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Strong convergence theorem for totally quasi-ϕ-asymptotically nonexpansive multi-valued mappings under relaxed conditions

Shanguang Qian*

*Correspondence:
qiansg1975@126.com
School of Land Resources Engineering, Kunming University of Science and Technology, Kunming, Yunnan, P.R. China
Architectural Engineering Faculty, Kunming Metallurgy College, Kunming, Yunnan, P.R. China

Abstract

We construct a relaxed hybrid shrinking iteration algorithm for approximating common fixed points of a countable family of totally quasi-ϕ-asymptotically nonexpansive multi-valued mappings. A strong convergence theorem for solving generalized mixed equilibrium problems is established in the framework of Banach spaces under relaxed conditions. Since there is no need to impose a uniformity assumption on the involved mappings and no need to compute complex series in the iteration process, the results improve those of the authors with related interests.

MSC: 47J06; 47J25

Keywords: strong convergence; generalized mixed equilibrium problems; totally quasi-ϕ-asymptotically nonexpansive multi-valued mappings

1 Introduction

Throughout this paper we assume that E is a real Banach space with its dual E^*, C is a nonempty closed convex subset of E and $J : E \to 2^{E^*}$ is the *normalized duality mapping* defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E.$$

In the sequel, we use $F(T)$ to denote the set of fixed points of a mapping T.

Definition 1.1 [1] (1) A multi-valued mapping $T : C \to 2^C$ is said to be *totally quasi-ϕ-asymptotically nonexpansive*, if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ with $\zeta(0) = 0$ such that

$$\phi(p, w_n) \le \phi(p, x) + \nu_n \zeta\bigl(\phi(p, x)\bigr) + \mu_n, \quad \forall n \ge 1, x \in C, w \in T^n x, p \in F(T), \tag{1.1}$$

where $\phi : E \times E \to \mathbb{R}^+ \cup \{0\}$ denotes the *Lyapunov functional* defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.2}$$

It is obvious from the definition of ϕ that

$$\left(\|x\| - \|y\|\right)^2 \leq \phi(x,y) \leq \left(\|x\| + \|y\|\right)^2 \tag{1.3}$$

and

$$\phi\left(x, J^{-1}\left(\lambda Jy + (1-\lambda)Jz\right)\right) \leq \lambda\phi(x,y) + (1-\lambda)\phi(x,z), \quad \forall x,y \in E, \lambda \in [0,1]. \tag{1.4}$$

(2) A countable family of multi-valued mappings $\{T_i\}: C \to C$ said to be *uniformly totally quasi-ϕ-asymptotically nonexpansive*, if $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ with $\zeta(0) = 0$ such that

$$\phi(p, w_{n,i}) \leq \phi(p,x) + \nu_n \zeta\left(\phi(p,x)\right) + \mu_n, \quad \forall n \geq 1, w_{n,i} \in T_i^n x, i \geq 1, x \in C, p \in F. \tag{1.5}$$

(3) A totally quasi-ϕ-asymptotically nonexpansive multi-valued mapping $T: C \to 2^C$ is said to be *uniformly L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\|w_n - s_n\| \leq L\|x - y\|, \quad \forall n \geq 1, x,y \in C, w_n \in T^n x, s_n \in T^n y. \tag{1.6}$$

Let $\theta : C \times C \to \mathbb{R}$ be a bifunction, $\psi : C \to \mathbb{R}$ a real valued function and $A : C \to E^*$ a nonlinear mapping. The so-called *generalized mixed equilibrium problem GMEP* is to find an $u \in C$ such that

$$\theta(u,y) + \langle Au, y-u\rangle + \psi(y) - \psi(u) \geq 0, \quad \forall y \in C, \tag{1.7}$$

whose set of solutions is denoted by Ω.

