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Extra-gradient methods for solving split feasibility and fixed point problems

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Abstract

The purpose of this paper is to study the extra-gradient methods for solving split feasibility and fixed point problems involved in pseudo-contractive mappings in real Hilbert spaces. We propose an Ishikawa-type extra-gradient iterative algorithm for finding a solution of the split feasibility and fixed point problems involved in pseudo-contractive mappings with Lipschitz assumption. Moreover, we also suggest a Mann-type extra-gradient iterative algorithm for finding a solution of the split feasibility and fixed point problems involved in pseudo-contractive mappings without Lipschitz assumption. It is proven that under suitable conditions, the sequences generated by the proposed iterative algorithms converge weakly to a solution of the split feasibility and fixed point problems. The results presented in this paper extend and improve some corresponding ones in the literature.

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Keywords: Ishikawa-type iterative algorithm; Mann-type iterative algorithm; extra-gradient methods; split feasibility problems; fixed point problems; pseudo-contractive mappings

1 Introduction

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$ be two nonempty closed convex sets. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with its adjoint A^*. Let $S : \mathcal{H}_2 \to \mathcal{H}_2$ and $T : \mathcal{H}_1 \to \mathcal{H}_1$ be two nonlinear mappings.

The purpose of this paper is to study the following split feasibility and fixed point problems:

$$\text{Find } x^* \in C \cap \operatorname{Fix}(T) \text{ such that } Ax^* \in Q \cap \operatorname{Fix}(S). \tag{1.1}$$

We use Γ to denote the set of solutions of (1.1), that is,

$$\Gamma = \{x^* : x^* \in C \cap \operatorname{Fix}(T), Ax^* \in Q \cap \operatorname{Fix}(S)\}.$$

In the sequel, we assume $\Gamma \neq \emptyset$.

A special case of the split feasibility and fixed point problems is the split feasibility problem (SFP):

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q. \tag{1.2}$$

We use Γ_0 to denote the set of solutions of (1.2), that is,

$$\Gamma_0 = \{x^* : x^* \in C, Ax^* \in Q\}.$$

The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. Recently, it has been found that the SFP can be applied to study intensity-modulated radiation therapy (IMRT) [3–5]. In the recent past, a wide variety of iterative algorithms have been used in signal processing and image reconstruction and for solving the SFP; see, for example, [2, 4, 6–10] and the references therein (see also [11–13] for relevant projection methods for solving image recovery problems).

The original algorithm given in [1] involves the computation of the inverse A^{-1} (assuming the existence of the inverse of A), and thus does not become popular. A seemingly more popular algorithm that solves the SFP is the CQ algorithm presented by Byrne [2]:

$$x_{n+1} = P_C\bigl(I - \gamma A^*(I - P_Q)A\bigr)x_n, \quad n \geq 0, \tag{1.3}$$

where the initial guess $x_0 \in \mathcal{H}_1$ and $\gamma \in (0, \frac{2}{\lambda})$, with λ being the largest eigenvalue of the matrix A^*A. Algorithm (1.3) is found to be a gradient-projection method (GPM) in convex minimization. It is also a special case of the proximal forward-backward splitting method [14]. The CQ algorithm only involves the computations of the projections P_C and P_Q onto the sets C and Q, respectively.

Many authors have also made a continuation of the study on the CQ algorithm and its variant form, refer to [15–21]. In 2010, Xu [15] applied a Mann-type iterative algorithm to the SFP and proposed an average CQ algorithm which was proven to be weakly convergent to a solution of the SFP. He derived a weak convergence result, which shows that for suitable choices of iterative paraments, the sequence of iterative algorithm solutions can converge weakly to an exact solution of the SFP.

On the other hand, in 1976, to study the saddle point problem, Korpelevich [22] introduced the so-called extra-gradient method:

$$\begin{cases} y_n = P_C(x_n - \lambda \mathcal{A} x_n), \\ x_{n+1} = P_C(x_n - \lambda \mathcal{A} y_n), \quad n \geq 0, \end{cases}$$

where $\lambda > 0$, operator $\mathcal{A}$ is both strongly monotone and Lipschitz continuous.

Very recently, Ceng *et al.* [23] studied extra-gradient method for finding a common element of the solution set Γ_0 of the SFP and the set $\mathrm{Fix}(S)$ of fixed points of a nonexpansive mapping S in the setting of infinite-dimensional Hilbert spaces. Motivated and inspired by Nadezhkina and Takahashi [24], the authors proposed an iterative algorithm in the following manner:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = P_C(I - \lambda_n \nabla f)x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S P_C(x_n - \lambda_n \nabla f(y_n)), \quad n \geq 0, \end{cases} \tag{1.4}$$

where $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0, \frac{1}{\|A\|^2})$ and $\{\beta_n\} \subset [c,d]$ for some $c,d \in (0,1)$. The authors proved that the sequences generated by (1.4) converge weakly to an element $x \in \Gamma_0 \cap \operatorname{Fix}(S)$.

In 2014, Yao *et al.* [25] studied the split feasibility and fixed point problems. They constructed an iterative algorithm in the following way:

$$\begin{cases} u_n = P_C(\alpha_n u + (1-\alpha_n)(x_n - \delta A^*(I - SP_Q)Ax_n)), \\ x_{n+1} = (1-\beta_n)u_n + \beta_n T((1-\gamma_n)u_n + \gamma_n Tu_n), \quad n \geq 0, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three real number sequences in $(0,1)$ and δ is a constant in $(0, \frac{1}{\|A\|^2})$. The authors proved that the sequences generated by (1.5) converge strongly to a solution of the split feasibility and fixed point problems.

In this paper, motivated by the work of Ceng *et al.* [23], Yao *et al.* [25], we propose an Ishikawa-type extra-gradient iterative algorithm for finding a solution of the split feasibility and fixed point problems involved in pseudo-contractive mappings with Lipschitz assumption. On the other hand, we also suggest a Mann-type extra-gradient iterative algorithm for finding a solution of the split feasibility and fixed point problems involved in pseudo-contractive mappings without Lipschitz assumption. We establish weak convergence theorems for the sequences generated by the proposed iterative algorithms. Our results substantially improve and develop the corresponding results in [15, 23–25]; for example, [15], Theorem 3.6, [23], Theorem 3.2, [24], Theorem 3.1 and [25], Theorem 3.2. It is noteworthy that our results are new and novel.

2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of $\mathcal{H}$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, that is,

$$\omega_w(x_n) = \{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Projections are an important tool for our work in this paper. Recall that the (nearest point or metric) projection from $\mathcal{H}$ onto C, denoted by P_C, is defined in such a way that, for each $x \in \mathcal{H}$, $P_C x$ is the unique point in C with the property

$$\|x - P_C x\| = \min\{\|x - y\| : y \in C\}.$$

Some important properties of projections are gathered in the following proposition.

Proposition 2.1 *Given $x \in \mathcal{H}$ and $z \in C$,*

(1) $z = P_C x \Leftrightarrow \langle x - z, y - z \rangle \leq 0$ *for all* $y \in C$;
(2) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$ *for all* $y \in C$;
(3) $\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2$ *for all* $y \in \mathcal{H}$, *which hence implies that* P_C *is nonexpansive.*

We also need other sorts of nonlinear operators which are stated as follows.

