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Existence of solutions for a class of operator equations

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Abstract

In this paper we deal with the existence and multiplicity of nontrivial solutions to a class of operator equation. By using infinite dimensional Morse theory, we establish some conditions which guarantee that the equation has many nontrivial solutions.

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1 Introduction

Let $E = C[0, 1]$ be the usual real Banach space with the norm $\|u\|_0 = \max_{t \in [0, 1]} |u(t)|$ for all $u \in C[0, 1]$, and $H = L^2[0, 1]$ be the usual real Hilbert space with the inner product (\cdot, \cdot) and the norm $\|\cdot\|$. Obviously, E is embedded continuously into H , denoted by $E \hookrightarrow H$.

This paper is concerned with the existence of nontrivial solutions for the following operator equation of the form

$$u = K^2 f u, \tag{1.1}$$

where $f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ is continuous and that f'_x , the first-order derivative of f in x , is also continuous on \mathbb{R}^1 , $\mathbf{f} : E \rightarrow E$ is defined as $\mathbf{f}u(t) = f(u(t))$, $\forall u \in E$; $K : H \rightarrow E \hookrightarrow H$ is a compact symmetric positive linear operator with $0 \notin \sigma_p(K)$, where $\sigma_p(K)$ denotes all the eigenvalues of K . In recent years, there have been many papers to study the existence of nontrivial solutions on higher order boundary value problems, see [1–9]. In [5], by using spectral theory and the fixed point theorem, Li established some conditions for f to guarantee that the problem has a unique solution. In a later paper [4], by applying the strongly monotone operator principle and the critical point theory, some new existence theorems on unique, at least one nontrivial and infinitely many solutions were established. Motivated by the above papers, in this paper, we try to discuss equation (1.1) by using Morse theory. Specifically, we consider the existence and multiplicity of the solutions for (1.1) and obtain at least two nontrivial solutions, three nontrivial solutions and five nontrivial solutions, respectively. And then we apply the abstract results to a fourth-order boundary value problem.

In this paper, we consider the existence of solutions to equation (1.1) by applying Morse theory. Our methods are different from those in the literature mentioned above. As is well

known, this kind of theory is based on deformation lemmas. In general the functional needs to satisfy a compactness condition. In this article, we use the Palais-Smale (PS) condition: Let D be a real Banach space, $J \in C^1(D, \mathbb{R}^1)$. If any sequence $\{v_k\}_1^\infty \subset D$ for which $\{J(v_k)\}$ is bounded and $dJ(v_k) \rightarrow \theta$ in D as $k \rightarrow \infty$ possesses a convergent subsequence, then we say J satisfies the Palais-Smale (PS) condition.

The paper is organized as follows. In Section 2, we present some preliminary knowledge about Morse theory. In Section 3, we apply Morse theory to give the proofs of Theorems 3.1-3.3 and provide some examples to illustrate the results.

2 Preliminary

Assume that $\{\lambda_k\}_1^\infty$ is the sequence of all eigenvalues of K , where each eigenvalue is repeated according to its multiplicity, and $\{e_k\}_1^\infty \subset E$ is the corresponding orthonormal eigenvector sequence in H . In the following, we outline some preliminary knowledge about Morse theory, which will be used in the proofs of our main results. Please refer to [10] for more details.

Let \mathcal{H} be a real Hilbert space with the norm $\|\cdot\|$ and the inner product (\cdot, \cdot) , $J \in C^1(\mathcal{H}, \mathbb{R}^1)$. Suppose that (X, Y) is a pair of topological spaces and $H_q(X, Y)$ is the q th singular relative homology group with coefficients in an Abelian group G . Also $\beta_q = \text{rank } H_q(X, Y)$ is called the q -dimension Betti number. Let p be an isolated critical point of J with $J(p) = c$, $c \in \mathbb{R}^1$, and U be a neighborhood of p in which J has no critical points except p . The group

$$C_q(J, p) = H_q(J_c \cap U, (J_c \setminus \{p\}) \cap U), \quad q = 0, 1, 2, \dots$$

is called the q th critical group of J at p , where $J_c = \{u \in \mathcal{H} : J(u) \leq c\}$. We call the dimension of a negative space corresponding to the spectral decomposition of $d^2J(p)$ the Morse index of p , denoted by $\text{ind}(J, p)$ (it can be ∞). And p is called a nondegenerate critical point if $d^2J(p)$ has a bounded inverse.