In 2012, Chang *et al.* [1] used the following hybrid shrinking iteration algorithm finding a common element of the set of solutions for a GMEP, the set of solutions for variational inequality problems, and the set of common fixed points for a countable family of multi-valued total quasi-ϕ-asymptotically nonexpansive mappings in a real uniformly smooth and strictly convex Banach space with Kadec-Klee property:

$$\begin{cases} x_0 \in C; \quad C_0 = C, \\ y_n = J^{-1}[\alpha_n Jx_n + (1-\alpha_n)Jz_n], \\ z_n = J^{-1}[\beta_{n,0}Jx_n + \sum_{i=1}^{\infty}\beta_{n,i}Jw_{n,i}], \\ u_n \in C \text{ such that } \forall y \in C, \\ \theta(u_n,y) + \langle Au_n, y-u_n\rangle + \psi(y) - \psi(u_n) + \frac{1}{r_n}\langle y-u_n, Ju_n - Jy_n\rangle \geq 0, \\ C_{n+1} = \{v \in C_n : \phi(v,u_n) \leq \phi(v,x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \geq 0, \end{cases} \tag{1.8}$$

where $\{T_i\}: C \to 2^C$ is a countable family of closed and *uniformly* totally quasi-ϕ-asymptotically nonexpansive multi-valued mappings; $w_{n,i} \in T_i^n x_n$, $\forall n \geq 1$, $i \geq 1$, $\xi_n := \nu_n \sup_{p\in F}\zeta(\phi(p,x_n)) + \mu_n$, $\Pi_{C_{n+1}}$ is the *generalized projection* (see (2.1)) of E onto C_{n+1}. Their results not only generalized the corresponding results of [2–19] from single-valued mappings to multi-valued mappings, but they also improved and extended the main results of Homaeipour and Razani [20].

However, it is obviously a quite strong condition that the involved multi-valued mappings are assumed to be uniformly $(\{\nu_n\}, \{\mu_n\}, \zeta)$-totally quasi-$\phi$-asymptotically nonexpansive. In addition, the accurate computation of the series $\sum_{i=1}^{\infty} \beta_{n,i} J w_{n,i}$ at each step of the iteration process is not easily attainable, which leads to gradually increasing errors.

Inspired and motivated by the study mentioned above, in this paper, we use a relaxed hybrid iteration algorithm for approximating common fixed points of a countable family of multi-valued totally quasi-ϕ-asymptotically nonexpansive mappings and obtain a strong convergence theorem under some suitable conditions. The results improve those of Chang *et al.* [1].

2 Preliminaries

We say that a Banach space E is *strictly convex* if the following implication holds for $x, y \in E$:

$$\|x\| = \|y\| = 1, \quad x \neq y \quad \Rightarrow \quad \left\| \frac{x+y}{2} \right\| < 1. \tag{2.1}$$

E is also said to be *uniformly convex* if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \quad \|x - y\| \geq \epsilon \quad \Rightarrow \quad \left\| \frac{x+y}{2} \right\| \leq 1 - \delta. \tag{2.2}$$

It is well known that if E is a uniformly convex Banach space, then E is reflexive and strictly convex. A Banach space E is said to be *smooth* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.3}$$

exists for each $x, y \in S(E) := \{x \in E : \|x\| = 1\}$. E is said to be *uniformly smooth* if the limit (2.3) is attained uniformly for $x, y \in S(E)$.

Following Alber [21], the *generalized projection* $\Pi_C : E \to C$ is defined by

$$\Pi_C = \arg \inf_{y \in C} \phi(y, x), \quad \forall x \in E. \tag{2.4}$$

Lemma 2.1 [21] *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Then the following conclusions hold*:

(1) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$ *for all* $x \in C$ *and* $y \in E$;

(2) *If* $x \in E$ *and* $z \in C$, *then* $z = \Pi_C x \Leftrightarrow \langle z - y, Jx - Jz \rangle \geq 0$, $\forall y \in C$;

(3) *For* $x, y \in E$, $\phi(x, y) = 0$ *if and only if* $x = y$.

Remark 2.2 The following basic properties for a Banach space E can be found in Cioranescu [22].

(i) If E is uniformly smooth, then J is uniformly continuous on each bounded subset of E;

(ii) If E is reflexive and strictly convex, then J^{-1} is norm-weak-continuous;

(iii) If E is a smooth, strictly convex and reflexive Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is single valued, one-to-one and onto;

(iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex;

(v) Each uniformly convex Banach space E has the *Kadec-Klee property*, *i.e.*, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$ as $n \to \infty$.