Definition 2.1 A nonlinear operator $T:\mathcal{H}\to\mathcal{H}$ is said to be

(1) L-Lipschitzian if there exists $L>0$ such that

$$\|Tx-Ty\|\le L\|x-y\|,\quad \forall x,y\in\mathcal{H};$$

if $L=1$, we call T nonexpansive;

(2) firmly nonexpansive if $2T-I$ is nonexpansive, or equivalently,

$$\langle x-y,Tx-Ty\rangle\ge\|Tx-Ty\|^2,\quad \forall x,y\in\mathcal{H};$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T=\frac{1}{2}(I+S),$$

where $S:\mathcal{H}\to\mathcal{H}$ is nonexpansive;

(3) monotone if

$$\langle x-y,Tx-Ty\rangle\ge 0,\quad \forall x,y\in\mathcal{H};$$

(4) β-strongly monotone, with $\beta>0$, if

$$\langle x-y,Tx-Ty\rangle\ge\beta\|x-y\|^2,\quad \forall x,y\in\mathcal{H};$$

(5) ν-inverse strongly monotone (ν-ism), with $\nu>0$, if

$$\langle x-y,Tx-Ty\rangle\ge\nu\|Tx-Ty\|^2,\quad \forall x,y\in\mathcal{H}.$$

Inverse strongly monotone (also referred to as co-coercive) operators have been widely applied in solving practical problems in various fields, for instance, in traffic assignment problems; see, for example, [26, 27].

It is well known that metric projection $P_C:\mathcal{H}\to C$ is firmly nonexpansive, that is,

$$\begin{aligned}&\langle x-y,P_Cx-P_Cy\rangle\ge\|P_Cx-P_Cy\|^2\\ &\quad\Leftrightarrow\quad \|P_Cx-P_Cy\|^2\le\|x-y\|^2-\big\|(I-P_C)x-(I-P_C)y\big\|^2,\quad \forall x,y\in\mathcal{H}.\end{aligned}\tag{2.1}$$

For all $x,y\in\mathcal{H}$, the following conclusions hold:

$$\big\|tx+(1-t)y\big\|^2=t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|^2,\quad t\in[0,1]\tag{2.2}$$

and

$$\|x+y\|^2=\|x\|^2+2\langle x,y\rangle+\|y\|^2.\tag{2.3}$$

On the other hand, in a real Hilbert space $\mathcal{H}$, a mapping $T:C\to C$ is called *pseudo-contractive* if

$$\langle Tx-Ty,x-y\rangle\le\|x-y\|^2,\quad \forall x,y\in C.$$

It is well known that T is a pseudo-contractive mapping if and only if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{2.4}$$

The notation $\mathrm{Fix}(T)$ denotes the set of fixed points of the mapping T, that is, $\mathrm{Fix}(T) = \{x \in \mathcal{H} : Tx = x\}$.

Proposition 2.2 [6] *Let $T : \mathcal{H} \to \mathcal{H}$ be a given mapping.*
(1) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.*
(2) *If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.*
(3) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0,1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

Proposition 2.3 *Let T be a pseudo-contractive mapping with the nonempty fixed point set* $\mathrm{Fix}(T)$, *then the following conclusion holds*:

$$\langle Ty - y, Ty - x^* \rangle \le \|Ty - y\|^2, \quad \forall y \in C, \forall x^* \in \mathrm{Fix}(T).$$

Proof From the definition of a pseudo-contractive mapping T, we have

$$\begin{aligned} \langle Ty - y, Ty - x^* \rangle &= \|Ty - y\|^2 + \langle Ty - y, y - x^* \rangle \\ &= \|Ty - y\|^2 + \langle Ty - x^*, y - x^* \rangle - \|y - x^*\|^2 \\ &\le \|Ty - y\|^2. \end{aligned}$$
□

Generally speaking, pseudo-contractive mappings are assumed to be L-Lipschitzian with $L > 1$. Next, to overcome the L-Lipschitzian property, we assume that the pseudo-contractive mapping T satisfies the following condition:

$$\langle Ty - y, Ty - x^* \rangle \le 0, \quad \forall y \in C, \forall x^* \in \mathrm{Fix}(T). \tag{2.5}$$

The following demiclosedness principle for pseudo-contractive mappings will often be used in the sequel.

Lemma 2.1 [28] *Let $\mathcal{H}$ be a real Hilbert space, C be a closed convex subset of $\mathcal{H}$. Let $T : C \to C$ be a continuous pseudo-contractive mapping. Then*
(1) $\mathrm{Fix}(T)$ *is a closed convex subset of C*;
(2) $(I - T)$ *is demiclosed at zero.*

The following result is useful when we prove weak convergence of a sequence.

Lemma 2.2 [29] *Let $\mathcal{H}$ be a Hilbert space and $\{x_n\}$ be a bounded sequence in $\mathcal{H}$ such that there exists a nonempty closed convex set C of $\mathcal{H}$ satisfying*:
(1) *for every $w \in C$, $\lim_{n\to\infty} \|x_n - w\|$ exists*;
(2) *each weak-cluster point of the sequence $\{x_n\}$ is in C.*
Then $\{x_n\}$ converges weakly to a point in C.

We can use fixed point algorithms to solve the SFP on the basis of the following observation.

Let $\lambda > 0$ and assume that x^* solves the SFP. Then $Ax^* \in Q$, which implies that $(I - P_Q)Ax^* = 0$, and thus, $\lambda(I - P_Q)Ax^* = 0$. Hence, we have the fixed point equation $x^* = (I - \lambda A^*(I - P_Q)A)x^*$. Requiring that $x^* \in C$, we consider the fixed point equation

$$x^* = P_C\bigl(I - \lambda A^*(I - P_Q)A\bigr)x^* = P_C(I - \lambda\nabla f)x^*. \tag{2.6}$$

It is proven in [15] that the solutions of the fixed point equation (2.6) are exactly the solutions of the SFP; namely, for given $x^* \in \mathcal{H}_1$, x^* solves the SFP if and only if x^* solves the fixed point equation (2.6).

3 Ishikawa-type extra-gradient iterative algorithm involved in pseudo-contractive mappings with Lipschitz assumption

We are now in a position to propose an Ishikawa-type extra-gradient iterative algorithm for solving the split feasibility and fixed point problems involved in pseudo-contractive mappings with Lipschitz assumption.

Theorem 3.1 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$ be two nonempty closed convex sets. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $S : Q \to Q$ be a nonexpansive mapping and let $T : C \to C$ be an L-Lipschitzian pseudo-contractive mapping with $L > 1$. For $x_0 \in \mathcal{H}_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by the following Ishikawa-type extra-gradient iterative algorithm*:

$$\begin{cases} y_n = P_C(x_n - \lambda_n A^*(I - SP_Q)Ax_n), \\ z_n = P_C(x_n - \lambda_n A^*(I - SP_Q)Ay_n), \\ w_n = (1 - \alpha_n)z_n + \alpha_n Tz_n, \\ x_{n+1} = (1 - \beta_n)z_n + \beta_n Tw_n, \quad n \geq 0, \end{cases} \tag{3.1}$$

where $\{\lambda_n\} \subset (0, \frac{1}{2\|A\|^2})$ and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ such that $0 < a < \beta_n < c < \alpha_n < b < \frac{1}{\sqrt{1+L^2}+1}$.