Let A be a bounded self-adjoint operator defined on \mathcal{H} . According to its spectral decomposition, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_0 \oplus \mathcal{H}_-$, where $\mathcal{H}_\pm, \mathcal{H}_0$ are invariant subspaces corresponding to the positive/negative, and zero spectrum of A respectively. We shall study the number of critical points of the functional

$$J(u) = \frac{1}{2}(Au, u) + g(u),$$

or equivalently, the number of solutions of the operator equation

$$Au + dg(u) = 0.$$

The following assumptions are given:

- (i) $A_\pm = A|_{\mathcal{H}_\pm}$ has a bounded inverse on \mathcal{H}_\pm .
- (ii) $\gamma = \dim(\mathcal{H}_0 \oplus \mathcal{H}_-) < \infty$.
- (iii) $g \in C^1(\mathcal{H}, \mathbb{R}^1)$ has a bounded and compact differential $dg(x)$. In addition, if $\dim \mathcal{H}_0 \neq 0$, we assume

$$g(v) \rightarrow -\infty \quad \text{as } \|v\| \rightarrow \infty, v \in \mathcal{H}_0.$$

Lemma 2.1 [10] *Under assumptions (i), (ii) and (iii), we have that*

- (1) *J satisfies the (PS) condition, and*
- (2) *$H_q(\mathcal{H}, J_a) = \delta_{qr}G$ for $-a$ large enough, as $J_a \cap K = \emptyset$.*

Lemma 2.2 [10] *Under assumptions (i), (ii) and (iii), if J has critical points $\{p_i\}_{i=1}^n$ with*

$$\gamma \notin \bigcup_{i=1}^n [m_-(p_i), m_-(p_i) + m_0(p_i)],$$

where $m_-(p) = \text{index}(J, p)$ and $m_0(p) = \dim \ker d^2J(p)$, then J has a critical p_0 different from p_1, \dots, p_n with $C_r(J, p_0) \neq 0$.

Lemma 2.3 [10] *Under assumptions (i), (ii) and (iii), iff has a nondegenerate critical point p_0 with Morse index $m_-(p_0) \neq \gamma$, then f has a critical point $p_1 \neq p_0$. Moreover, if*

$$m_0(p_1) \leq |m_-(p_0) - \gamma|,$$

then f has one more critical point $p_2 \neq p_0, p_1$.

Remark 2.1 In Lemmas 2.1 and 2.2, if $\dim \mathcal{H}_0 = 0$, the boundedness of dg can be replaced by the following condition:

$$\|dg(u)\| = o(\|u\|) \quad \text{as } \|u\| \rightarrow \infty.$$

Lemma 2.4 [10] *Let $J \in C^2(H, \mathbb{R}^1)$ be a function satisfying the (PS) condition. Assume that $dJ = I - T$, where T is a compact mapping, and that p_0 is an isolated critical point of J. Then we have*

$$\text{ind}(dJ, p_0) = \sum_{q=0}^{\infty} (-1)^q \text{rank } C_q(J, p_0).$$

Lemma 2.5 [10] *Assume that $J \in C^2(H, \mathbb{R}^1)$ is bounded from below, satisfies the (PS) condition. Suppose that $dJ = I - T$ is a compact vector field, and p_0 is an isolated critical point but not the global minimum with $\text{index}(dJ, p_0) = \pm 1$. Then J has at least three critical points.*

3 Proofs of main results

In this section, we will prove the main results.

Lemma 3.1 *Suppose that $\{v_k\}_1^\infty \subset \mathcal{H}$ is bounded and that $dJ(v_k) = (I - K\mathbf{f}K)v_k \rightarrow \theta$ in \mathcal{H} as $k \rightarrow \infty$. Then J satisfies the (PS) condition.*

Proof Since $K : \mathcal{H} \rightarrow E$ is completely continuous, $\mathbf{f} : E \rightarrow E$ is bounded and continuous, and $v_k - K\mathbf{f}Kv_k \rightarrow \theta$ as $k \rightarrow \infty$, we have that $\{v_k\}_1^\infty$ has a convergent subsequence. Thus J satisfies the (PS) condition. □

Theorem 3.1 *Assume that f satisfies the condition*

$$(H_1) \quad f \in C^1(\mathbb{R}^1) \text{ with } f'_x(x) \geq 0 \text{ for all } x \in \mathbb{R}^1 \text{ and } \limsup_{x \rightarrow 0} f(x)/x < 1/\lambda_1^2;$$

(H₂) there exist $\mu \in (0, 1/2)$ and $R > 0$ such that $0 < F(x) \leq \mu x f(x)$ for all $|x| \geq R$, where $F(x) = \int_0^x f(y) dy$.