Lemma 2.3 [6] *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and C be a nonempty closed convex subset of E. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in C such that $x_n \to p$ and $\phi(x_n, y_n) \to 0$, where ϕ is the function defined by* (1.2), *then $y_n \to p$.*

Lemma 2.4 [1] *Let E and C be the same as in Lemma* 2.3. *Let $T : C \to C$ be a closed and totally quasi-ϕ-asymptotically nonexpansive multi-valued mappings with nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that $\nu_n, \mu_n \to 0$ and $\zeta(0) = 0$. If $\mu_1 = 0$, then the fixed point set $F(T)$ of T is a closed and convex subset of C.*

Lemma 2.5 [6] *Let E be a real uniformly convex Banach space and let $B_r(0)$ be the closed ball of E with center at the origin and radius $r > 0$. Then for any for any sequence $\{x_i\} \subset B_r(0)$ and for any sequence $\{\lambda_i\}$ of positive numbers with $\sum_{i=1}^{\infty} \lambda_i = 1$, there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that such that for any positive integer $i \neq 1$, the following hold*:

$$\left\| \sum_{i=1}^{\infty} \lambda_i x_i \right\|^2 \leq \sum_{i=1}^{\infty} \lambda_i \|x_i\|^2 - \lambda_1 \lambda_i g\bigl(\|x_1 - x_i\|\bigr) \tag{2.5}$$

and, for all $x \in E$,

$$\phi\left(x, J^{-1}\left(\sum_{i=1}^{\infty} \lambda_i J x_i\right)\right) \leq \sum_{i=1}^{\infty} \lambda_i \phi(x, x_i) - \lambda_1 \lambda_i g\bigl(\|Jx_1 - Jx_i\|\bigr). \tag{2.6}$$

Assume that, to obtain the solution of GMEP, the function $\psi : C \to \mathbb{R}$ is convex and lower semi-continuous, the nonlinear mapping $A : C \to E^*$ is continuous and monotone, and the bifunction $\theta : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A_1) $\theta(x, x) = 0$;
(A_2) θ is monotone, *i.e.*, $\theta(x, y) + \theta(y, x) \leq 0$;
(A_3) $\limsup_{t \downarrow 0} \theta(x + t(z - x), y) \leq \theta(x, y)$;
(A_4) the mapping $y \mapsto \theta(x, y)$ is convex and lower semicontinuous.

Lemma 2.6 [16] *Let E be a smooth, strictly convex, and reflexive Banach space, and C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a continuous and monotone mapping, $\psi : C \to \mathbb{R}$ a lower semi-continuous and convex function, and $\theta : C \times C \to \mathbb{R}$ a bifunction satisfying the conditions* (A_1)-(A_4). *Let $r > 0$ and $x \in E$. Then the following hold*:

(1) *There exists an $u \in C$ such that*

$$\theta(u, y) + \langle Au, y - u\rangle + \psi(y) - \psi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C.$$

(2) *A mapping $\kappa_r : C \to C$ is defined by*

$$\kappa_r(x) = \left\{ u \in C : \theta(u, y) + \langle Au, y - u\rangle + \psi(y) - \psi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0 \right\}.$$

Then the mapping κ_r has the following properties:

(i) κ_r *is single-valued*;

(ii) κ_r *a firmly nonexpansive-type mapping, i.e.*,

$$\langle \kappa_r z - \kappa_r y, J\kappa_r z - J\kappa_r y\rangle \leq \langle \kappa_r z - \kappa_r y, Jz - Jy\rangle;$$

(iii) $F(\kappa_r) = \Omega = \tilde{F}(\kappa_r)$;

(iv) Ω *is a closed convex set of* C;

(v) $\phi(p, \kappa_r z) + \phi(\kappa_r z, z) \leq \phi(p, z)$, $\forall p \in F(\kappa_r)$, $z \in E$,

where $\tilde{F}(\kappa_r)$ *denotes the set of asymptotic fixed points of* κ_r, *i.e.*,

$$\tilde{F}(\kappa_r) := \bigl\{x \in C : \exists\{x_n\} \subset C, s.t., x_n \rightharpoonup x, \|x_n - \kappa_r x_n\| \to 0\ (n \to \infty)\bigr\}.$$

Lemma 2.7 [23] *The unique solutions to the positive integer equation*

$$n = i_n + \frac{(m_n - 1)m_n}{2}, \qquad m_n \geq i_n, \quad n = 1, 2, \ldots, \tag{2.7}$$

are

$$i_n = n - \frac{(m_n - 1)m_n}{2}, \qquad m_n = -\left[\frac{1}{2} - \sqrt{2n + \frac{1}{4}}\right], \quad n = 1, 2, \ldots, \tag{2.8}$$

where $[x]$ *denotes the maximal integer that is not larger than* x.