Then the sequence $\{x_n\}$ generated by algorithm (3.1) *converges weakly to an element of* Γ.

Proof Taking $x^* \in \Gamma$, we have $x^* \in C \cap \operatorname{Fix}(T)$ and $Ax^* \in Q \cap \operatorname{Fix}(S)$. For simplicity, we write $\nabla f^S = A^*(I - SP_Q)A$, $v_n = P_Q Ax_n$, $u_n = x_n - \lambda_n A^*(I - SP_Q)Ax_n$ for all $n \geq 0$. Thus, we have $y_n = P_C u_n$ for all $n \geq 0$. By (2.1), we get

$$\begin{aligned} \|Sv_n - Ax^*\|^2 &= \|SP_Q Ax_n - SP_Q Ax^*\|^2 \\ &\leq \|P_Q Ax_n - P_Q Ax^*\|^2 \\ &\leq \|Ax_n - Ax^*\|^2 - \|v_n - Ax_n\|^2. \end{aligned} \tag{3.2}$$

Since P_C is nonexpansive, using (2.3), we get

$$\begin{aligned} \|y_n - x^*\|^2 &= \|P_C u_n - x^*\|^2 \leq \|u_n - x^*\|^2 \\ &= \|x_n - x^*\|^2 + 2\lambda_n\langle x_n - x^*, A^*(Sv_n - Ax_n)\rangle + \lambda_n^2\|A^*(Sv_n - Ax_n)\|^2. \end{aligned} \tag{3.3}$$

Since A is a linear operator with its adjoint A^*, we have

$$\begin{aligned}\langle x_n - x^*, A^*(Sv_n - Ax_n)\rangle &= \langle Ax_n - Ax^*, Sv_n - Ax_n\rangle \\ &= \langle Ax_n - Ax^* + Sv_n - Ax_n - (Sv_n - Ax_n), Sv_n - Ax_n\rangle \\ &= \langle Sv_n - Ax^*, Sv_n - Ax_n\rangle - \|Sv_n - Ax_n\|^2. \end{aligned} \tag{3.4}$$

Again using (2.3), we obtain

$$\langle Sv_n - Ax^*, Sv_n - Ax_n\rangle = \frac{1}{2}\left(\|Sv_n - Ax^*\|^2 + \|Sv_n - Ax_n\|^2 - \|Ax_n - Ax^*\|^2\right). \tag{3.5}$$

From (3.2), (3.4) and (3.5), we get

$$\begin{aligned}&\langle x_n - x^*, A^*(Sv_n - Ax_n)\rangle \\ &\quad= \frac{1}{2}\left(\|Sv_n - Ax^*\|^2 + \|Sv_n - Ax_n\|^2 - \|Ax_n - Ax^*\|^2\right) - \|Sv_n - Ax_n\|^2 \\ &\quad\le \frac{1}{2}\left(\|Ax_n - Ax^*\|^2 - \|v_n - Ax_n\|^2 + \|Sv_n - Ax_n\|^2 - \|Ax_n - Ax^*\|^2\right) \\ &\qquad - \|Sv_n - Ax_n\|^2 \\ &\quad= -\frac{1}{2}\|v_n - Ax_n\|^2 - \frac{1}{2}\|Sv_n - Ax_n\|^2. \end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.3) and by the assumption of $\{\lambda_n\}$, we deduce

$$\begin{aligned}&\|y_n - x^*\|^2 \\ &\quad\le \|x_n - x^*\|^2 + 2\lambda_n\left(-\frac{1}{2}\|v_n - Ax_n\|^2 - \frac{1}{2}\|Sv_n - Ax_n\|^2\right) + \lambda_n^2\|A\|^2\|Sv_n - Ax_n\|^2 \\ &\quad= \|x_n - x^*\|^2 - \lambda_n\|v_n - Ax_n\|^2 - \lambda_n\left(1 - \lambda_n\|A\|^2\right)\|Sv_n - Ax_n\|^2 \\ &\quad\le \|x_n - x^*\|^2. \end{aligned} \tag{3.7}$$

Since S and P_Q are nonexpansive, we know that composition operator SP_Q is still nonexpansive. By Proposition 2.2(1) the complement $I - SP_Q$ is $\frac{1}{2}$-ism. Therefore, it is easy to see that $\nabla f^S = A^*(I - SP_Q)A$ is $\frac{1}{2\|A\|^2}$-ism, that is,

$$\langle x - y, \nabla f^S(x) - \nabla f^S(y)\rangle \ge \frac{1}{2\|A\|^2}\|\nabla f^S(x) - \nabla f^S(y)\|^2. \tag{3.8}$$

This together with Proposition 2.1(2) implies that

$$\begin{aligned}&\|z_n - x^*\|^2 \\ &\quad\le \|x_n - \lambda_n\nabla f^S(y_n) - x^*\|^2 - \|x_n - \lambda_n\nabla f^S(y_n) - z_n\|^2 \\ &\quad= \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\langle\nabla f^S(y_n), x^* - z_n\rangle \\ &\quad= \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\left(\langle\nabla f^S(y_n) - \nabla f^S(x^*), x^* - y_n\rangle\right. \\ &\qquad \left. + \langle\nabla f^S(x^*), x^* - y_n\rangle + \langle\nabla f^S(y_n), y_n - z_n\rangle\right) \end{aligned}$$

$$
\begin{aligned}
&\le \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\langle \nabla f^S(y_n), y_n - z_n\rangle \\
&= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 + 2\langle x_n - \lambda_n \nabla f^S(y_n) - y_n, z_n - y_n\rangle.
\end{aligned}
$$

Further, by Proposition 2.1(1) and (3.8), we have

$$
\begin{aligned}
&\langle x_n - \lambda_n \nabla f^S(y_n) - y_n, z_n - y_n\rangle \\
&\quad = \langle x_n - \lambda_n \nabla f^S(x_n) - y_n, z_n - y_n\rangle + \lambda_n\langle \nabla f^S(x_n) - \nabla f^S(y_n), z_n - y_n\rangle \\
&\quad \le \lambda_n\langle \nabla f^S(x_n) - \nabla f^S(y_n), z_n - y_n\rangle \\
&\quad \le \lambda_n \|\nabla f^S(x_n) - \nabla f^S(y_n)\| \|z_n - y_n\| \\
&\quad \le 2\lambda_n \|A\|^2 \|x_n - y_n\| \|z_n - y_n\|.
\end{aligned}
$$