Then equation (1.1) possesses at least two nontrivial solutions.

Proof By condition (H₁), there exist $\varepsilon \in (0, 1)$ and $\delta > 0$ such that

$$F(x) \leq \frac{1}{2\lambda_1}(1 - \varepsilon)|x|^2, \quad x \in [-\delta, \delta]. \tag{3.1}$$

Let $\rho \leq \delta/M_1$, where $M_1 = C(\sum_{k=1}^\infty \lambda_k)^{1/2}$. Then it follows from [4] that $\|Kv\|_0 \leq M_1\|v\| \leq \delta$ for all $v \in B_\rho$. Hence by (3.1) we have

$$\begin{aligned} J(v) &= \frac{1}{2}\|v\|^2 - \int_0^1 F(Kv(t)) dt \\ &\geq \frac{1}{2}\|v\|^2 - \frac{1}{2\lambda_1^2}(1 - \varepsilon) \int_0^1 |Kv(t)|^2 dt \\ &= \frac{1}{2}\|v\|^2 - \frac{1}{2\lambda_1^2}(1 - \varepsilon)(Kv, v) \\ &\geq \frac{1}{2}\|v\|^2 - \frac{1}{2\lambda_1^2}(1 - \varepsilon)\lambda_1^2\|v\|^2 = \frac{1}{2}\varepsilon\|v\|^2, \quad v \in B_\rho, \end{aligned}$$

that is,

$$J(v) \geq \frac{1}{2}\varepsilon\|v\|^2, \quad v \in B_\rho. \tag{3.2}$$

It follows from (3.2) that θ is a local minimum.

We now find two nontrivial solutions. Let us define

$$f_+(x) = \begin{cases} f(x), & x \geq 0, \\ 0, & x < 0, \end{cases}$$

and

$$J_+(v) = \frac{1}{2}\|v\|^2 - \int_0^1 F_+(Kv(t)) dt,$$

where $F_+(x) = \int_0^x f_+(y) dy$.

By condition (H₂), there exist $C_1, C_2 > 0$ such that

$$F_+(x) \geq C_1|x|^{1/\mu} - C_2, \quad x \in \mathbb{R}^1.$$

Thus,

$$\begin{aligned} J_+(\tau v) &= \frac{1}{2}\tau^2\|v\|^2 - \int_0^1 F_+(\tau Kv(t)) dt \\ &\leq \frac{1}{2}\tau^2\|v\|^2 - C_1\tau^{1/\mu}\|Kv\|_{1/\mu}^{1/\mu} + C_2, \quad v \in H. \end{aligned}$$

This implies that $\lim_{\tau \rightarrow +\infty} J_+(\tau e_1) = -\infty$.

Now we shall prove that J_+ satisfies the (PS) condition on H . Let $\{v_k\}_1^\infty \subset H$ with $|J_+(v_k)| \leq \beta$ for all $k \in \mathbb{N} \setminus \{0\}$ and some $\beta > 0$, and $dJ_+(v_k) = (I - Kf_+K)v_k \rightarrow \theta$ as $k \rightarrow \infty$. By Lemma 3.1, we only claim that $\{v_k\}_1^\infty$ is bounded.

In fact, notice that

$$(dJ_+(v_k), v_k) = (v_k - Kf_+Kv_k, v_k) = \|v_k\|^2 - \int_0^1 f_+(Kv_k(t))Kv_k(t) dt.$$

According to condition (H_2) , there exists $C_3 > 0$ such that

$$F_+(x) \leq \mu x f_+(x) + C_3, \quad x \in \mathbb{R}^1,$$

thus, we have

$$\begin{aligned} \beta &\geq J_+(v_k) = \frac{1}{2} \|v_k\|^2 - \int_0^1 F_+(Kv_k(t)) dt \\ &\geq \frac{1}{2} \|v_k\|^2 - \mu \int_0^1 f_+(Kv_k(t))Kv_k(t) dt - C_3 \\ &= (1/2 - \mu) \|v_k\|^2 + \mu (dJ_+(v_k), v_k) - C_3 \\ &\geq (1/2 - \mu) \|v_k\|^2 - \mu \|dJ_+(v_k)\| \|v_k\| - C_3, \quad k \in \mathbb{N} \setminus \{0\}. \end{aligned}$$