3 Main results

Recall that a multi-valued mapping $T : C \to 2^C$ is said to be closed, if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $w_n \in Tx_n$ with $w_n \to y$ as $n \to \infty$, then $y \in Tx$.

Theorem 3.1 *Let* E *be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property and* C *a nonempty closed convex subset of* E. *Let* $\theta : C \times C \to \mathbb{R}$ *be a bifunction satisfying the conditions* (A_1)-(A_4), $A : C \to E^*$ *a continuous and monotone mapping, and* $\psi : C \to \mathbb{R}$ *a lower semi-continuous and convex function. Let* $\{T_i\} : C \to 2^C$ *be a countable family of closed and totally quasi-*ϕ*-asymptotically nonexpansive multi-valued mappings with nonnegative real sequences* $\{\nu_n^{(i)}\}$, $\{\mu_n^{(i)}\}$ *satisfying* $\nu_n^{(i)} \to 0$ *and* $\mu_n^{(i)} \to 0$ (*as* $n \to \infty$ *and for each* $i \geq 1$) *and a strictly increasing and continuous function* $\zeta : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ *satisfying condition* (1.1) *and each* T_i *is uniformly* L_i*-Lipschitz continuous with* $\mu_1^{(i)} = 0$. *Let* $\{\alpha_i\}$ *be a sequence in* $[0, 1)$ *and* $\{\beta_i\}$ *be a sequence in* $(0, 1)$. *Let* $\{x_n\}$ *be the sequence generated by*

$$\begin{cases} x_1 \in C; \quad C_1 = C, \\ y_n = J^{-1}[\alpha_{i_n} Jx_n + (1 - \alpha_{i_n})Jz_n], \\ z_n = J^{-1}[\beta_{i_n} Jx_n + (1 - \beta_{i_n})Jw_{m_n}^{(i_n)}], \\ u_n \in C \text{ such that } \forall y \in C, \\ \theta(u_n, y) + \langle Au_n, y - u_n\rangle + \psi(y) - \psi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $w_{m_n}^{(i_n)} \in T_{i_n}^{m_n} x_n$, $\forall n \geq 1$, $\xi_n := \nu_{m_n}^{(i_n)} \sup_{p\in F} \zeta_{i_n}(\phi(p,x_n)) + \mu_{m_n}^{(i_n)}$, $\Pi_{C_{n+1}}$ *is the generalized projection of* E *onto* C_{n+1}; *and* i_n *and* m_n *are the solutions to the positive integer equation*: $n = i_n + \frac{(m_n-1)m_n}{2}$ ($m_n \geq i_n$, $n = 1,2,\ldots$), *that is, for each* $n \geq 1$, *there exist unique* i_n *and* m_n *such that*

$$\begin{aligned} &i_1 = 1, \quad i_2 = 1, \quad i_3 = 2, \quad i_4 = 1, \quad i_5 = 2, \\ &i_6 = 3, \quad i_7 = 1, \quad i_8 = 2, \quad \ldots; \\ &m_1 = 1, \quad m_2 = 2, \quad m_3 = 2, \quad m_4 = 3, \quad m_5 = 3, \\ &m_6 = 3, \quad m_7 = 4, \quad m_8 = 4, \quad \ldots. \end{aligned}$$

If $G := F \cap \Omega \neq \emptyset$ *and* $F := \bigcap_{i=1}^{\infty} F(T_i)$ *is bounded, then* $\{x_n\}$ *converges strongly to* $\Pi_G x_1$.

Proof Two functions $\tau : C \times C \to \mathbb{R}$ and $\kappa_r : C \to C$ are defined by

$$\begin{aligned} &\tau(x,y) = \theta(x,y) + \langle Ax, y-x\rangle + \psi(y) - \psi(x); \\ &\kappa_r(x) = \left\{ u \in C : \tau(u,y) + \frac{1}{r}\langle y-u, Ju-Jx\rangle \geq 0, \forall y \in C \right\}. \end{aligned}$$

By Lemma 2.6, we know that the function τ satisfies the conditions (A_1)-(A_4) and κ_r has the properties (i)-(v). Therefore, (3.1) can be rewritten as

$$\begin{cases} x_1 \in C; \quad C_1 = C, \\ y_n = J^{-1}[\alpha_{i_n} Jx_n + (1-\alpha_{i_n})Jz_n], \\ z_n = J^{-1}[\beta_{i_n} Jx_n + (1-\beta_{i_n})Jw_{m_n}^{(i_n)}], \\ u_n \in C \text{ such that } \tau(u_n,y) + \frac{1}{r_n}\langle y-u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v,u_n) \leq \phi(v,x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N}. \end{cases} \tag{3.2}$$

We divide the proof into several steps.