By the assumption of $\{\lambda_n\}$, we obtain

$$
\begin{aligned}
&\|z_n - x^*\|^2 \\
&\quad \le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 + 2\langle x_n - \lambda_n \nabla f^S(y_n) - y_n, z_n - y_n\rangle \\
&\quad \le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 + 4\lambda_n \|A\|^2 \|x_n - y_n\| \|z_n - y_n\| \\
&\quad \le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 + \|z_n - y_n\|^2 + 4\lambda_n^2 \|A\|^4 \|x_n - y_n\|^2 \\
&\quad = \|x_n - x^*\|^2 - (1 - 4\lambda_n^2\|A\|^4)\|x_n - y_n\|^2 \\
&\quad \le \|x_n - x^*\|^2. \qquad (3.9)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\|z_n - x^*\|^2 &\le \|x_n - x^*\|^2 - (1 - 4\lambda_n^2\|A\|^4)\|z_n - y_n\|^2 \\
&\le \|x_n - x^*\|^2.
\end{aligned}
$$

From (2.4), we have

$$
\|Tz_n - x^*\|^2 \le \|z_n - x^*\|^2 + \|z_n - Tz_n\|^2 \qquad (3.10)
$$

and

$$
\begin{aligned}
\|Tw_n - x^*\|^2 &= \|T((1-\alpha_n)z_n + \alpha_n Tz_n) - x^*\|^2 \\
&\le \|(1-\alpha_n)(z_n - x^*) + \alpha_n(Tz_n - x^*)\|^2 \\
&\quad + \|(1-\alpha_n)z_n + \alpha_n Tz_n - T((1-\alpha_n)z_n + \alpha_n Tz_n)\|^2. \qquad (3.11)
\end{aligned}
$$

Applying equality (2.2), we have

$$
\begin{aligned}
&\|(1-\alpha_n)z_n + \alpha_n Tz_n - T((1-\alpha_n)z_n + \alpha_n Tz_n)\|^2 \\
&\quad = \|(1-\alpha_n)(z_n - T((1-\alpha_n)z_n + \alpha_n Tz_n)) + \alpha_n(Tz_n - T((1-\alpha_n)z_n + \alpha_n Tz_n))\|^2
\end{aligned}
$$

$$\begin{aligned}
&= (1-\alpha_n)\big\|z_n - T\big((1-\alpha_n)z_n + \alpha_n Tz_n\big)\big\|^2 + \alpha_n\big\|Tz_n - T\big((1-\alpha_n)z_n + \alpha_n Tz_n\big)\big\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|z_n - Tz_n\|^2. \qquad (3.12)
\end{aligned}$$

Since T is L-Lipschitzian and $z_n - ((1-\alpha_n)z_n + \alpha_n Tz_n) = \alpha_n(z_n - Tz_n)$, by (3.12), we have

$$\begin{aligned}
&\big\|(1-\alpha_n)z_n + \alpha_n Tz_n - T\big((1-\alpha_n)z_n + \alpha_n Tz_n\big)\big\|^2 \\
&\quad \le (1-\alpha_n)\big\|z_n - T\big((1-\alpha_n)z_n + \alpha_n Tz_n\big)\big\|^2 + \alpha_n^3 L^2\|z_n - Tz_n\|^2 \\
&\qquad - \alpha_n(1-\alpha_n)\|z_n - Tz_n\|^2 \\
&\quad = (1-\alpha_n)\big\|z_n - T\big((1-\alpha_n)z_n + \alpha_n Tz_n\big)\big\|^2 + \big(\alpha_n^3 L^2 + \alpha_n^2 - \alpha_n\big)\|z_n - Tz_n\|^2. \qquad (3.13)
\end{aligned}$$

By (2.2) and (3.10), we have

$$\begin{aligned}
&\big\|(1-\alpha_n)\big(z_n - x^*\big) + \alpha_n\big(Tz_n - x^*\big)\big\|^2 \\
&\quad = (1-\alpha_n)\big\|z_n - x^*\big\|^2 + \alpha_n\big\|Tz_n - x^*\big\|^2 - \alpha_n(1-\alpha_n)\|z_n - Tz_n\|^2 \\
&\quad \le (1-\alpha_n)\big\|z_n - x^*\big\|^2 + \alpha_n\big(\big\|z_n - x^*\big\|^2 + \|z_n - Tz_n\|^2\big) - \alpha_n(1-\alpha_n)\|z_n - Tz_n\|^2 \\
&\quad = \big\|z_n - x^*\big\|^2 + \alpha_n^2\|z_n - Tz_n\|^2. \qquad (3.14)
\end{aligned}$$

From (3.11), (3.13) and (3.14), we deduce

$$\begin{aligned}
\big\|Tw_n - x^*\big\|^2 &= \big\|T\big((1-\alpha_n)z_n + \alpha_n Tz_n\big) - x^*\big\|^2 \\
&\le \big\|z_n - x^*\big\|^2 + (1-\alpha_n)\big\|z_n - T\big((1-\alpha_n)z_n + \alpha_n Tz_n\big)\big\|^2 \\
&\quad - \alpha_n\big(1 - 2\alpha_n - \alpha_n^2 L^2\big)\|z_n - Tz_n\|^2. \qquad (3.15)
\end{aligned}$$

Since $\alpha_n < b < \frac{1}{\sqrt{1+L^2}+1}$, we derive that

$$1 - 2\alpha_n - \alpha_n^2 L^2 > 0, \quad n \ge 0.$$

This together with (3.15) implies that

$$\begin{aligned}
\big\|Tw_n - x^*\big\|^2 &= \big\|T\big((1-\alpha_n)z_n + \alpha_n Tz_n\big) - x^*\big\|^2 \\
&\le \big\|z_n - x^*\big\|^2 + (1-\alpha_n)\big\|z_n - T\big((1-\alpha_n)z_n + \alpha_n Tz_n\big)\big\|^2. \qquad (3.16)
\end{aligned}$$

By (2.2), (3.1) and (3.16), we have

$$\begin{aligned}
\big\|x_{n+1} - x^*\big\|^2 &= \big\|(1-\beta_n)z_n + \beta_n Tw_n - x^*\big\|^2 \\
&= \big\|(1-\beta_n)z_n + \beta_n T\big((1-\alpha_n)z_n + \alpha_n Tz_n\big) - x^*\big\|^2 \\
&= (1-\beta_n)\big\|z_n - x^*\big\|^2 + \beta_n\big\|T\big((1-\alpha_n)z_n + \alpha_n Tz_n\big) - x^*\big\|^2 \\
&\quad - \beta_n(1-\beta_n)\big\|z_n - T\big((1-\alpha_n)z_n + \alpha_n Tz_n\big)\big\|^2 \\
&\le \big\|z_n - x^*\big\|^2 - \beta_n(\alpha_n - \beta_n)\big\|z_n - T\big((1-\alpha_n)z_n + \alpha_n Tz_n\big)\big\|^2 \\
&\le \big\|z_n - x^*\big\|^2. \qquad (3.17)
\end{aligned}$$

This together with (3.9) implies that

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|$$

for every $x^* \in \Gamma$ and for all $n \geq 0$. Thus, $\{x_n\}$ generated by algorithm (3.1) is the Féjer-monotone with respect to Γ. So, we obtain $\lim_{n\to\infty} \|x_n - x^*\|$ exists immediately, this implies that $\{x_n\}$ is bounded, the sequence $\{\|x_n - x^*\|\}$ is monotonically decreasing. Additionally, we get the boundedness of $\{y_n\}$ and $\{z_n\}$ from (3.7) and (3.9) immediately.

