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A topological degree for operators of generalized (S_+) type

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Abstract

As an extension of the Leray-Schauder degree, we introduce a topological degree theory for a class of demicontinuous operators of generalized (S_+) type in real reflexive Banach spaces, based on the recent Berkovits degree. Using the degree theory, we show that the Borsuk theorem holds true for this class. Moreover, we study the Dirichlet boundary value problem involving the p-Laplacian by way of an abstract Hammerstein equation.

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1 Introduction

Topological degree theory may be one of the most effective tools in solving nonlinear equations. As a measure of the number of solutions of equation $Fx = h$ for a fixed h, the degree has fundamental properties such as existence, normalization, additivity, and homotopy invariance. The most powerful one that the value of the degree is invariant under appropriate perturbations plays a crucial role in the study of nonlinear differential and integral equations.

Brouwer [1] initiated a topological degree for continuous maps in the Euclidean space. Leray and Schauder [2] developed the degree theory for compact operators in infinite-dimensional Banach spaces. Since then numerous generalizations and applications have been investigated in various ways of approach; see, *e.g.*, [3–6]. Browder [7, 8] introduced a topological degree for nonlinear operators of monotone type in reflexive Banach spaces, where the Galerkin method is used to apply the Brouwer degree. Berkovits [9, 10] gave a new construction of the Browder degree, based on the Leray-Schauder degree. In this point of view, he studied in [11] an extension of the Leray-Schauder degree for operators of generalized monotone type.

We consider an abstract Hammerstein equation of the form

$$u + S \circ Tu = 0,$$

where X is a reflexive Banach space with dual space X^* and $T: X \to X^*$ and $S: X^* \to X$ are operators of monotone type. In the case where S is linear, the solvability of abstract Hammerstein equation was systematically dealt with in [12], with application to Hammerstein integral equations. When S is quasimonotone and T satisfies condition (S_+), it was studied in [11].

In the present paper, our goal is to study the Berkovits degree theory for demicontinuous operators of generalized (S_+) type in real reflexive Banach spaces. To do this, we first observe a class of not necessarily bounded operators satisfying a generalized condition (S_+) with respect to T that contains abstract Hammerstein operators; see Lemma 2.3 below. As the next step for the construction of a new degree, we show that a demicontinuous operator can be reduced to some bounded operator on a suitable domain. Based on the Berkovits degree in [11], we introduce a topological degree for a wider class of demicontinuous operators satisfying a generalized condition (S_+) with respect to T. For the case of unbounded operators of class (S_+), we refer to [9]. Applying the degree theory, we prove that the Borsuk theorem holds true for this class. When a given boundary value problem is transformed to the corresponding integral equation, it may be often written in the form of abstract Hammerstein equation. As an example, we investigate the Dirichlet boundary value problem involving the p-Laplacian by using our degree theory in terms of the Hammerstein equation.

This paper is organized as follows. In Section 2, we introduce some classes of operators of generalized (S_+) type and present some elementary facts which will be later needed. For the construction of a new degree, we show that demicontinuous homotopies can be reduced to bounded homotopies on a suitable domain. In Section 3, we prove that a topological degree for demicontinuous operators satisfying a generalized condition (S_+) with respect to T is well defined and satisfies some of fundamental properties. Using the degree theory, we show that the Borsuk theorem is still valid for our class. In Section 4, we study the solvability of elliptic boundary value problem by way of an abstract Hammerstein equation.

2 Some classes of operators

Let X and Y be two real Banach spaces. Given a nonempty subset Ω of X, let $\overline{\Omega}$ and $\partial\Omega$ denote the closure and the boundary of Ω in X, respectively. Let $B_r(a)$ denote the open ball in X of radius $r > 0$ centered at a. The symbol $\to$ ($\rightharpoonup$) stands for strong (weak) convergence.

An operator $F\colon \Omega \subset X \to Y$ is said to be *bounded* if it takes any bounded set into a bounded set; F is said to be *locally bounded* if for each $u \in \Omega$ there exists a neighborhood U of u such that the set $F(U)$ is bounded. F is said to be *demicontinuous* if for each $u \in \Omega$ and any sequence (u_k) in Ω, $u_k \to u$ implies $Fu_k \rightharpoonup Fu$; F is said to be *compact* if it is continuous and the image of any bounded set is relatively compact.

Let X be a real reflexive Banach space with dual space X^*. The symbol $\langle\cdot,\cdot\rangle_X$ denotes the usual dual paring between X^* and X in this order. In the reflexive case where the bidual space X^{**} is identified with X, we sometimes write $\langle y, x\rangle$ for $\langle x, y\rangle_{X^*}$ for $x \in X$ and $y \in X^*$.

We say that an operator $F\colon \Omega \subset X \to X^*$ satisfies condition (S_+) if for any sequence (u_k) in Ω with $u_k \rightharpoonup u$ and $\limsup\langle Fu_k, u_k - u\rangle \leq 0$, we have $u_k \to u$; F is said to be *quasimonotone* if for any sequence (u_k) in Ω with $u_k \rightharpoonup u$, we have $\limsup\langle Fu_k, u_k - u\rangle \geq 0$.

For any operator $F\colon \Omega \subset X \to X$ and any bounded operator $T\colon \Omega_1 \subset X \to X^*$ such that $\Omega \subset \Omega_1$, we say that F satisfies condition $(S_+)_T$ if for any sequence (u_k) in Ω with $u_k \rightharpoonup u$, $y_k := Tu_k \rightharpoonup y$ and $\limsup\langle Fu_k, y_k - y\rangle \leq 0$, we have $u_k \to u$; we say that F has the property $(QM)_T$ if for any sequence (u_k) in Ω with $u_k \rightharpoonup u$, $y_k := Tu_k \rightharpoonup y$, we have $\limsup\langle Fu_k, y_k - y\rangle \geq 0$.

