to C$ be a completely continuous and uniformly L-Lipschitzian asymptotically hemicontractive mapping with sequence $\{a_n\}$ in $[1,\infty)$ such that $\sum_{n=1}^{\infty}(a_n - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ be such that $\frac{1-\beta_n}{1-\alpha_n} \le \frac{R}{2}$ where $R = (\pi - 2\eta)\tan(\eta)$ and $\alpha_n, \beta_n \in [\varepsilon, b]$ for some $\varepsilon > 0$ and $b \in (0, \frac{\sqrt{R^2+4RL^2-4L^2}-R}{2L^2})$. Given $x_1 \in C$, define the iteration scheme $\{x_n\}$ by*

$$
\begin{aligned}
x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n T^n y_n, \\
y_n &= (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \ge 1.
\end{aligned}
$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof Since T is completely continuous, $\{Tx_n\}$ has a convergent subsequence in C. By using Lemma 3.5, we can show that $\{x_n\}$ has a convergent subsequence, say $x_{n_k} \to q \in C$. Hence $q \in F(T)$ by (14) and the continuity of T. It follows from (15) and (16) that

$$
\rho^2(x_{n+1}, p) \le \rho^2(x_n, p) + (a_n - 1)M.
$$

Since $\sum_{n=1}^{\infty}(a_n - 1) < \infty$, by Lemma 2.4 we have $x_n \to q$. This completes the proof. □

As consequences of Theorem 3.6, we obtain the following.

Corollary 3.7 *Let $\kappa > 0$ and (X,ρ) be a* $\mathrm{CAT}(\kappa)$ *space with* $\operatorname{diam}(X) \le \frac{\pi/2-\eta}{\sqrt{\kappa}}$ *for some $\eta \in (0,\pi/2)$. Let C be a nonempty closed convex subset of X, and $T: C \to C$ be a completely continuous and uniformly L-Lipschitzian asymptotically demicontractive mapping with sequence $\{a_n\}$ in $[1,\infty)$ such that $\sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ be such that $\frac{1-\beta_n}{1-\alpha_n} \le \frac{R}{2}$ where $R = (\pi - 2\eta)\tan(\eta)$ and $\alpha_n, \beta_n \in [\varepsilon, b]$ for some $\varepsilon > 0$ and $b \in (0, \frac{\sqrt{R^2+4RL^2-4L^2}-R}{2L^2})$. Given $x_1 \in C$, define the iteration scheme $\{x_n\}$ by*

$$
\begin{aligned}
x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n T^n y_n, \\
y_n &= (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \ge 1.
\end{aligned}
$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Corollary 3.8 (Theorem 11 of [29]) *Let (X,ρ) be a* $\mathrm{CAT}(0)$ *space, let C be a nonempty bounded closed convex subset of X, and let $T: C \to C$ be a completely continuous and uniformly L-Lipschitzian asymptotically hemicontractive mapping with sequence $\{a_n\}$ in $[1,\infty)$ such that $\sum_{n=1}^{\infty}(a_n - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ be such that $\varepsilon \le \alpha_n \le \beta_n \le b$ for some $\varepsilon > 0$ and $b \in (0, \frac{\sqrt{1+L^2}-1}{L^2})$. Given $x_1 \in C$, define the iteration scheme $\{x_n\}$ by*

$$
\begin{aligned}
x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n T^n y_n, \\
y_n &= (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \ge 1.
\end{aligned}
$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Competing interests
The author declares that there is no conflict of interests regarding the publication of this article.

Acknowledgements
The author thanks a referee for his/her careful reading and valuable comments and suggestions which led to the present form of the paper. This research was supported by Chiang Mai University.

Received: 5 August 2015 Accepted: 8 October 2015 Published online: 15 October 2015