Returning to (3.9) and (3.17), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \|z_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - \left(1 - 4\lambda_n^2\|A\|^4\right)\|x_n - y_n\|^2. \end{aligned}$$

Hence,

$$\left(1 - 4\lambda_n^2\|A\|^4\right)\|x_n - y_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2,$$

which implies that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.18}$$

Similarly, we have

$$\lim_{n\to\infty} \|z_n - y_n\| = 0.$$

From (3.7) and (3.18), we have

$$\begin{aligned} &\lambda_n\|v_n - Ax_n\|^2 + \lambda_n\left(1 - \lambda_n\|A\|^2\right)\|Sv_n - Ax_n\|^2 \\ &\quad \leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 \\ &\quad \leq \left(\|x_n - x^*\| + \|y_n - x^*\|\right)\|x_n - y_n\|, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|v_n - Ax_n\| = \lim_{n\to\infty} \|Sv_n - Ax_n\| = 0.$$

So,

$$\lim_{n\to\infty} \|v_n - Sv_n\| = 0.$$

From (3.9) and (3.17), we deduce

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \|z_n - x^*\|^2 - \beta_n(\alpha_n - \beta_n)\left\|z_n - T\left((1-\alpha_n)z_n + \alpha_n Tz_n\right)\right\|^2 \\ &\leq \|x_n - x^*\|^2 - \beta_n(\alpha_n - \beta_n)\left\|z_n - T\left((1-\alpha_n)z_n + \alpha_n Tz_n\right)\right\|^2. \end{aligned}$$

It follows that

$$\beta_n(\alpha_n - \beta_n)\|z_n - T((1-\alpha_n)z_n + \alpha_n Tz_n)\|^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

Therefore,

$$\lim_{n\to\infty} \|z_n - T((1-\alpha_n)z_n + \alpha_n Tz_n)\| = 0. \tag{3.19}$$

Observe that

$$\begin{aligned}\|z_n - Tz_n\| &\le \|z_n - T((1-\alpha_n)z_n + \alpha_n Tz_n)\| + \|T((1-\alpha_n)z_n + \alpha_n Tz_n) - Tz_n\| \\ &\le \|z_n - T((1-\alpha_n)z_n + \alpha_n Tz_n)\| + \alpha_n L\|z_n - Tz_n\|.\end{aligned}$$

Thus,

$$\|z_n - Tz_n\| \le \frac{1}{1-\alpha_n L}\|z_n - T((1-\alpha_n)z_n + \alpha_n Tz_n)\|.$$

This together with (3.19) implies that

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0.$$

Using the firm nonexpansiveness of P_C, (2.1) and (3.3), we have

$$\begin{aligned}\|y_n - x^*\|^2 = \|P_C u_n - x^*\|^2 &\le \|u_n - x^*\|^2 - \|P_C u_n - u_n\|^2 \\ &\le \|x_n - x^*\|^2 - \|y_n - u_n\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\|y_n - u_n\|^2 &\le \|x_n - x^*\|^2 - \|y_n - x^*\|^2 \\ &\le (\|x_n - x^*\| + \|y_n - x^*\|)\|x_n - y_n\|.\end{aligned}$$

From (3.18), we deduce

$$\lim_{n\to\infty} \|y_n - u_n\| = 0.$$

Since the sequence $\{x_n\}$ is bounded, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \hat{x}$. Consequently, we derive from the above conclusions that

$$\begin{cases} x_{n_i} \rightharpoonup \hat{x}, \\ y_{n_i} \rightharpoonup \hat{x}, \\ u_{n_i} \rightharpoonup \hat{x}, \\ z_{n_i} \rightharpoonup \hat{x} \end{cases} \quad \text{and} \quad \begin{cases} Ax_{n_i} \rightharpoonup A\hat{x}, \\ v_{n_i} \rightharpoonup A\hat{x}. \end{cases} \tag{3.20}$$

Applying Lemma 2.1, we deduce

$$\hat{x} \in \mathrm{Fix}(T) \quad \text{and} \quad A\hat{x} \in \mathrm{Fix}(S).$$

Note that $y_{n_i} = P_C u_{n_i} \in C$ and $v_{n_i} = P_Q A x_{n_i}$. From (3.20), we deduce

$$\hat{x} \in C \quad \text{and} \quad A\hat{x} \in Q.$$

To this end, we deduce

$$\hat{x} \in C \cap \operatorname{Fix}(T) \quad \text{and} \quad A\hat{x} \in Q \cap \operatorname{Fix}(S).$$

That is to say, $\hat{x} \in \Gamma$. This shows that $\omega_w(x_n) \subset \Gamma$. Since the $\lim_{n\to\infty} \|x_n - x^*\|$ exists for every $x^* \in \Gamma$, the weak convergence of the whole sequence $\{x_n\}$ follows by applying Lemma 2.2. This completes the proof. □

Remark 3.1 Theorem 3.1 improves, extends and develops [15], Theorem 3.6, [23], Theorem 3.2, [24], Theorem 3.1 and [25], Theorem 3.2 in the following aspects.

- Theorem 3.1 extends the extra-gradient method due to Nadezhkina and Takahashi [24], Theorem 3.1.
- The corresponding iterative algorithms in [15], Theorem 3.6 and [23], Theorem 3.2 are extended for developing our Ishikawa-type extra-gradient iterative algorithm involved in pseudo-contractive mappings with Lipschitz assumption in Theorem 3.1.
- The technique of proving weak convergence in Theorem 3.1 is different from those in [15], Theorem 3.6 and [23], Theorem 3.2 because our technique depends only on the demiclosedness principle for pseudo-contractive mappings in Hilbert spaces.
- The problem of finding an element of Γ is more general than the problem of finding a solution of the SFP in [15], Theorem 3.6 and the problem of finding an element of $\Gamma_0 \cap \operatorname{Fix}(S)$ with $S : C \to C$ being a nonexpansive mapping in [23], Theorem 3.2.
- Algorithm 3.1 of Yao *et al.* [25] is extended to develop the Ishikawa-type extra-gradient iterative algorithm in our Theorem 3.1 by virtue of the extra-gradient method.

Furthermore, we can immediately obtain the following weak convergence results.

Corollary 3.1 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$ be two nonempty closed convex sets. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $S : Q \to Q$ be a nonexpansive mapping and let $T : C \to C$ be an L-Lipschitzian pseudo-contractive mapping with $L > 1$. For $x_0 \in \mathcal{H}_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by the following Ishikawa-type iterative algorithm*:

$$\begin{cases} y_n = P_C(x_n - \lambda_n A^*(I - SP_Q)Ax_n), \\ z_n = (1-\alpha_n)y_n + \alpha_n T y_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T z_n, \quad n \geq 0, \end{cases} \tag{3.21}$$

where $\{\lambda_n\} \subset (0, \frac{1}{\|A\|^2})$ and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ such that $0 < a < \beta_n < c < \alpha_n < b < \frac{1}{\sqrt{1+L^2}+1}$.