We consider the following classes of operators:

$$
\begin{aligned}
&\mathcal{F}_1(\Omega)\\
&\quad := \{F\colon \Omega \subset X \to X^* | F \text{ is bounded, demicontinuous and satisfies condition } (S_+)\},\\
&\mathcal{F}_{T,B}(\Omega)\\
&\quad := \{F\colon \Omega \subset X \to X | F \text{ is bounded, demicontinuous and satisfies condition } (S_+)_T\},\\
&\mathcal{F}_T(\Omega) := \{F\colon \Omega \subset X \to X | F \text{ is demicontinuous and satisfies condition } (S_+)_T\},
\end{aligned}
$$

for any $\Omega \subset D_F$ and any $T \in \mathcal{F}_1(\Omega)$, where D_F denotes the domain of F.

Let

$$
\begin{aligned}
&\mathcal{F}_{S_+}(X) := \{F \in \mathcal{F}_1(\overline{G}) | G \in \mathcal{O}\},\\
&\mathcal{F}_B(X) := \{F \in \mathcal{F}_{T,B}(\overline{G}) | G \in \mathcal{O}, T \in \mathcal{F}_1(\overline{G})\},\\
&\mathcal{F}(X) := \{F \in \mathcal{F}_T(\overline{G}) | G \in \mathcal{O}, T \in \mathcal{F}_1(\overline{G})\},
\end{aligned}
$$

where $\mathcal{O}$ denotes the collection of all bounded open sets in X. Here, $T \in \mathcal{F}_1(\overline{G})$ is called an *essential inner map* to F.

First, we establish the relationship between operators satisfying $(S_+)_T$ and $(QM)_T$.

Lemma 2.1 *Let $T : \overline{G} \to X^*$ be a bounded operator, where G is a bounded open set in a real reflexive Banach space X. Then it has the following properties*:

(a) *If $F\colon \overline{G} \to X$ is locally bounded and satisfies condition $(S_+)_T$ and T is continuous, then F has the property $(QM)_T$.*

(b) *If $F\colon \overline{G} \to X$ has the property $(QM)_T$, then for any sequence (u_k) in $\overline{G}$ with $u_k \rightharpoonup u$ and $y_k := Tu_k \rightharpoonup y$, we have*

$$\liminf \langle Fu_k, y_k - y\rangle \geq 0.$$

(c) *If $F_1, F_2\colon \overline{G} \to X$ have the property $(QM)_T$, then so do $F_1 + F_2$ and αF_1 for any positive number α.*

(d) *If $F\colon \overline{G} \to X$ satisfies condition $(S_+)_T$ and $S\colon \overline{G} \to X$ has the property $(QM)_T$, then $F+S$ satisfies condition $(S_+)_T$.*

Proof (a) Assume to the contrary that there exists a sequence (u_k) in $\overline{G}$ with $u_k \rightharpoonup u$, $y_k := Tu_k \rightharpoonup y$ and

$$\limsup \langle Fu_k, y_k - y\rangle < 0. \tag{2.1}$$

Then condition $(S_+)_T$ on F implies that $u_k \to u$ and hence, by the continuity of T, $y_k = Tu_k \to y$. Since F is locally bounded, the set $F(B_r(u))$ is bounded in X for some positive number r. By the boundedness of the sequence (Fu_k), we get

$$\lim \langle Fu_k, y_k - y\rangle = 0,$$

which is a contradiction to our assumption (2.1). We conclude that the operator F has the property $(QM)_T$.

(b) If $q := \liminf\langle Fu_k, y_k - y\rangle < 0$ for some sequence (u_k) in $\overline{G}$ with $u_k \rightharpoonup u$ and $y_k := Tu_k \rightharpoonup y$, then there is a subsequence (u_j) of (u_k) such that

$$\limsup\langle Fu_j, y_j - y\rangle = \lim\langle Fu_j, y_j - y\rangle = q < 0$$

and thus F does not have the property $(QM)_T$.

(c) Let (u_k) be any sequence in $\overline{G}$ with $u_k \rightharpoonup u$ and $y_k := Tu_k \rightharpoonup y$. Since F_1 and F_2 have the property $(QM)_T$, we have by (b)

$$\limsup\bigl\langle (F_1 + F_2)u_k, y_k - y\bigr\rangle \geq \limsup\langle F_1u_k, y_k - y\rangle + \liminf\langle F_2u_k, y_k - y\rangle \geq 0.$$

Therefore, the sum $F_1 + F_2$ has the property $(QM)_T$. If $\alpha > 0$, then it is clear that αF_1 has the property $(QM)_T$.

(d) Let (u_k) be any sequence in $\overline{G}$ such that

$$u_k \rightharpoonup u, \qquad y_k := Tu_k \rightharpoonup y, \quad \text{and} \quad \limsup\bigl\langle (F + S)u_k, y_k - y\bigr\rangle \leq 0.$$

Then we have by (b)

$$\begin{aligned}\limsup\langle Fu_k, y_k - y\rangle &\leq \limsup\langle Fu_k, y_k - y\rangle + \liminf\langle Su_k, y_k - y\rangle \\ &\leq \limsup\bigl\langle (F + S)u_k, y_k - y\bigr\rangle \\ &\leq 0.\end{aligned}$$

Since F satisfies condition $(S_+)_T$, we obtain that $u_k \to u$. Therefore, the operator $F + S$ satisfies condition $(S_+)_T$. This completes the proof. □

Remark 2.2 Note that each demicontinuous operator $F : \overline{G} \to X$ is locally bounded. In particular, if $F \in \mathcal{F}_T(\overline{G})$ and T is continuous, then F has the property $(QM)_T$.

The following result shows that the Hammerstein operator of the form $I + S \circ T$ belongs to the class $\mathcal{F}(X)$; see [11], Lemma 2.2, for the class $\mathcal{F}_B(X)$.