References

1. Kirk, WA: Geodesic geometry and fixed point theory. In: Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003). Colecc. Abierta, vol. 64, pp. 195-225. Univ. Sevilla Secr. Publ., Seville (2003)
2. Kirk, WA: Geodesic geometry and fixed point theory II. In: International Conference on Fixed Point Theory and Applications, pp. 113-142. Yokohama Publ., Yokohama (2004)
3. Dhompongsa, S, Kaewkhao, A, Panyanak, B: Lim's theorems for multivalued mappings in CAT(0) spaces. J. Math. Anal. Appl. **312**, 478-487 (2005)
4. Dhompongsa, S, Kirk, WA, Sims, B: Fixed points of uniformly Lipschitzian mappings. Nonlinear Anal. **65**, 762-772 (2006)
5. Chaoha, P, Phon-on, A: A note on fixed point sets in CAT(0) spaces. J. Math. Anal. Appl. **320**, 983-987 (2006)
6. Fujiwara, K, Nagano, K, Shioya, T: Fixed point sets of parabolic isometries of CAT(0)-spaces. Comment. Math. Helv. **81**, 305-335 (2006)
7. Leustean, L: A quadratic rate of asymptotic regularity for CAT(0) spaces. J. Math. Anal. Appl. **325**, 386-399 (2007)
8. Shahzad, N, Markin, J: Invariant approximations for commuting mappings in CAT(0) and hyperconvex spaces. J. Math. Anal. Appl. **337**, 1457-1464 (2008)
9. Dhompongsa, S, Panyanak, B: On Δ-convergence theorems in CAT(0) spaces. Comput. Math. Appl. **56**, 2572-2579 (2008)
10. Espinola, R, Fernandez-Leon, A: CAT(k)-Spaces, weak convergence and fixed points. J. Math. Anal. Appl. **353**, 410-427 (2009)
11. Shahzad, N: Fixed point results for multimaps in CAT(0) spaces. Topol. Appl. **156**, 997-1001 (2009)
12. Razani, A, Salahifard, H: Invariant approximation for CAT(0) spaces. Nonlinear Anal. **72**, 2421-2425 (2010)
13. Saejung, S: Halpern's iteration in CAT(0) spaces. Fixed Point Theory Appl. **2010**, Article ID 471781 (2010)
14. Kakavandi, BA, Amini, M: Duality and subdifferential for convex functions on complete CAT(0) metric spaces. Nonlinear Anal. **73**, 3450-3455 (2010)
15. Abkar, A, Eslamian, E: Common fixed point results in CAT(0) spaces. Nonlinear Anal. **74**, 1835-1840 (2011)
16. Khan, AR, Khamsi, MA, Fukhar-ud-din, H: Strong convergence of a general iteration scheme in CAT(0) spaces. Nonlinear Anal. **74**, 783-791 (2011)
17. Cho, YJ, Ciric, L, Wang, SH: Convergence theorems for nonexpansive semigroups in CAT(0) spaces. Nonlinear Anal. **74**, 6050-6059 (2011)
18. Lin, LJ, Chuang, CS, Yu, ZT: Fixed point theorems and Δ-convergence theorems for generalized hybrid mappings on CAT(0) spaces. Fixed Point Theory Appl. **2011**, Article ID 49 (2011)
19. Piatek, B: Halpern iteration in CAT(κ) spaces. Acta Math. Sin. Engl. Ser. **27**, 635-646 (2011)
20. Abbas, M, Kadelburg, Z, Sahu, DR: Fixed point theorems for Lipschitzian type mappings in CAT(0) spaces. Math. Comput. Model. **55**, 1418-1427 (2012)
21. Bacak, M, Searston, I, Sims, B: Alternating projections in CAT(0) spaces. J. Math. Anal. Appl. **385**, 599-607 (2012)
22. Chang, SS, Wang, L, Lee, HWJ, Chan, CK, Yang, L: Demiclosed principle and Δ-convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces. Appl. Math. Comput. **219**, 2611-2617 (2012)
23. He, JS, Fang, DH, Lopez, G, Li, C: Mann's algorithm for nonexpansive mappings in CAT(κ) spaces. Nonlinear Anal. **75**, 445-452 (2012)
24. Kimura, Y, Sato, K: Convergence of subsets of a complete geodesic space with curvature bounded above. Nonlinear Anal. **75**, 5079-5085 (2012)
25. Wangkeeree, R, Preechasilp, P: Viscosity approximation methods for nonexpansive mappings in CAT(0) spaces. J. Inequal. Appl. **2013**, Article ID 93 (2013)
26. Goebel, K, Kirk, WA: A fixed point theorem for transformations whose iterates have uniform Lipschitz constant. Stud. Math. **47**, 137-140 (1973)
27. Schu, J: Iterative construction of fixed points of asymptotically nonexpansive mappings. J. Math. Anal. Appl. **158**, 407-413 (1991)
28. Qihou, L: Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings. Nonlinear Anal. **26**, 1835-1842 (1996)
29. Kim, KS: Some convergence theorems for contractive type mappings in CAT(0) spaces. Abstr. Appl. Anal. **2013**, Article ID 381715 (2013)
30. Bridson, M, Haefliger, A: Metric Spaces of Non-Positive Curvature. Springer, Berlin (1999)
31. Kirk, WA, Shahzad, N: Fixed Point Theory in Distance Spaces. Springer, Cham (2014)
32. Bruhat, F, Tits, J: Groupes réductifs sur un corps local. I. Données radicielles valuées. Publ. Math. Inst. Hautes Études Sci. **41**, 5-251 (1972)
33. Ohta, S: Convexities of metric spaces. Geom. Dedic. **125**, 225-250 (2007)
34. Ishikawa, S: Fixed points by a new iteration method. Proc. Am. Math. Soc. **44**, 147-150 (1974)
35. Mann, WR: Mean value methods in iteration. Proc. Am. Math. Soc. **4**, 506-510 (1953)