Then the sequence $\{x_n\}$ generated by algorithm (3.21) converges weakly to an element of Γ.

Proof Taking $x^* \in \Gamma$, we have $x^* \in C \cap \operatorname{Fix}(T)$ and $Ax^* \in Q \cap \operatorname{Fix}(S)$. For simplicity, we write $v_n = P_Q A x_n$, $u_n = x_n - \lambda_n A^*(I - SP_Q)Ax_n$ for all $n \geq 0$. Thus, we have $y_n = P_C u_n$ for all

$n \geq 0$. Similarly to Theorem 3.1, we have

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \lambda_n\|v_n - Ax_n\|^2 - \lambda_n(1 - \lambda_n\|A\|^2)\|Sv_n - Ax_n\|^2 \tag{3.22}$$

and

$$\|x_{n+1} - x^*\|^2 \leq \|y_n - x^*\|^2. \tag{3.23}$$

Thus, the boundedness of the sequence $\{x_n\}$ yields our result.

From (3.22) and (3.23), we have

$$\lim_{n\to\infty} \|v_n - Ax_n\| = \lim_{n\to\infty} \|Sv_n - Ax_n\| = 0. \tag{3.24}$$

So,

$$\lim_{n\to\infty} \|v_n - Sv_n\| = 0.$$

Using the firm nonexpansiveness of P_C, we have

$$\begin{aligned}\|y_n - x^*\|^2 = \|P_C u_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|P_C u_n - u_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \|y_n - u_n\|^2.\end{aligned}$$

This together with (3.23) implies that

$$\|y_n - u_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

Hence,

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{3.25}$$

By setting $u_n = x_n - \lambda_n A^*(I - SP_Q)Ax_n$, it follows from (3.24) that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0.$$

This together with (3.25) implies that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

As in the proof of Theorem 3.1, we have $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$.

Therefore, all the conditions in Theorem 3.1 are satisfied. The conclusion of Corollary 3.1 can be obtained from Theorem 3.1 immediately. □

Corollary 3.2 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$ be two nonempty closed convex sets. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $T : C \to C$ be an L-Lipschitzian pseudo-contractive mapping with $L > 1$ such that $\Gamma_0 \cap \mathrm{Fix}(T) \neq \emptyset$. For*

$x_0 \in \mathcal{H}_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by the following Ishikawa-type extragradient iterative algorithm:

$$\begin{cases} y_n = P_C(x_n - \lambda_n A^*(I - P_Q)Ax_n), \\ z_n = P_C(x_n - \lambda_n A^*(I - P_Q)Ay_n), \\ w_n = (1 - \alpha_n)z_n + \alpha_n Tz_n, \\ x_{n+1} = (1 - \beta_n)z_n + \beta_n Tw_n, \quad n \geq 0, \end{cases} \tag{3.26}$$

where $\{\lambda_n\} \subset (0, \frac{1}{\|A\|^2})$ and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ such that $0 < a < \beta_n < c < \alpha_n < b < \frac{1}{\sqrt{1+L^2}+1}$.

Then the sequence $\{x_n\}$ generated by algorithm (3.26) *converges weakly to an element of* $\Gamma_0 \cap \mathrm{Fix}(T)$.

Remark 3.2 Corollary 3.2 improves, extends and develops [15], Theorem 3.6 and [23], Theorem 3.2 in the following aspects.

- Compared with [23], Theorem 3.2, Corollary 3.2 is essentially coincident with [23], Theorem 3.2 whenever $\alpha_n = 0$ and T is a nonexpansive mapping in the scheme (3.26).
- The problem of finding an element of $\Gamma_0 \cap \mathrm{Fix}(T)$ with $T : C \to C$ being a pseudo-contractive mapping is more general than the problem of finding a solution of the SFP in [15], Theorem 3.6 and the problem of finding an element of $\Gamma_0 \cap \mathrm{Fix}(S)$ with $S : C \to C$ being a nonexpansive mapping in [23], Theorem 3.2.

4 Mann-type extra-gradient iterative algorithm involved in pseudo-contractive mappings without Lipschitz assumption

We are now in a position to propose a Mann-type extra-gradient iterative algorithm for solving the split feasibility and fixed point problems involved in pseudo-contractive mappings without Lipschitz assumption.

Theorem 4.1 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$ be two nonempty closed convex sets. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $S : Q \to Q$ be a nonexpansive mapping and let $T : C \to C$ be a continuous pseudo-contractive mapping. For $x_0 \in \mathcal{H}_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by the following Mann-type extra-gradient iterative algorithm*:

$$\begin{cases} y_n = P_C(x_n - \lambda_n A^*(I - SP_Q)Ax_n), \\ z_n = P_C(x_n - \lambda_n A^*(I - SP_Q)Ay_n), \\ x_{n+1} = (1 - \alpha_n)z_n + \alpha_n Tz_n, \quad n \geq 0, \end{cases} \tag{4.1}$$

where $\{\lambda_n\} \subset (0, \frac{1}{2\|A\|^2})$ and $\{\alpha_n\} \subset (0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$.

Then the sequence $\{x_n\}$ generated by algorithm (4.1) *converges weakly to an element of* Γ.

Proof Taking $x^* \in \Gamma$, we have $x^* \in C \cap \mathrm{Fix}(T)$ and $Ax^* \in Q \cap \mathrm{Fix}(S)$. For simplicity, we write $u_n = x_n - \lambda_n A^*(I - SP_Q)Ax_n$ for all $n \geq 0$. Thus, we have $y_n = P_C u_n$ for all $n \geq 0$. As is proven in Theorem 3.1,

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \lambda_n \|v_n - Ax_n\|^2 - \lambda_n\bigl(1 - \lambda_n\|A\|^2\bigr)\|Sv_n - Ax_n\|^2 \tag{4.2}$$

and

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \left(1 - 4\lambda_n^2\|A\|^4\right)\|x_n - y_n\|^2. \tag{4.3}$$

Similarly, we have

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \left(1 - 4\lambda_n^2\|A\|^4\right)\|z_n - y_n\|^2.$$

From (2.2), (2.5), (4.1) and (4.3), we obtain that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)z_n + \alpha_n Tz_n - x^*\|^2 \\
&= (1-\alpha_n)\|z_n - x^*\|^2 + \alpha_n\|Tz_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|z_n - Tz_n\|^2 \\
&= (1-\alpha_n)\|z_n - x^*\|^2 + \alpha_n\langle Tz_n - z_n, Tz_n - x^*\rangle \\
&\quad + \alpha_n\langle z_n - x^*, Tz_n - x^*\rangle - \alpha_n(1-\alpha_n)\|z_n - Tz_n\|^2 \\
&\leq \|z_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|z_n - Tz_n\|^2 \\
&\leq \|x_n - x^*\|^2 - \left(1 - 4\lambda_n^2\|A\|^4\right)\|x_n - y_n\|^2 - \alpha_n(1-\alpha_n)\|z_n - Tz_n\|^2.
\end{aligned} \tag{4.4}$$

It follows from the assumption of $\{\lambda_n\}$ that

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|.$$

As the same argument of Theorem 3.1, the boundedness of the sequence $\{x_n\}$ yields our result.