Lemma 2.3 *Suppose that $T \in \mathcal{F}_1(\overline{G})$ is continuous and $S \colon D_S \subset X^* \to X$ is demicontinuous such that $T(\overline{G}) \subset D_S$, where G is a bounded open set in a real reflexive Banach space X. Then the following statements are true*:

(a) *If S is quasimonotone, then $I + S \circ T \in \mathcal{F}_T(\overline{G})$, where I denotes the identity operator.*

(b) *If S satisfies condition (S_+), then $S \circ T \in \mathcal{F}_T(\overline{G})$.*

Proof (a) Set $F := I + S \circ T$. Let (u_k) be any sequence in $\overline{G}$ such that

$$u_k \rightharpoonup u, \qquad y_k := Tu_k \rightharpoonup y, \quad \text{and} \quad \limsup\langle Fu_k, y_k - y\rangle \leq 0. \tag{2.2}$$

Since the sequence $(\langle Tu_k, u_k - u\rangle)$ is bounded in $\mathbb{R}$, there is a subsequence (u_j) of (u_k) such that $\lim\langle Tu_j, u_j - u\rangle$ exists. In view of $X^{**} \cong X$, we know that

$$\lim\langle u_j, y_j - y\rangle_{X^*} = \lim\langle Tu_j, u_j - u\rangle_X. \tag{2.3}$$

By the quasimonotonicity of S, (2.2), and (2.3), we get

$$\begin{aligned} 0 &\le \limsup \langle Sy_j, y_j - y\rangle \\ &= \limsup \langle u_j + Sy_j, y_j - y\rangle - \lim \langle u_j, y_j - y\rangle \\ &\le -\lim \langle Tu_j, u_j - u\rangle. \end{aligned}$$

Since T satisfies condition (S_+), we have $u_j \to u$. By the convergence principle in [6], Proposition 10.13, the entire sequence (u_k) converges strongly to u. Thus, the operator F satisfies condition $(S_+)_T$. Since F is demicontinuous on $\overline{G}$, we conclude that $F \in \mathcal{F}_T(\overline{G})$.

(b) Let (u_k) be any sequence in $\overline{G}$ such that

$$u_k \rightharpoonup u, \qquad y_k := Tu_k \rightharpoonup y, \quad \text{and} \quad \limsup \langle S \circ Tu_k, y_k - y\rangle \le 0.$$

Since S satisfies condition (S_+), it follows that $y_k \to y$. Since $\lim\langle Tu_k, u_k - u\rangle = 0$ and T satisfies condition (S_+), we have $u_k \to u$. Consequently, we obtain that $S \circ T \in \mathcal{F}_T(\overline{G})$. This completes the proof. □

We give a simple example of an operator which satisfies condition $(S_+)_T$ but not condition (S_+); see also [11], Example 3.3.

Example 2.4 Let $(X, \langle\cdot,\cdot\rangle)$ be an infinite-dimensional real Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. If we define a linear operator $T : X \to X$ by setting

$$Te_n = e_n + (-1)^n e_{n+(-1)^n} \quad \text{for } n \in \mathbb{N},$$

then the operator $F := T \circ T$ satisfies condition $(S_+)_T$. However, F does not satisfy condition (S_+).

Proof From $\langle Tu, u\rangle = \|u\|^2$ for all $u \in X$ it follows that T is bounded, continuous and satisfies condition (S_+). In virtue of Lemma 2.3(b), the operator $F = T \circ T$ satisfies condition $(S_+)_T$. If we take a sequence (u_k), where $u_k = e_{2k}$, then it is easy to see that $u_k \rightharpoonup 0$ and

$$\limsup \langle Fu_k, u_k\rangle = \limsup \langle 2e_{2k+1}, e_{2k}\rangle = 0,$$

but the sequence (u_k) does not converge strongly. Thus, F does not satisfy condition (S_+). This completes the proof. □

For a bounded operator $T: \overline{G} \subset X \to X^*$, we say that a homotopy $H: [0,1] \times \overline{G} \to X$ satisfies condition $(S_+)_T$ if for any sequence (t_k, u_k) in $[0,1] \times \overline{G}$ such that

$$u_k \rightharpoonup u, \qquad t_k \to t, \qquad y_k := Tu_k \rightharpoonup y, \quad \text{and} \quad \limsup \bigl\langle H(t_k, u_k), y_k - y\bigr\rangle \le 0,$$

we have $u_k \to u$.

The following result tells us that each affine homotopy with a common essential inner map satisfies condition $(S_+)_T$.

Lemma 2.5 *Let G be a bounded open subset of a real reflexive Banach space X, and let $T \in \mathcal{F}_1(\overline{G})$ be continuous. If $F, S \in \mathcal{F}_T(\overline{G})$, then an affine homotopy $H\colon [0,1] \times \overline{G} \to X$ defined by*

$$H(t,u) := (1-t)Fu + tSu \quad \text{for } (t,u) \in [0,1] \times \overline{G}$$

satisfies condition $(S_+)_T$.

In this case, the homotopy is called an *admissible affine homotopy* with the common essential inner map T.

Proof of Lemma 2.5 Let (u_k) be any sequence in $\overline{G}$ and (t_k) any sequence in $[0,1]$ such that

$$u_k \rightharpoonup u, \qquad t_k \to t, \qquad y_k := Tu_k \rightharpoonup y, \quad \text{and} \quad \limsup \bigl\langle H(t_k,u_k), y_k - y \bigr\rangle \le 0.$$

Note that

$$\bigl\langle H(t_k,u_k), y_k - y \bigr\rangle = (1-t_k)\langle Fu_k, y_k - y\rangle + t_k \langle Su_k, y_k - y\rangle.$$

If $t = 1$, then it follows from $S \in \mathcal{F}_T(\overline{G})$ that

$$\limsup \langle Su_k, y_k - y\rangle = \limsup \bigl\langle H(t_k,u_k), y_k - y \bigr\rangle \le 0$$

implies $u_k \to u$. If $t \in [0,1)$, then the property $(QM)_T$ of S, in view of Lemma 2.1, implies that

$$\begin{aligned} (1-t)\limsup \langle Fu_k, y_k - y\rangle &\le (1-t)\limsup \langle Fu_k, y_k - y\rangle + t \liminf \langle Su_k, y_k - y\rangle \\ &\le \limsup \bigl\langle H(t_k,u_k), y_k - y \bigr\rangle \\ &\le 0. \end{aligned}$$

Since F satisfies condition $(S_+)_T$, we have $u_k \to u$. In both cases, we have shown that $u_k \to u$. This completes the proof. □

For our aim, we make the following observation to reduce suitably the domain of a demicontinuous homotopy.