Since $dJ_+(v_k) \rightarrow \theta$ as $k \rightarrow \infty$, there exists $N_0 \in \mathbb{N} \setminus \{0\}$ such that

$$\beta \geq (1/2 - \mu) \|v_k\|^2 - \|v_k\| - C_3, \quad k > N_0.$$

Thus $\{v_k\}_1^\infty \subset H$ is bounded. Again, $J_+ \in C^2(H, \mathbb{R}^1)$ satisfies the (PS) condition. We also have

$$J_+(se_1) \rightarrow -\infty, \quad s \rightarrow +\infty.$$

On the other hand, by the same way to (3.2), we have

$$J_+|_{\partial B_\rho} > 0.$$

The mountain pass lemma is applied to obtain a critical point $v_+ \in H$ of J_+ , with critical value $c_+ > 0$, which satisfies

$$Kv_+ = Kf_+Kv_+.$$

By the positive property of K , we have $Kv_+ \geq 0$, so v_+ is again a critical point of J .

Analogously, we define

$$f_-(x) = \begin{cases} f(x), & x \leq 0, \\ 0, & x > 0, \end{cases}$$

and then obtain a critical point v_- of J with critical value $c_- > 0$. The proof is completed. \square

Theorem 3.2 *Assume that:*

- (H₃) $f(0) = 0$ and $0 \leq f'(0) < 1/\lambda_1^2$;
- (H₄) $f'(u) > 0$ and strictly increasing in u for $u > 0$;
- (H₅) $f'(\infty) = \lim_{|u| \rightarrow \infty} f'(u)$ exists and lies in $(1/\lambda_1^2, 1/\lambda_2^2)$.

Then (1.1) has at least three distinct solutions.

Proof Define a functional $J : H \rightarrow \mathbb{R}$ as

$$J(v) = \frac{1}{2} \|v\|^2 - \int_0^1 F(Kv(t)) dt, \quad v \in H.$$

First, it is obvious that θ is a solution, which is also a strict local minimum of the functional J on H .

By (H₃), for all $\varepsilon > 0$, there exists $\delta > 0$, $0 < |x| < \delta$, such that $|f(x)| < (1/\lambda_1^2 - \varepsilon)|x|$, we have $\|Kv\|_0 \leq C\|v\|$. Thus

$$\begin{aligned} J(u) &= \frac{1}{2} \|v\|^2 - \int_0^1 F(Kv(t)) dt \\ &\geq \frac{1}{2} \|v\|^2 - \frac{1}{2} (1/\lambda_1^2 - \varepsilon) \int_0^1 |Kv(t)|^2 dt \\ &\geq \frac{1}{2} \|v\|^2 - \frac{1}{2} (1/\lambda_1^2 - \varepsilon) \lambda_1^2 \|v\|^2 \\ &= \varepsilon \lambda_1 \|v\|^2 > 0, \quad 0 < \|u\| < \delta/C. \end{aligned}$$

Modify f to be a new function

$$\widehat{f}(x) = \begin{cases} f(x), & x \geq 0, \\ 0, & x < 0, \end{cases}$$

and consider a new functional

$$\widehat{J}(v) = \frac{1}{2} \|v\|^2 - \int_0^1 \widehat{F}(Kv(t)) dt,$$

where $\widehat{F}(x) = \int_0^x \widehat{f}(s) ds$. It is easily seen that θ is also a strict local minimum of \widehat{J} , which is a C^1 functional with the (PS) condition.