(I) F and C_n ($\forall n \geq 1$) both are closed and convex subsets in C.

In fact, it follows from Lemma 2.4 that each $F(T_i)$ is a closed and convex subset of C, so is F. In addition, with C_1 ($= C$) being closed and convex, we may assume that C_n is closed and convex for some $n \geq 2$. In view of the definition of ϕ we have

$$C_{n+1} = \{v \in C : \varphi(v) \leq a\} \cap C_n,$$

where $\varphi(v) = 2\langle v, Jx_n - Jy_n\rangle$ and $a = \|x_n\|^2 - \|y_n\|^2 + \xi_n$. This shows that C_{n+1} is closed and convex.

(II) G is a subset of $\bigcap_{n=1}^{\infty} C_n$.

It is obvious that $G \subset C_1$. Suppose that $G \subset C_n$ for some $n \geq 2$. Since $u_n = \kappa_{r_n} y_n$, by Lemma 2.6, it is easily shown that κ_{r_n} is quasi-ϕ-nonexpansive. Hence, for any $p \in G \subset C_n$, it follows from (1.4) that

$$\begin{aligned} \phi(p,u_n) &= \phi(p,\kappa_{r_n} y_n) \leq \phi(p,y_n) = \phi\bigl(p, J^{-1}\bigl[\alpha_n Jx_n + (1-\alpha_n)Jx_n\bigr]\bigr) \\ &\leq \alpha_n \phi(p,x_n) + (1-\alpha_n)\phi(p,z_n). \end{aligned} \tag{3.3}$$

Furthermore, it follows from Lemma 2.5 that for any $p \in G \subset C_n$, $w_{m_n}^{(i_n)} \in T_{i_n}^{m_n} x_n$, we have

$$\begin{aligned}
\phi(p,z_n) &= \phi\bigl(p, J^{-1}\bigl[\beta_{i_n} Jx_n + (1-\beta_{i_n}) Jw_{m_n}^{(i_n)}\bigr]\bigr) \\
&\le \beta_{i_n}\phi(p,x_n) + (1-\beta_{i_n})\phi\bigl(p, w_{m_n}^{(i_n)}\bigr) - \beta_{i_n}(1-\beta_{i_n}) g\bigl(\bigl\|Jx_n - Jw_{m_n}^{(i_n)}\bigr\|\bigr) \\
&\le \beta_{i_n}\phi(p,x_n) + (1-\beta_{i_n})\bigl[\phi(p,x_n) + \nu_{m_n}^{(i_n)}\zeta_{i_n}\bigl(\phi(p,x_n)\bigr) + \mu_{m_n}^{(i_n)}\bigr] \\
&\quad - \beta_{i_n}(1-\beta_{i_n}) g\bigl(\bigl\|Jx_n - Jw_{m_n}^{(i_n)}\bigr\|\bigr) \\
&\le \phi(p,x_n) + \nu_{m_n}^{(i_n)} \sup_{p\in F}\zeta_{i_n}\bigl(\phi(p,x_n)\bigr) + \mu_{m_n}^{(i_n)} - \beta_{i_n}(1-\beta_{i_n}) g\bigl(\bigl\|Jx_n - Jw_{m_n}^{(i_n)}\bigr\|\bigr) \\
&= \phi(p,x_n) + \xi_n - \beta_{i_n}(1-\beta_{i_n}) g\bigl(\bigl\|Jx_n - Jw_{m_n}^{(i_n)}\bigr\|\bigr).
\end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3) and simplifying it, we have

$$\begin{aligned}
\phi(p,u_n) \le \phi(p,y_n) &\le \phi(p,x_n) + (1-\alpha_{i_n})\xi_n - (1-\alpha_{i_n})\beta_{i_n}(1-\beta_{i_n}) g\bigl(\bigl\|Jx_n - Jw_{m_n}^{(i_n)}\bigr\|\bigr) \\
&\le \phi(p,x_n) + \xi_n - (1-\alpha_{i_n})\beta_{i_n}(1-\beta_{i_n}) g\bigl(\bigl\|Jx_n - Jw_{m_n}^{(i_n)}\bigr\|\bigr) \\
&\le \phi(p,x_n) + \xi_n.
\end{aligned} \tag{3.5}$$

This implies that $p \in C_{n+1}$, and so $G \subset C_{n+1}$.