Theorem 2.6 *Let G be an open subset of a real reflexive Banach space X, and let Y be a real normed space. Suppose that $H\colon [0,1] \times \overline{G} \to Y$ is a demicontinuous homotopy. If $S \subset G$ is a nonempty compact set, then there exists an open set G_0 and a positive constant R such that*

(a) *$S \subset G_0 \subset G$ and*
(b) *$\|H(t,u)\| \le R$ for all $t \in [0,1]$ and all $u \in \overline{G}_0$.*

If, in addition, the sets G and S are symmetric with respect to the origin $0 \in S$, then G_0 is also symmetric.

Proof Let S be a nonempty compact set with $S \subset G$, and let

$$D_n := \left\{ u \in X \,\middle|\, \operatorname{dist}(u,S) < \frac{1}{n} \right\} \quad \text{for } n \in \mathbb{N}.$$

The compactness of S enables us to write it in the form

$$D_n = \left\{ u \in X \,\middle|\, \|u - z\| < \frac{1}{n} \text{ for some } z \in S \right\}.$$

Setting $G_n := D_n \cap G$, we see that G_n is open and $S \subset G_n \subset G$, that is, (a) holds for each G_n. We now prove that at least one of the sets G_n possesses property (b). If none of the sets G_n satisfies (b), we find sequences (t_n) in $[0,1]$ and (u_n) in $\overline{G}_n$ such that

$$\left\| H(t_n, u_n) \right\| > n. \tag{2.4}$$

In view of $u_n \in \overline{D}_n$, we can choose a sequence (z_n) in S such that $\|u_n - z_n\| \le 2/n$. By the compactness of the set S, there exists a subsequence (z_k) of (z_n) which converges to some $z \in S$. Hence it follows from the inequality

$$\|u_k - z\| \le \|u_k - z_k\| + \|z_k - z\|$$

that $u_k \to z$. We may suppose that $t_k \to t \in [0,1]$. It follows from the demicontinuity of H that $H(t_k, u_k) \rightharpoonup H(t,z)$, which contradicts (2.4), by noting that every weakly convergent sequence in the normed space Y is bounded. Therefore, at least one of the sets G_n satisfies (a) and (b), say G_{n_0}. Set $G_0 := G_{n_0}$.

Next, to show that G_0 is symmetric, let $u \in G_0$, then there exists $z \in S$ such that $\|u - z\| < 1/n_0$. Since $\|(-u) - (-z)\| < 1/n_0$, it follows from $-u \in G$ and $-z \in S$ that $-u \in G_0$. This completes the proof. □

We show that any demicontinuous operator satisfying condition $(S_+)_T$ is proper on closed bounded sets; see [9], Lemma 2.5, for the case of class (S_+).

Lemma 2.7 *Let G be a bounded open set in a real reflexive Banach space X, and let $H : [0,1] \times \overline{G} \to X$ be a demicontinuous homotopy satisfying condition $(S_+)_T$, where $T : \overline{G} \to X^*$ is bounded. For any compact set $A \subset X$,*

$$K := \left\{ u \in \overline{G} | H(t,u) \in A \textit{ for some } t \in [0,1] \right\}$$

is a compact subset of X.

Proof Let (u_k) be any sequence in K. Then there exists a sequence (t_k) in $[0,1]$ such that $H(t_k, u_k) \in A$ for all $k \in \mathbb{N}$. Since A is compact, we can choose a subsequence (u_j) of (u_k) in $\overline{G}$ and a subsequence (t_j) of (t_k) in $[0,1]$ such that $H(t_j, u_j) \to w \in A$. By the boundedness of the set G and the map T, we may suppose, without loss of generality, that

$$u_j \rightharpoonup u \in X, \qquad y_j := Tu_j \rightharpoonup y \in X^*, \quad \text{and} \quad t_j \to t \in [0,1].$$

Since we have $\lim\langle H(t_j,u_j),y_j-y\rangle = 0$, the assumptions on the homotopy H imply that $u_j \to u$ and $H(t_j,u_j) \rightharpoonup H(t,u)$. Consequently, we have $H(t,u) = w \in A$ and $u \in \overline{G}$, which means that $u \in K$. Thus, the set K is compact. This completes the proof. □

Corollary 2.8 *Suppose that $F\colon \overline{G} \to X$ is demicontinuous and satisfies condition $(S_+)_T$, where G is a bounded open subset of X and T is bounded on $\overline{G}$. For every $h \notin F(\partial G)$, there exists an open set G_0 such that $F^{-1}(h) \subset G_0 \subset G$ and F is bounded on $\overline{G}_0$.*

Proof Let $h \notin F(\partial G)$. By Lemma 2.7, $F^{-1}(h)$ is a compact subset of X and $F^{-1}(h) \subset G$. Applying Theorem 2.6 with the constant homotopy F and $S = F^{-1}(h)$, there exists an open set G_0 such that $F^{-1}(h) \subset G_0 \subset G$ and F is bounded on $\overline{G}_0$. □

Corollary 2.9 *Let G be a bounded open symmetric subset of X with respect to the origin $0 \in G$. Suppose that $F\colon \overline{G} \to X$ is odd, demicontinuous and satisfies condition $(S_+)_T$ with $0 \notin F(\partial G)$, where T is bounded on $\overline{G}$. Then there exists a symmetric open set G_0 such that $F^{-1}(0) \subset G_0 \subset G$ and F is bounded on $\overline{G}_0$.*

Proof Since F is odd on $\overline{G}$ and $0 \notin F(\partial G)$, it is obvious from Lemma 2.7 that $0 \in F^{-1}(0) \subset G$ and $F^{-1}(0)$ is symmetric and compact. By Theorem 2.6, there exists a symmetric open set G_0 such that $F^{-1}(0) \subset G_0 \subset G$ and F is bounded on $\overline{G}_0$. □

3 Degree theory

In this section, we extend the degree theory of Berkovits to all demicontinuous operators satisfying condition $(S_+)_T$ in the class $\mathcal{F}(X)$, and this is used to establish the Borsuk theorem.