Indeed,

$$\begin{aligned} \widehat{J}(v) &= \frac{1}{2} \|v\|^2 - \int_0^1 \widehat{F}(Kv(t)) dt \\ &= \frac{1}{2} \|v\|^2 - \int_0^{2\pi} \widehat{F}(Kv(t)) dt, \quad v \in H. \end{aligned}$$

It is well known that θ is also a strict local minimum of \widehat{J} . We will demonstrate that \widehat{J} satisfies the (PS) condition as follows. Suppose $\{v_n\}_1^\infty \subset H$ such that $\widehat{J}(v_n)$ is bounded and

$\widehat{J}(v_n) \rightarrow 0$ as $n \rightarrow \infty$. We derive

$$o(1)\|v_n^-\| = \widehat{J}(v_n, v_n^-) = \frac{1}{2}\|v_n^-\|^2 - \int_0^{2\pi} \widehat{f}(Kv_n(t))Kv_n^-(t) dt.$$

Hence $\{v_n^-\}$ is bounded. In the following, we will show that $\{v_n^-\}$ is also bounded by contradiction. Setting $u_n = \frac{v_n^+}{\|v_n^+\|}$, $\widehat{J}(v_n^+) = \widehat{J}(v_n) - \widehat{J}(v_n^-)$, and $\widehat{J}(v_n^-)$ is bounded. By (H_5) , there exists $C > 0$ such that $|f(x)| < 1/\lambda_2^2|x| + C$, we have

$$\begin{aligned} \frac{1}{2} &= \frac{\widehat{J}(v_n^+)}{\|v_n^+\|^2} + \int_0^1 \frac{\widehat{F}(Kv_n^+)}{\|v_n^+\|^2} dt \\ &\leq o(1) + \frac{1}{2\lambda_2} \int_0^1 |Ku_n(t)|^2 dt + \frac{C}{\|v_n^+\|} \int_0^1 Ku_n(t) dt \\ &= o(1) + \frac{\lambda_1}{2\lambda_2} \int_0^1 |u_n(t)|^2 dt + \frac{C}{\|v_n^+\|} \int_0^1 Ku_n(t) dt. \end{aligned}$$

Thus $u_n \neq 0$, $n = 1, 2, \dots$, and

$$\begin{aligned} \int_0^1 u_n e_1 dt &= (u_n, e_1) = \left(\frac{\widehat{J}(v_n)}{\|v_n^+\|}, e_1 \right) - \left(\frac{v_n^-}{\|v_n^+\|}, e_1 \right) + \int_0^1 \frac{\widehat{f}(Kv_n)}{\|v_n^+\|} e_1 dt \\ &= o(1) + \int_0^1 \frac{\widehat{f}(Kv_n)}{\|u_n^+\|} Ke_1 dt \\ &\geq o(1) + (1/\lambda_1^2 + \varepsilon)\lambda_1^2 \int_0^1 \frac{v_n^+(t)}{\|v_n^+\|} e_1 dt - C \int_0^1 \frac{e_1}{\|v_n^+\|} dt \\ &= o(1) + (1/\lambda_1 + \varepsilon)\lambda_1^2 \int_0^1 u_n e_1 dt. \end{aligned}$$

Hence, we have $\varepsilon \int_0^1 u e_1 dt \leq 0$ is a contradiction.

Since \widehat{J} is unbounded from below, along the ray $u_s = se_1(t)$, $s > 0$. Indeed,

$$\begin{aligned} \widehat{J}(u_s) &= \frac{1}{2}\|u_s\|^2 - \int_0^1 \widehat{F}(u_s) dt \\ &= \frac{1}{2}\|u_s\|^2 - \int_0^1 \widehat{F}(u_s) dt \\ &\leq \frac{1}{2}\|u_s\|^2 - \left(\frac{1}{2\lambda_1^2} + \varepsilon \right) \int_0^1 (Ku_s(t))^2 dt + C \int_0^1 Ku_s(t) dt \\ &= -\varepsilon\lambda_1^2 s^2 + C\lambda_1 \int_0^1 e_1 dt. \end{aligned}$$

The mountain pass lemma is applied to obtain a critical point $u_0 \neq \theta$ of \widehat{J} which solves the equation

$$u(t) = K^2 \widehat{f}(u(t)), \quad t \in [0, 1].$$

Since $\widehat{f} \geq 0$, by the maximum principle, $u_0 \geq 0$, hence u_0 is a critical point of J .