(III) $x_n \to x^* \in C$ as $n \to \infty$.

In fact, since $x_n = \Pi_{C_n} x_1$, from Lemma 2.1(2) we have $\langle x_n - y, Jx_1 - Jx_n\rangle \ge 0$, $\forall y \in C_n$. Again since $F \subset \bigcap_{n=1}^{\infty} C_n$, we have $\langle x_n - p, Jx_1 - Jx_n\rangle \ge 0$, $\forall p \in F$. It follows from Lemma 2.1(1) that for each $p \in F$ and for each $n \ge 1$,

$$\phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1) \le \phi(p, x_1) - \phi(p, x_n) \le \phi(p, x_1),$$

which implies that $\{\phi(x_n, x_1)\}$ is bounded, so is $\{x_n\}$. Since for all $n \ge 1$, $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have $\phi(x_n, x_1) \le \phi(x_{n+1}, x_1)$. This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing, hence the limit

$$\lim_{n\to\infty} \phi(x_n, x_1) \text{ exists.}$$

Since E is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x^* \in C$ as $i \to \infty$. Since C_n is closed and convex and $C_{n+1} \subset C_n$, this implies that C_n is weakly closed and $x^* \in C_n$ for each $n \ge 1$. In view of $x_{n_i} = \Pi_{C_{n_i}} x_1$, we have

$$\phi(x_{n_i}, x_1) \le \phi\bigl(x^*, x_1\bigr), \quad \forall i \ge 1.$$

Since the norm $\|\cdot\|$ is weakly lower semi-continuous, we have

$$\begin{aligned}
\liminf_{i\to\infty} \phi(x_{n_i}, x_1) &= \liminf_{i\to\infty} \bigl(\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_1\rangle + \|x_1\|^2\bigr) \\
&\ge \bigl\|x^*\bigr\|^2 - 2\bigl\langle x^*, Jx_1\bigr\rangle + \|x_1\|^2 \\
&= \phi\bigl(x^*, x_1\bigr)
\end{aligned}$$

and so

$$\phi(x^*,x_1) \le \liminf_{i\to\infty} \phi(x_{n_i},x_1) \le \limsup_{i\to\infty} \phi(x_{n_i},x_1) \le \phi(x^*,x_1).$$

This implies that $\lim_{i\to\infty} \phi(x_{n_i},x_1) = \phi(x^*,x_1)$, and so $\|x_{n_i}\| \to \|x^*\|$ as $i \to \infty$. Since $x_{n_i} \rightharpoonup x^*$, by virtue of *Kadec-Klee property* of E, we obtain

$$\lim_{i\to\infty} x_{n_i} = x^*.$$

Since $\{\phi(x_n,x_1)\}$ is convergent, this, together with $\lim_{i\to\infty} \phi(x_{n_i},x_1) = \phi(x^*,x_1)$, shows that $\lim_{n\to\infty} \phi(x_n,x_1) = \phi(x^*,x_1)$. If there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to y$ as $j \to \infty$, then from Lemma 2.1(1) we have

$$\begin{aligned}
\phi(x^*,y) &= \lim_{i,j\to\infty} \phi(x_{n_i},x_{n_j}) = \lim_{i,j\to\infty} \phi(x_{n_i},\Pi_{C_{n_j}}x_1) \\
&\le \lim_{i,j\to\infty} \bigl(\phi(x_{n_i},x_1) - \phi(\Pi_{C_{n_j}}x_1,x_1)\bigr) \\
&= \lim_{i,j\to\infty} \bigl(\phi(x_{n_i},x_1) - \phi(x_{n_j},x_1)\bigr) \\
&= \phi(x^*,x_1) - \phi(x^*,x_1) = 0,
\end{aligned}$$

that is, $x^* = y$ and so

$$\lim_{n\to\infty} x_n = x^*. \tag{3.6}$$

(IV) x^* is a member of F.