In what follows, X will always be an infinite-dimensional real reflexive separable Banach space which has been renormed so that both X and X^* are locally uniformly convex.

We first introduce the topological degree for the class $\mathcal{F}_B(X)$ due to Berkovits [11]. For the details on the class $\mathcal{F}_{S_+}(X)$, we refer to [9, 10].

Theorem 3.1 *There exists a unique degree function*

$$d_B : \bigl\{(F,G,h) | G \in \mathcal{O}, T \in \mathcal{F}_1(\overline{G}), F \in \mathcal{F}_{T,B}(\overline{G}), h \notin F(\partial G)\bigr\} \to \mathbb{Z}$$

that satisfies the following properties:

(a) (Existence) *If $d_B(F,G,h) \neq 0$, then the equation $Fu = h$ has a solution in G.*

(b) (Additivity) *Let $F \in \mathcal{F}_{T,B}(\overline{G})$. If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F(\overline{G} \setminus (G_1 \cup G_2))$, then we have*

$$d_B(F,G,h) = d_B(F,G_1,h) + d_B(F,G_2,h).$$

(c) (Homotopy invariance) *If $H\colon [0,1] \times \overline{G} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and $h : [0,1] \to X$ is a continuous path in X such that $h(t) \notin H(t,\partial G)$ for all $t \in [0,1]$, then the value of $d_B(H(t,\cdot),G,h(t))$ is constant for all $t \in [0,1]$.*

(d) (Normalization) *For any $h \in G$, we have $d_B(I,G,h) = +1$.*

Lemma 3.2 *Let $F \in \mathcal{F}_T(\overline{G})$ be an operator, where G is a bounded open set in X and $T \in \mathcal{F}_1(\overline{G})$. Suppose that for $i = 1, 2$, G_i is an open subset of G such that*

$$F^{-1}(h) \subset G_i \subset G \quad \text{and} \quad F \text{ is bounded on } \overline{G}_i.$$

Then the degree $d_B(F, G_i, h)$ is well defined for $i = 1, 2$ and

$$d_B(F, G_1, h) = d_B(F, G_2, h).$$

Proof For $i = 1, 2$, since $F \in \mathcal{F}_{T,B}(\overline{G}_i)$ and $h \notin F(\partial G_i)$, the degree $d_B(F, G_i, h)$ is well defined and $F^{-1}(h) \subset G_1 \cap G_2 \subset G_i$ implies $h \notin F(\overline{G}_i \setminus (G_1 \cap G_2))$. Applying Theorem 3.1(b) twice, we get

$$d_B(F, G_1, h) = d_B(F, G_1 \cap G_2, h) = d_B(F, G_2, h). \qquad \square$$

Definition 3.3 Let

$$M = \{(F, G, h) | G \in \mathcal{O}, T \in \mathcal{F}_1(\overline{G}), F \in \mathcal{F}_T(\overline{G}), h \notin F(\partial G)\}.$$

Then we define a degree function $d\colon M \to \mathbb{Z}$ as follows:

$$d(F, G, h) := d_B(F|_{\overline{G}_0}, G_0, h),$$

where G_0 is any open subset of G with $F^{-1}(h) \subset G_0$ and F is bounded on $\overline{G}_0$, according to Corollary 2.8. Here, $F|_{\overline{G}_0}$ denotes the restriction of F to $\overline{G}_0$.

In view of Lemma 3.2, the degree d does not depend on the choice of the set G_0. Especially, if F is bounded on $\overline{G}$, then we may take $G_0 = G$ and $d(F, G, h) = d_B(F, G, h)$, which means that d and d_B coincide on $\mathcal{F}_{T,B}(\overline{G})$.

Theorem 3.4 *The above degree d for the class $\mathcal{F}(X)$ has the following properties*:

(a) (Existence) *If $d(F, G, h) \neq 0$, then the equation $Fu = h$ has a solution in G.*

(b) (Additivity) *Let $F \in \mathcal{F}_T(\overline{G})$. If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F(\overline{G} \setminus (G_1 \cup G_2))$, then we have*

$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h).$$

(c) (Homotopy invariance) *Suppose that $H\colon [0,1] \times \overline{G} \to X$ is an admissible affine homotopy with a common continuous essential inner map and $h : [0,1] \to X$ is a continuous path in X such that $h(t) \notin H(t, \partial G)$ for all $t \in [0,1]$. Then the value of $d(H(t, \cdot), G, h(t))$ is constant for all $t \in [0,1]$.*

(d) (Normalization) *For any $h \in G$, we have $d(I, G, h) = +1$.*

(e) (Boundary dependence) *If $F, S \in \mathcal{F}_T(\overline{G})$ coincide on ∂G and $h \notin F(\partial G)$, then*

$$d(F, G, h) = d(S, G, h).$$

Proof (a) If $d(F,G,h) \neq 0$, then we have by Definition 3.3

$$d_B(F,G_0,h) \neq 0$$

for a suitable open set $G_0 \subset G$. By Theorem 3.1(a), the equation $Fu = h$ has a solution in G_0 which also belongs to G.