Now we shall prove that $I - K^2 f'(u_0(t))$ has a bounded inverse operator on H , i.e., u_0 is a nondegenerate critical point of J . Since u_0 satisfies (1.1), it is also a solution of the equation

$$u_0(t) = K^2 g(t)u_0(t), \quad t \in [0, 1],$$

where $g(t) = \int_0^1 f'(su_0(t)) ds$. Let $\mu_1 > \mu_2 > \dots$ be eigenvalues of the problem

$$K^2 f'(u_0(t))w(t) = \mu w(t), \quad t \in [0, 1].$$

We shall prove that $\mu_1 > 1 > \mu_2$. This implies the invertibility of the operator $K^2 f'(u_0(t))$. In fact, according to assumption (H_4) , we have $g(t) < f'(u_0(t))$, $t \in [0, 1]$, such that

$$1/\mu_1 = \min \frac{(w, w)}{\int_0^1 K^2 f'(u_0(t))w^2 dt} < \min \frac{(w, w)}{\int_0^1 K^2 g(t)w^2 dt} \leq 1.$$

Again, by assumptions (H_4) and (H_5) , we have $f'(u_0(t)) < 1/\lambda_2^2$, $t \in [0, 1]$. According to the Rayleigh quotient characterization of the eigenvalues

$$1/\mu_2 = \sup_{E_1} \inf_{w \in E_1^\perp} \frac{(w, w)}{\int_0^1 K^2 f'(u_0(t))w^2 dt} \geq \frac{\lambda_2^2}{1 - \lambda_2^2 \varepsilon} \sup_{E_1} \inf_{w \in E_1^\perp} \frac{(w, w)}{\int_0^{2\pi} K^2 w^2 dt} > 1,$$

where E_1 is any one-dimensional subspace in H .

The Morse identity yields an odd number of critical points. Therefore there are at least three solutions of (1.1). The proof is completed. □

Theorem 3.3 *Assume that:*

(H_6) $f(0) = 0$ and $1/\lambda_2^2 < f'(0) < 1/\lambda_3^2$;

(H_7) $f'(\infty) = \lim_{x \rightarrow \pm\infty} f'(x)$ and $f'(\infty)^{-1} \notin \sigma_p(K^2)$ with $f'(\infty) > 1/\lambda_3^2$;

(H_8) $|f(x)| < 1$ and $0 \leq f'(x) < 1/\lambda_3^2$ in the interval $[-c, c]$, where $c = \max_{t \in [0, 1]} \varphi(t)$, and $\varphi(t)$ is the solution of the equation

$$\varphi(t) = K^2 1.$$

Then (1.1) possesses at least five nontrivial solutions.

Proof Define

$$\widehat{f}(x) = \begin{cases} f(c), & x > c, \\ f(x), & |x| \leq c, \\ f(-c), & x < -c, \end{cases}$$

and let

$$\widehat{J}(v) = \frac{1}{2} \|v\|^2 - \int_0^1 \widehat{F}(Kv(t)) dt,$$

where $\widehat{F}(x) = \int_0^x \widehat{f}(y) dy$. The truncated problem

$$u(t) = K\widehat{f}(u(t)) \tag{3.3}$$

possesses at least three solutions θ, u_1, u_2 because there are two pairs of subsolution and supersolution $[\varepsilon e_1, \varphi]$ and $[-\varphi, -\varepsilon e_1]$, where e_1 is the first eigenfunction of K and $\varepsilon > 0$ is a small enough constant.

In fact, one may assume that $\underline{u}(x), \overline{u}(x)$ is a pair of sub- and supersolution of (1.1) with $\underline{u}(x) < \overline{u}(x)$ without loss of generality. Define a new function

$$\widehat{f}(u) = \begin{cases} f(\overline{u}(t)), & u > \overline{u}(t), \\ f(u), & \underline{u}(x) \leq u \leq \overline{u}(t), \\ f(\underline{u}(t)), & u < \underline{u}(t). \end{cases}$$

By definition, $\widehat{f}(x) \in C(\mathbb{R}^1)$ is bounded and satisfies $f(u) = \widehat{f}(u)$ for $\underline{u}(x) \leq u \leq \overline{u}(t)$. Let $\widehat{F}(x) = \int_0^x \widehat{f}(y) dy$. Then $\widehat{F} \in C^1(\mathbb{R}^1)$, and the functional

$$\widehat{J}(v) = \frac{1}{2} \|v\|^2 - \int_0^1 \widehat{F}(Ku(t)) dt$$

defined on H is bounded from below and satisfies the (PS) condition. Hence there is a minimum u_0 which satisfies $d\widehat{J}(u_0) = \theta$. Since \underline{u} is a strict sub-solution,

$$K(u_0 - \underline{u})(t) \geq 0, \quad \text{but not identical to 0 in } t \in [0, 1].$$