Set $\mathcal{K}_i = \{k \ge 1 : k = i_k + \frac{(m_k-1)m_k}{2}, m_k \ge i_k, m_k \in \mathbb{N}\}$ for each $i \ge 1$. Note that $\nu_{m_k}^{(i_k)} = \nu_{m_k}^{(i)}$, $\mu_{m_k}^{(i_k)} = \mu_{m_k}^{(i)}$, and $\zeta_{i_k} = \zeta_i$ whenever $k \in \mathcal{K}_i$ for each $i \ge 1$. For example, by Lemma 2.7 and the definition of $\mathcal{K}_1$, we have $\mathcal{K}_1 = \{1,2,4,7,11,16,\ldots\}$ and $i_1 = i_2 = i_4 = i_7 = i_{11} = i_{16} = \cdots = 1$. Then we have

$$\xi_k = \nu_{m_k}^{(i)} \sup_{p\in F} \zeta_i\bigl(\phi(p,x_k)\bigr) + \mu_{m_k}^{(i)}, \quad \forall k \in \mathcal{K}_i. \tag{3.7}$$

Note that $\{m_k\}_{k\in\mathcal{K}_i} = \{i, i+1, i+2, \ldots\}$, *i.e.*, $m_k \uparrow \infty$ as $\mathcal{K}_i \ni k \to \infty$. It follows from (3.6) and (3.7) that

$$\lim_{k\to\infty} \xi_k = 0. \tag{3.8}$$

Since $x_{n+1} \in C_{n+1}$, it follows from (3.1), (3.6), and (3.8) that

$$\phi(x_{k+1},y_k) \le \phi(x_{k+1},x_k) + \xi_k \to 0 \tag{3.9}$$

as $\mathcal{K}_i \ni k \to \infty$. Since $x_k \to x^*$, it follows from (3.9) and Lemma 2.3 that

$$\lim_{\mathcal{K}_i\ni k\to\infty} y_k = x^*. \tag{3.10}$$

Note that $w_{m_k}^{(i_k)} = w_{m_k}^{(i)}$, $T_{i_k}^{m_k} = T_i^{m_k}$, $\alpha_{i_k} = \alpha_i$, and $\beta_{i_k} = \beta_i$ whenever $k \in \mathcal{K}_i$ for each $i \geq 1$. From (3.5), for any $p \in F$ and $w_{m_k}^{(i)} \in T_i^{m_k} x_k$, $\forall k \in \mathcal{K}_i$, we have

$$\phi(p, y_k) \leq \phi(p, x_k) + \xi_k - (1 - \alpha_i)\beta_i(1 - \beta_i)g\left(\left\|Jx_k - Jw_{m_k}^{(i)}\right\|\right),$$

that is,

$$(1 - \alpha_i)\beta_i(1 - \beta_i)g\left(\left\|Jx_k - Jw_{m_k}^{(i)}\right\|\right) \leq \phi(p, x_k) + \xi_k - \phi(p, y_k) \to 0 \quad (\mathcal{K}_i \ni k \to \infty).$$

This, together with assumption conditions imposed on the sequence $\{\alpha_i\}$ and $\{\beta_i\}$, shows that $\lim_{\mathcal{K}_i \ni k \to \infty} g(\|Jx_k - Jw_{m_k}^{(i)}\|) = 0$. In view of property of g, we have

$$\lim_{\mathcal{K}_i \ni k \to \infty} \left\|Jx_k - Jw_{m_k}^{(i)}\right\| = 0.$$

In addition, $Jx_k \to Jx^*$ implies that $\lim_{\mathcal{K}_i \ni k \to \infty} Jw_{m_k}^{(i)} = Jx^*$. From Remark 2.2(ii) it yields, as $\mathcal{K}_i \ni k \to \infty$,

$$w_{m_k}^{(i)} \rightharpoonup x^*, \quad \forall i \geq 1. \tag{3.11}$$

Again, since, for each $i \geq 1$, as $\mathcal{K}_i \ni k \to \infty$,

$$\left|\left\|w_{m_k}^{(i)}\right\| - \left\|x^*\right\|\right| = \left|\left\|Jw_{m_k}^{(i)}\right\| - \left\|Jx^*\right\|\right| \leq \left\|Jw_{m_k}^{(i)} - Jx^*\right\| \to 0,$$

this, together with (3.11) and the *Kadec-Klee property* of E, shows that

$$\lim_{\mathcal{K}_i \ni k \to \infty} w_{m_k}^{(i)} = x^*, \quad \forall i \geq 1. \tag{3.12}$$