(b) Let $K = \{u \in G\colon Fu = h\}$ and $K_i = \{u \in G_i\colon Fu = h\}$ for $i = 1,2$. Note by hypotheses that K is the disjoint union of the sets K_1 and K_2. By Corollary 2.8, there exist open sets G_{0i} such that $K_i \subset G_{0i} \subset G_i$, and F is bounded on $\overline{G}_{0i}$ for $i = 1,2$. Set $G_0 := G_{01} \cup G_{02}$. Obviously, F is bounded on $\overline{G}_0$ and $K \subset G_0 \subset G$, and so $h \notin F(\overline{G}_0 \setminus (G_{01} \cup G_{02}))$. Hence it follows from Definition 3.3 and Theorem 3.1(b) that

$$\begin{aligned} d(F,G,h) &= d_B(F|_{\overline{G}_0},G_0,h) \\ &= d_B(F|_{\overline{G}_{01}},G_{01},h) + d_B(F|_{\overline{G}_{02}},G_{02},h) \\ &= d(F,G_1,h) + d(F,G_2,h). \end{aligned}$$

(c) Note that $A = \{h(t) \in X | t \in [0,1]\}$ is a compact subset of X. By Lemma 2.7,

$$S = \bigl\{u \in \overline{G} | H(t,u) \in A \text{ for some } t \in [0,1]\bigr\}$$

is a compact subset of X. In particular, we have $S \subset G$ in view of $h(t) \notin H(t,\partial G)$ for all $t \in [0,1]$. According to Theorem 2.6, there exists an open set G_0 such that $S \subset G_0 \subset G$ and H is bounded on $[0,1] \times \overline{G}_0$. This implies that $h(t) \notin H(t,\partial G_0)$ and

$$d\bigl(H(t,\cdot),G,h(t)\bigr) = d_B\bigl(H(t,\cdot),G_0,h(t)\bigr) \quad \text{for each } t \in [0,1].$$

By Theorem 3.1(c), we conclude that the value of $d(H(t,\cdot),G,h(t))$ is constant for all $t \in [0,1]$.

(d) Since the identity operator I is bounded, it is an immediate consequence of Theorem 3.1(d). Actually, the identity operator $I = J^{-1} \circ J$ belongs to $\mathcal{F}_J(\overline{G})$, in view of Lemma 2.3, where J denotes the duality operator. It is known in, *e.g.*, [12] that $J : X \to X^*$ is bounded, continuous and satisfies condition (S_+) and $J^{-1} : X^* \to X$ is continuous and satisfies condition (S_+).

(e) Consider an affine homotopy $H : [0,1] \times \overline{G} \to X$ given by

$$H(t,u) = (1-t)Fu + tSu \quad \text{for } (t,u) \in [0,1] \times \overline{G}.$$

As $h \notin F(\partial G) = H(t,\partial G)$ for all $t \in [0,1]$, the homotopy invariance property (c) of the degree d implies that

$$d(F,G,h) = d(S,G,h).$$

This completes the proof. □

For the next aim, we need the Borsuk theorem for operators in $\mathcal{F}_B(X)$ taken from [11], Theorem 8.1.

Lemma 3.5 *Let G be a bounded open subset of X which is symmetric with respect to the origin $0 \in G$, and let $T \in \mathcal{F}_1(\overline{G})$ be continuous. If $F \in \mathcal{F}_{T,B}(\overline{G})$ is odd on ∂G such that $0 \notin F(\partial G)$, then $d_B(F, G, 0)$ is an odd number.*

Now we give a new version of the Borsuk theorem for operators in $\mathcal{F}(X)$.

Theorem 3.6 *Let G be a bounded open set in X which is symmetric with respect to $0 \in G$. Suppose that $T \in \mathcal{F}_1(\overline{G})$ is continuous and odd on $\overline{G}$ and $F \in \mathcal{F}_T(\overline{G})$ is odd on ∂G with $0 \notin F(\partial G)$. Then $d(F, G, 0)$ is an odd number, and the equation $Fu = 0$ has at least one solution in G.*

Proof Let $P\colon \overline{G} \to X$ be an operator defined by

$$Pu := \frac{1}{2}\bigl(Fu - F(-u)\bigr) \quad \text{for } u \in \overline{G}.$$

Then P is odd and demicontinuous on $\overline{G}$. Since F is odd on ∂G and $0 \notin F(\partial G)$, it is clear that $0 \notin P(\partial G)$. To show that P satisfies condition $(S_+)_T$, let (u_k) be any sequence in $\overline{G}$ such that

$$u_k \rightharpoonup u, \qquad y_k := Tu_k \rightharpoonup y, \quad \text{and} \quad \limsup \langle Pu_k, y_k - y\rangle \leq 0.$$

Since T is odd and continuous on $\overline{G}$ and $F \in \mathcal{F}_T(\overline{G})$, we have by Lemma 2.1(a) and (b)

$$\begin{aligned} 0 &\geq \limsup \langle Pu_k, y_k - y\rangle \\ &\geq \frac{1}{2}\limsup \langle Fu_k, y_k - y\rangle + \frac{1}{2}\liminf \bigl\langle F(-u_k), T(-u_k) - (-y)\bigr\rangle \\ &\geq \frac{1}{2}\limsup \langle Fu_k, y_k - y\rangle. \end{aligned}$$

Since F satisfies condition $(S_+)_T$, this implies that $u_k \to u$ and thus P satisfies condition $(S_+)_T$. In view of Corollary 2.9, we can choose a symmetric open subset G_0 of G such that

$$P^{-1}(0) \subset G_0 \subset G \quad \text{and} \quad P \text{ is bounded on } \overline{G}_0.$$

Since F and P coincide on ∂G, we have by Theorem 3.4(e)

$$d(F, G, 0) = d(P, G, 0). \tag{3.1}$$

It follows from Theorem 3.4(b) that

$$d(P, G, 0) = d(P, G_0, 0) = d_B(P|_{\overline{G}_0}, G_0, 0). \tag{3.2}$$

Since the restriction $P|_{\overline{G}_0} \in \mathcal{F}_{T,B}(\overline{G}_0)$ is odd on ∂G_0, Lemma 3.5 says that

$$d_B(P|_{\overline{G}_0}, G_0, 0) \text{ is odd.}$$

Combining this with (3.1) and (3.2), we conclude that $d(F, G, 0)$ is an odd number. By Theorem 3.4(a), the equation $Fu = 0$ has a solution in G. This completes the proof. □

4 Application

In this section, we study the Dirichlet boundary value problem based on the degree theory in Section 3.