Returning to (4.4), we have

$$\begin{aligned}
&\alpha_n(1-\alpha_n)\|z_n - Tz_n\|^2 + \left(1 - 4\lambda_n^2\|A\|^4\right)\|x_n - y_n\|^2 \\
&\quad \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.
\end{aligned}$$

Therefore, by the assumption of $\{\alpha_n\}$, we have

$$\lim_{n\to\infty}\|z_n - Tz_n\| = 0,$$

and by the assumption of $\{\lambda_n\}$, we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{4.5}$$

Similarly, we have

$$\lim_{n\to\infty}\|z_n - y_n\| = 0.$$

Returning to (4.2), we have

$$\begin{aligned}
&\lambda_n\|v_n - Ax_n\|^2 + \lambda_n\left(1 - \lambda_n\|A\|^2\right)\|Sv_n - Ax_n\|^2 \\
&\quad \leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 \\
&\quad \leq \left(\|x_n - x^*\| + \|y_n - x^*\|\right)\|x_n - y_n\|.
\end{aligned}$$

From (4.5) and by the assumption of $\{\lambda_n\}$, we have

$$\lim_{n\to\infty} \|v_n - Ax_n\| = \lim_{n\to\infty} \|Sv_n - Ax_n\| = 0. \tag{4.6}$$

So,

$$\lim_{n\to\infty} \|v_n - Sv_n\| = 0.$$

By setting $u_n = x_n - \lambda_n A^*(I - SP_Q)Ax_n$, it follows from (4.6) that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0.$$

This together with (4.5) implies that

$$\lim_{n\to\infty} \|u_n - y_n\| = 0.$$

Therefore, all the conditions in Theorem 3.1 are satisfied. The conclusion of Theorem 4.1 can be obtained from Theorem 3.1 immediately. □

Remark 4.1 Theorem 4.1 improves, extends and develops [15], Theorem 3.6, [23], Theorem 3.2, [24], Theorem 3.1 and [25], Theorem 3.2 in the following aspects.

- Theorem 4.1 extends the extra-gradient method due to Nadezhkina and Takahashi [24], Theorem 3.1.
- The corresponding iterative algorithms in [15], Theorem 3.6 and [23], Theorem 3.2 are extended for developing our Mann-type extra-gradient iterative algorithm involved in pseudo-contractive mappings without Lipschitz assumption in Theorem 4.1.
- The technique of proving weak convergence in Theorem 4.1 is different from those in [15], Theorem 3.6 and [23], Theorem 3.2 because our technique depends on the demiclosedness principle for pseudo-contractive mappings and bases on condition (2.5) in Hilbert spaces.
- The problem of finding an element of Γ is more general than the problem of finding a solution of the SFP in [15], Theorem 3.6 and the problem of finding an element of $\Gamma_0 \cap \mathrm{Fix}(S)$ with $S : C \to C$ being a nonexpansive mapping in [23], Theorem 3.2.
- In Algorithm 3.1 of [25], Yao *et al.* proposed the following iterative algorithm:

$$\begin{aligned} u_n &= P_C\big(\alpha_n u + (1-\alpha_n)\big(x_n - \delta A^*(I - SP_Q)Ax_n\big)\big), \\ x_{n+1} &= (1-\beta_n)u_n + \beta_n T\big((1-\gamma_n)u_n + \gamma_n Tu_n\big), \quad n \geq 0. \end{aligned}$$

  Via replacing the first iterative step by the extra-gradient method and replacing the second iterative step by the Mann-type iterative algorithm, we obtain the Mann-type extra-gradient iterative algorithm (4.1) in Theorem 4.1.

Utilizing Theorem 4.1, we have the following two new results in the setting of real Hilbert spaces.

Corollary 4.1 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$ be two nonempty closed convex sets. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $S : Q \to Q$ be a nonexpansive mapping and let $T : C \to C$ be a continuous pseudo-contractive mapping. For $x_0 \in \mathcal{H}_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by the following Mann-type iterative algorithm*:

$$\begin{cases} y_n = P_C(x_n - \lambda_n A^*(I - SP_Q)Ax_n), \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n Ty_n, \quad n \geq 0, \end{cases} \tag{4.7}$$

where $\{\lambda_n\} \subset (0, \frac{1}{\|A\|^2})$ and $\{\alpha_n\} \subset (0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$.

Then the sequence $\{x_n\}$ generated by algorithm (4.7) *converges weakly to an element of* Γ.

Proof Taking an $x^* \in \Gamma$, we have $x^* \in C \cap \mathrm{Fix}(T)$ and $Ax^* \in Q \cap \mathrm{Fix}(S)$. For simplicity, we write $u_n = x_n - \lambda_n A^*(I - SP_Q)Ax_n$ for all $n \geq 0$. Thus, we have $y_n = P_C u_n$ for all $n \geq 0$. Similarly to Theorem 4.1,

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0$$

and

$$\lim_{n\to\infty} \|v_n - Ax_n\| = \lim_{n\to\infty} \|Sv_n - Ax_n\| = \lim_{n\to\infty} \|v_n - Sv_n\| = 0.$$

Similarly to Corollary 3.1,

$$\lim_{n\to\infty} \|u_n - y_n\| = \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Therefore, all the conditions in Theorem 4.1 are satisfied. The conclusion of Corollary 4.1 can be obtained from Theorem 4.1 immediately. □

Corollary 4.2 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$ be two nonempty closed convex sets. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $T : C \to C$ be a continuous pseudo-contractive mapping such that $\Gamma_0 \cap \mathrm{Fix}(T) \neq \emptyset$. For $x_0 \in \mathcal{H}_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by the following Mann-type extra-gradient iterative algorithm*:

$$\begin{cases} y_n = P_C(x_n - \lambda_n A^*(I - P_Q)Ax_n), \\ z_n = P_C(x_n - \lambda_n A^*(I - P_Q)Ay_n), \\ x_{n+1} = (1 - \alpha_n)z_n + \alpha_n Tz_n, \quad n \geq 0, \end{cases} \tag{4.8}$$

where $\{\lambda_n\} \subset (0, \frac{1}{\|A\|^2})$ and $\{\alpha_n\} \subset (0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$.

Then the sequence $\{x_n\}$ generated by algorithm (4.8) *converges weakly to an element of* $\Gamma_0 \cap \mathrm{Fix}(T)$.

Remark 4.2 Corollary 4.2 improves, extends and develops [15], Theorem 3.6 and [23], Theorem 3.2 in the following aspects.

- Compared with [23], Theorem 3.2, Corollary 4.2 is essentially coincident with [23], Theorem 3.2 whenever T is a nonexpansive mapping. Hence our Corollary 4.2 includes [23], Theorem 3.2 as a special case.
- The problem of finding an element of $\Gamma_0 \cap \mathrm{Fix}(T)$ with $T : C \to C$ being a pseudo-contractive mapping is more general than the problem of finding a solution of the SFP in [15], Theorem 3.6 and the problem of finding an element of $\Gamma_0 \cap \mathrm{Fix}(S)$ with $S : C \to C$ being a nonexpansive mapping in [23], Theorem 3.2.