It follows from the maximum principle that $u_0 > \underline{u}$; similarly we have $\overline{u} > u_0$. By a weak version of the mountain pass lemma, there is a mountain pass point u_3 . Thus, we have

$$C_k(\widehat{J}, u_3) = \begin{cases} G, & k = 1, \\ 0, & k \neq 1. \end{cases}$$

But from (H_6) , it easy to see that

$$C_k(\widehat{J}, \theta) = \begin{cases} G, & k = 2, \\ 0, & k \neq 2. \end{cases}$$

Hence, $u_3 \neq \theta$. It follows from [10], Lemma 2.1, p.145, that

$$C_k(\widehat{J}, u_i) = \begin{cases} G, & k = 0, \\ 0, & k \neq 0, \end{cases}$$

where $i = 1, 2$. Noticing that \widehat{J} is bounded from below, we conclude that there is at least another critical point u_4 by using the Morse inequalities.

Obviously, all these critical points $u_i, i = 1, 2, 3, 4$, are solutions of problem (3.3).

On account of the first condition in (H_8) , in combination with the maximum principle, all solutions of (3.3) are bounded in the interval $[-c, c]$. Therefore they are solutions of

(1.1); moreover, all these solutions u , because of their ranges, are included in $[-c, c]$, and we conclude

$$\text{ind}(J, u) + \dim \ker(d^2J(u)) \leq 2 = \dim \bigoplus_{j=1}^2 (K^2 - \lambda_j I),$$

provided by the second condition in (H_8) .

Because of condition (H_7) , we learned from Lemma 2.2 with $\gamma > 2$. Therefore there exists another critical point u_5 , which yields the fifth nontrivial solution for problem (1.1). The proof is completed. \square

We now present some examples. Consider the following problem:

$$\begin{cases} u^{(4)}(t) = f(u(t)), & t \in [0, 1], \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0. \end{cases} \tag{3.4}$$

It is well known that for each $v \in E$, a solution in $C^2[0, 1]$ of the boundary value problem $-u''(t) = v(t)$ for all $t \in [0, 1]$ with $u(0) = u(1) = 0$ is equivalent to a solution in E of the following integral equation:

$$u(t) = \int_0^1 G(t, s)v(s) ds, \quad t \in [0, 1],$$

where $G : [0, 1] \times [0, 1] \rightarrow [0, +\infty)$ is the Green's function of the linear boundary value problem $-u''(t) = 0$ for all $t \in [0, 1]$ with $u(0) = u(1) = 0$, i.e.,

$$G(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Now we define the operator $T : E \rightarrow E$ as follows:

$$Tu(t) = \int_0^1 G(t, s)u(s) ds, \quad t \in [0, 1], u \in E.$$

It is easy to see that $T : E \rightarrow E$ is linear completely continuous. Then problem (3.4) is equivalent to the operator equation

$$u = T^2 \mathbf{f}u,$$

where $\mathbf{f}u(t) = f(u(t))$, $u \in E$.

Example 3.1 Let

$$f(u(t)) = u^3(t), \quad t \in [0, 1].$$

It is obvious that all the conditions of Theorem 3.1 are satisfied. Therefore, (3.4) has at least two nontrivial solutions in E .

Example 3.2 Let

$$f(u) = \begin{cases} 9\pi^3[u \arctan u - \frac{1}{2} \ln(u^2 + 1)] + (\pi^4 - 1)u, & u \geq 0, \\ -9\pi^3[u \arctan u - \frac{1}{2} \ln(u^2 + 1)] + (\pi^4 - 1)u, & u < 0. \end{cases}$$

We note that all the conditions of Theorem 3.2 are satisfied. It follows that (3.4) has at least three nontrivial solutions in E .

Example 3.3 Let

$$f(u(t)) = u^3(t) + 18\pi^4 u(t) - 30.7 \arctan u(t), \quad t \in [0, 1].$$

Since all the conditions of Theorem 3.3 are satisfied, (3.4) has at least five distinct solutions in E .

Competing interests

The author declares that he has no competing interests.

Author's contributions

The author contributed equally in writing this article. He read and approved the final manuscript.

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