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. Let $2 < p < N$ and set $p' = p/(p-1)$. We consider a nonlinear equation of the form

$$\begin{cases} -\Delta_p u = u + f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Δ_p is the p-Laplacian given by

$$\Delta_p u = \sum_{i=1}^{N} D_i\big(|D_i u|^{p-2} D_i u\big).$$

Assume that $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a real-valued function such that

(f1) f satisfies the Carathéodory condition, that is, $f(\cdot, \eta, \zeta)$ is measurable on Ω for all $(\eta, \zeta) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for almost all $x \in \Omega$.

(f2) f has the growth condition

$$\big|f(x, \eta, \zeta)\big| \le c\big(k(x) + |\eta|^{q-1} + |\zeta|^{q-1}\big)$$

for almost all $x \in \Omega$ and all $(\eta, \zeta) \in \mathbb{R} \times \mathbb{R}^N$, where c is a positive constant, $1 < q < p$, and $k \in L^{p'}(\Omega)$.

Let $W_0^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in the Sobolev space

$$W^{1,p}(\Omega) = \big\{u \in L^p(\Omega) | D_i u \in L^p(\Omega) \text{ for } i = 1, \ldots, N\big\},$$

equipped with the norm

$$\|u\|_{1,p} = \left(\|u\|_p^p + \sum_{i=1}^{N} \|D_i u\|_p^p\right)^{\frac{1}{p}},$$

where $\|\cdot\|_p$ stands for the norm on $L^p(\Omega)$. Due to the Poincaré inequality, the norm $\|\cdot\|_{1,p}$ on $W_0^{1,p}(\Omega)$ is equivalent to the norm $\|\cdot\|$ given by

$$\|u\| = \left(\sum_{i=1}^{N} \|D_i u\|_p^p\right)^{\frac{1}{p}} \quad \text{for } u \in W_0^{1,p}(\Omega). \tag{4.2}$$

Note that the Sobolev space $W_0^{1,p}(\Omega)$ is a uniformly convex Banach space and the embedding $I : W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact; see, *e.g.*, [12].

A point $u \in W_0^{1,p}(\Omega)$ is said to be a *weak solution* of (4.1) if

$$\sum_{i=1}^{N} \int_\Omega |D_i u|^{p-2} D_i u D_i \psi \, dx = \int_\Omega \big(u + f(x, u, \nabla u)\big)\psi \, dx \quad \text{for all } \psi \in W_0^{1,p}(\Omega).$$

Lemma 4.1 *Under assumptions* (f1) *and* (f2), *the operator* $S\colon W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ *setting by*

$$\langle Su, v\rangle = -\int_\Omega \bigl(u + f(x, u, \nabla u)\bigr) v \, dx \quad \text{for } u, v \in W_0^{1,p}(\Omega)$$

is compact.

Proof Let $X = W_0^{1,p}(\Omega)$ be the Sobolev space with the norm

$$\|u\|_X = \left(\|u\|_p^p + \sum_{i=1}^N \|D_i u\|_p^p \right)^{\frac{1}{p}}.$$

Let $\Phi : X \to L^{p'}(\Omega)$ be an operator defined by

$$\Phi u(x) := -f\bigl(x, u(x), \nabla u(x)\bigr) \quad \text{for } u \in X \text{ and } x \in \Omega.$$

We first show that the operator Φ is bounded and continuous. Note that the embedding $L^p(\Omega) \hookrightarrow L^{(q-1)p'}(\Omega)$ is continuous, that is,

$$\|u\|_{(q-1)p'} \le c_1 \|u\|_p \quad \text{for all } u \in L^p(\Omega), \tag{4.3}$$

where c_1 is a positive constant. For each $u \in X$, we have by the growth condition (f2) and (4.3)

$$\begin{aligned}
\|\Phi u\|_{p'} &= \left(\int_\Omega \bigl| f\bigl(x, u(x), \nabla u(x)\bigr)\bigr|^{p'} dx \right)^{\frac{1}{p'}} \\
&\le \text{const}\left(\int_\Omega \left(|k| + |u|^{q-1} + \sum_{i=1}^N |D_i u|^{q-1} \right)^{p'} dx \right)^{\frac{1}{p'}} \\
&\le \text{const}\left(\|k\|_{p'} + \|u\|_{(q-1)p'}^{q-1} + \sum_{i=1}^N \|D_i u\|_{(q-1)p'}^{q-1} \right) \\
&\le \text{const}\left(\|k\|_{p'} + \|u\|_p^{q-1} + \sum_{i=1}^N \|D_i u\|_p^{q-1} \right) \\
&\le \text{const}\bigl(\|k\|_{p'} + \|u\|_X^{q-1} \bigr).
\end{aligned}$$

This implies that Φ is bounded on X. To show that Φ is continuous, let $u_k \to u$ in X. Then $u_k \to u$ and $D_i u_k \to D_i u$ in $L^p(\Omega)$ for $i = 1, \dots, N$. Hence there exist a subsequence (u_j) of (u_k) and measurable functions v, w_i in $L^p(\Omega)$ for $i = 1, \dots, N$ such that

$$\begin{aligned}
&u_j(x) \to u(x) \quad \text{and} \quad D_i u_j(x) \to D_i u(x), \\
&\bigl|u_j(x)\bigr| \le v(x) \quad \text{and} \quad \bigl|D_i u_j(x)\bigr| \le w_i(x)
\end{aligned}$$

for almost all $x \in \Omega$ and for every $i \in \{1,\dots,N\}$ and all $j \in \mathbb{N}$. Since f satisfies the Carathéodory condition, we obtain that

$$f\big(x,u_j(x),\nabla u_j(x)\big) \to f\big(x,u(x),\nabla u(x)\big) \quad \text{for almost all } x \in \Omega.$$