Example 4.1 [25] Let $\mathcal{H} = \mathbf{R}$ with the inner product defined by $\langle x, y\rangle = xy$ for all $x, y \in \mathbf{R}$ and the absolute-valued norm $|\cdot|$. Let $C = [0, +\infty)$ and $Tx = x - 1 + \frac{4}{x+1}$ for all $x \in C$. Obviously, $\mathrm{Fix}(T) = 3$. It is easy to see that

$$\begin{aligned}\langle Tx - Ty, x - y\rangle &= \left\langle x - 1 + \frac{4}{x+1} - y + 1 - \frac{4}{y+1}, x - y\right\rangle \\ &\leq \left(1 - \frac{4}{(x+1)(y+1)}\right)|x - y|^2 \\ &\leq |x - y|^2\end{aligned}$$

and

$$\begin{aligned}|Tx - Ty| &\leq \left|x - 1 + \frac{4}{x+1} - y + 1 - \frac{4}{y+1}\right| \\ &\leq \left|1 - \frac{4}{(x+1)(y+1)}\right| |x - y| \\ &\leq 5|x - y|\end{aligned}$$

for all $x, y \in C$. But

$$\left|T\left(\frac{1}{4}\right) - T(0)\right| = \frac{11}{20} > \frac{1}{4}.$$

Thus, T is a Lipschitzian pseudo-contractive mapping but not a nonexpansive one.

The above example satisfies condition (2.5). Indeed, note that

$$\begin{aligned}\langle Tx - 3, x - 3\rangle &= \left\langle x - 1 + \frac{4}{x+1} - 3, x - 3\right\rangle \\ &\leq \left(1 - \frac{1}{x+1}\right)|x - 3|^2\end{aligned}$$

for all $x \in C$. Hence, we have

$$\begin{aligned}\langle Tx - x, Tx - 3\rangle &= |Tx - x|^2 + \langle Tx - 3, x - 3\rangle - \langle x - 3, x - 3\rangle \\ &\leq |Tx - x|^2 + \left(1 - \frac{1}{x+1}\right)|x - 3|^2 - |x - 3|^2 \\ &= |Tx - x|^2 - \frac{1}{x+1}|x - 3|^2 \leq |Tx - x|^2\end{aligned}$$

for all $x \in C$.

So, it follows that

$$\begin{aligned}
\langle Tx - x, Tx - 3\rangle &= |Tx - x|^2 + \langle Tx - 3, x - 3\rangle - \langle x - 3, x - 3\rangle \\
&\leq |Tx - x|^2 + \left(1 - \frac{1}{x+1}\right)|x - 3|^2 - |x - 3|^2 \\
&= |Tx - x|^2 - \frac{1}{x+1}|x - 3|^2 \\
&= \left(1 - \frac{4}{x+1}\right)^2 - \frac{1}{x+1}|x - 3|^2 \\
&= -\frac{x^3 - 6x^2 + 9x}{x^2 + 2x + 1} \leq 0
\end{aligned}$$

for all $x \in C$. So, it is reasonable that we introduce condition (2.5) in Theorem 4.1. Thus, we can use condition (2.5) to replace the Lipschitz assumption of pseudo-contractive mappings when we study a split feasibility problem or other problems involved in pseudo-contractive mappings.

5 Numerical example

In this section, we consider the following example to illustrate the theoretical result.

Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathbf{R}$ with the inner product defined by $\langle x, y\rangle = xy$ for all $x, y \in \mathbf{R}$ and the standard norm $|\cdot|$. Let $C = [0, +\infty)$ and $Tx = x - 1 + \frac{4}{x+1}$ for all $x \in C$. Let $Q = \mathbf{R}$ and $Sx = \frac{x}{3} + 1$ for all $x \in Q$. Let $Ax = \frac{1}{2}x$ for all $x \in \mathbf{R}$. Let $\lambda_n = 1$, $\alpha_n = \frac{1}{7}$, $\beta_n = \frac{1}{8}$. Let the sequence $\{x_n\}$ be generated iteratively by (3.1), then the sequence $\{x_n\}$ converges to 3.

Figure 1 The convergence of $\{x_n\}$ with initial value 8.

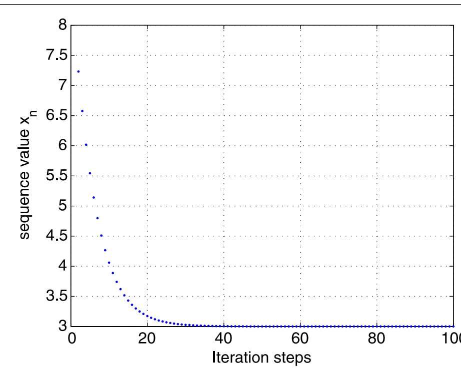

Figure 2 The convergence of $\{x_n\}$ with initial value –2.

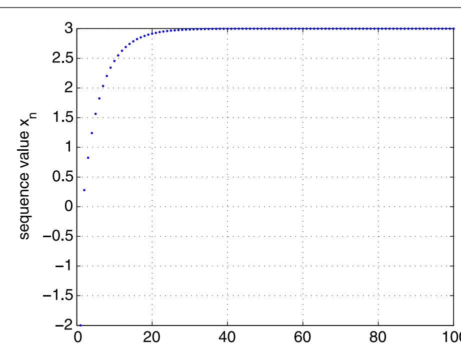

Table 1 The initial value is number 8

n	x_n
0	8.000000000000000
1	7.232040784873107
2	6.576740838659221
3	6.018450105660861
4	5.543605651619651
⋮	⋮
96	3.000000139089063
97	3.000000115627270
98	3.000000096123055
99	3.000000079908846
100	3.000000066429678

Table 2 The initial value is number –2

n	x_n
0	–2.000000000000000
1	0.278571428571429
2	0.825823030475218
3	1.240382351822075
4	1.565072721297795
⋮	⋮
96	2.999999933179721
97	2.999999944451085
98	2.999999953821176
99	2.999999961610703
100	2.999999968086279

Solution: It is easy to see that A is a bounded linear operator with its adjoint $A^* = A$, $\operatorname{Fix}(T) = 3$ and $\operatorname{Fix}(S) = \frac{3}{2}$. It can be observed that all the assumptions of Theorem 3.1 are satisfied. It is also easy to check that $\Gamma = \{3\}$. We now rewrite (3.1) as follows:

$$\begin{cases} y_n = P_C(\frac{5x_n}{6} + \frac{1}{2}), \\ z_n = P_C(x_n - \frac{y_n}{6} + \frac{1}{2}), \\ w_n = z_n + \frac{4}{7(z_n+1)} - \frac{1}{7}, \\ x_{n+1} = z_n + \frac{1}{14(z_n+1)} + \frac{1}{2z_n + \frac{8}{7(z_n+1)} + \frac{12}{7}} - \frac{1}{7}, \quad n \geq 0. \end{cases}$$

Choosing initial values $x_0 = 8$ and $x_0 = -2$ respectively, we see that figures (see Figures 1 and 2) and numerical results (see Tables 1 and 2) demonstrate Theorem 3.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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