It follows from (f2) and $v, w_i \in L^{(q-1)p'}(\Omega)$ that

$$\big|f\big(x,u_j(x),\nabla u_j(x)\big)\big| \le \text{const}\left(k(x) + \big|v(x)\big|^{q-1} + \sum_{i=1}^{N}\big|w_i(x)\big|^{q-1}\right)$$

for almost all $x \in \Omega$ and for all $j \in \mathbb{N}$ and

$$k + |v|^{q-1} + \sum_{i=1}^{N} |w_i|^{q-1} \in L^{p'}(\Omega).$$

Taking into account the identity

$$\|\Phi u_j - \Phi u\|_{p'}^{p'} = \int_\Omega \big|f\big(x,u_j(x),\nabla u_j(x)\big) - f\big(x,u(x),\nabla u(x)\big)\big|^{p'}\,dx,$$

the Lebesgue dominated convergence theorem implies that

$$\Phi u_j \to \Phi u \quad \text{in } L^{p'}(\Omega).$$

The convergence principle tells us that the entire sequence (Φu_k) converges to Φu in $L^{p'}(\Omega)$. We have just proved that Φ is continuous on X. Since the embedding $I : X \hookrightarrow L^p(\Omega)$ is compact, it is known that the adjoint operator $I^* : L^{p'}(\Omega) \to X^*$ is also compact. Therefore, the composition $I^* \circ \Phi : X \to X^*$ is compact. Moreover, considering the operator $J : X \to X^*$ given by

$$\langle Ju, v\rangle = -\int_\Omega uv\,dx \quad \text{for } u, v \in X,$$

it can be seen that J is compact, by noting that the embedding $i : L^p(\Omega) \hookrightarrow L^{p'}(\Omega)$ is continuous and $J = -I^* \circ i \circ I$. We conclude that $S = J + I^* \circ \Phi$ is compact. This completes the proof. □

Now we can show the solvability of the given boundary value problem involving the p-Laplacian by using the degree theory.

Theorem 4.2 *Under assumptions* (f1) *and* (f2), *problem* (4.1) *has a weak solution* u *in* $W_0^{1,p}(\Omega)$.

Proof Let $Y = W_0^{1,p}(\Omega)$ be the Sobolev space, and let $S : Y \to Y^*$ be as in Lemma 4.1. Define an operator $F : Y \to Y^*$ by the relation

$$\langle Fu, v\rangle = \sum_{i=1}^{N} \int_\Omega |D_i u|^{p-2} D_i u D_i v\,dx \quad \text{for } u, v \in Y.$$

Then $u \in Y$ is a weak solution of (4.1) if and only if

$$Fu = -Su. \tag{4.4}$$

It is known in [12], Proposition 26.10, that the operator $F : Y \to Y^*$ is bounded, continuous, and uniformly monotone. In particular, it is coercive and satisfies condition (S_+). Now let $X = Y^*$ and identify X^* with Y. By the main theorem on monotone operators due to Browder and Minty in [12], Theorem 26.A, the inverse operator $T := F^{-1} : X \to X^*$ is bounded, continuous and satisfies condition (S_+), where the last follows from the fact that F is continuous and satisfies condition (S_+) and T is bounded. Moreover, note by Lemma 4.1 that the operator $S : X^* \to X$ is bounded, continuous, and quasimonotone. Consequently, equation (4.4) is equivalent to

$$u = Tv \quad \text{and} \quad v + S \circ Tv = 0. \tag{4.5}$$

To solve equation (4.5), we will apply the degree theory for $\mathcal{F}(X)$. To do this, we first claim that the set

$$B := \{v \in X | v + tS \circ Tv = 0 \text{ for some } t \in [0,1]\}$$

is bounded. Indeed, let $v \in B$, that is, $v + tS \circ Tv = 0$ for some $t \in [0,1]$. Set $u := Tv$. Noting that the embeddings $L^p(\Omega) \hookrightarrow L^2(\Omega)$, $L^p(\Omega) \hookrightarrow L^q(\Omega)$, $L^p(\Omega) \hookrightarrow L^{(q-1)p'}(\Omega)$, and $Y \hookrightarrow L^p(\Omega)$ are continuous, we get by the growth condition (f2) the estimate

$$\begin{aligned} \|Tv\|^p &= \langle v, Tv\rangle = -t\langle S \circ Tv, Tv\rangle \\ &= t\int_\Omega \big(u + f(x,u,\nabla u)\big)u\,dx \\ &\le \text{const}\big(\|Tv\|^2 + \|Tv\|^q + \|Tv\|\big), \end{aligned}$$

where $\|\cdot\|$ denotes the equivalent norm on Y given by (4.2). From $p > 2$ and $p > q$ it follows that

$$\{Tv | v \in B\} \text{ is bounded.}$$

Since the operator S is bounded, it is obvious from (4.5) that the set B is bounded in X. We can now choose a positive constant R such that

$$\|v\|_X < R \quad \text{for all } v \in B.$$

This says that

$$v + tS \circ Tv \neq 0 \quad \text{for all } v \in \partial B_R(0) \text{ and all } t \in [0,1].$$

From Lemma 2.3 it follows that

$$I + S \circ T \in \mathcal{F}_T\big(\overline{B_R(0)}\big) \quad \text{and} \quad I = F \circ T \in \mathcal{F}_T\big(\overline{B_R(0)}\big).$$

Consider a homotopy $H\colon [0,1] \times \overline{B_R(0)} \to X$ given by

$$H(t,v) := v + tS \circ Tv \quad \text{for } (t,v) \in [0,1] \times \overline{B_R(0)}.$$

Applying the homotopy invariance and normalization property of the degree d stated in Theorem 3.4, we get

$$d\big(I + S \circ T, B_R(0), 0\big) = d\big(I, B_R(0), 0\big) = 1,$$

and hence there exists a point $v \in B_R(0)$ such that

$$v + S \circ Tv = 0.$$

We conclude that $u = Tv$ is a weak solution of (4.1). This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
KI conceived of the study and drafted the manuscript. HS participated in coordination. All authors approved the final manuscript.

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