For each $i \geq 1$, we now consider the sequence $\{s_{m_k}^{(i)}\}_{k \in \mathcal{K}_i}$ generated by

$$s_{m_{k+1}}^{(i)} \in T_i w_{m_k}^{(i)} \subset T_i^{m_{k+1}} x_k, \quad k \in \mathcal{K}_i, \forall i \geq 1. \tag{3.13}$$

By the assumptions that for each $i \geq 1$, T_i is uniformly L_i-Lipschitz continuous. Noting again that $\{m_k\}_{k \in \mathcal{K}_i} = \{i, i+1, i+2, \ldots\}$, *i.e.*, $m_{k+1} - 1 = m_k$ for all $k \in \mathcal{K}_i$, we then have

$$\begin{aligned}
\left\|s_{m_{k+1}}^{(i)} - w_{m_k}^{(i)}\right\| &\leq \left\|s_{m_{k+1}}^{(i)} - w_{m_{k+1}}^{(i)}\right\| + \left\|w_{m_{k+1}}^{(i)} - x_{k+1}\right\| \\
&\quad + \|x_{k+1} - x_k\| + \left\|x_k - w_{m_k}^{(i)}\right\| \\
&\leq (L_i + 1)\|x_{k+1} - x_k\| + \left\|w_{m_{k+1}}^{(i)} - x_{k+1}\right\| \\
&\quad + \left\|x_k - w_{m_k}^{(i)}\right\|.
\end{aligned} \tag{3.14}$$

From (3.12) and $x_k \to x^*$ we have $\lim_{\mathcal{K}_i \ni k \to \infty} \|s_{m_{k+1}}^{(i)} - w_{m_k}^{(i)}\| = 0$ and

$$\lim_{\mathcal{K}_i \ni k \to \infty} s_{m_{k+1}}^{(i)} = x^*, \quad \forall i \geq 1. \tag{3.15}$$

In view of the closedness of T_i, it follows from (3.12) and (3.13) that $x^* \in T_i x^*$ for each $i \geq 1$, namely $x^* \in F$.

(V) x^* is also a member of G.

Since $x_{n+1} = \Pi_{C_{n+1}} x_1$, it follows from (3.1) and (3.6) that

$$\phi(x_{k+1}, u_k) \le \phi(x_{k+1}, x_k) + \xi_k \to 0$$

as $\mathcal{K}_i \ni k \to \infty$. Since $x_k \to x^*$, by virtue of Lemma 2.1 we have

$$\lim_{\mathcal{K}_i \ni k \to \infty} u_k = x^*. \tag{3.16}$$

This, together with (3.10), shows that $\lim_{\mathcal{K}_i \ni k \to \infty} \|u_k - y_k\| = 0$ and $\lim_{\mathcal{K}_i \ni k \to \infty} \|Ju_k - Jy_k\| = 0$. By the assumption that $\{r_k\}_{k \in \mathcal{K}_i} \subset [a, \infty)$ for some $a > 0$, we have

$$\lim_{\mathcal{K}_i \ni k \to \infty} \frac{\|Ju_k - Jy_k\|}{r_k} = 0. \tag{3.17}$$

Since $\tau(u_k, y) + \frac{1}{r_k}\langle y - u_k, Ju_k - Jy_k\rangle \ge 0$, $\forall y \in C$, by condition (A_1), we have

$$\frac{1}{r_k}\langle y - u_k, Ju_k - Jy_k\rangle \ge -\tau(u_k, y) \ge \tau(y, u_k), \quad \forall y \in C. \tag{3.18}$$

By the assumption that the mapping $y \mapsto \tau(x, y)$ is convex and lower semi-continuous, letting $\mathcal{K}_i \ni k \to \infty$ in (3.18), from (3.16) and (3.17), we have $\tau(y, x^*) \le 0$, $\forall y \in C$.

For any $t \in (0,1]$ and any $y \in C$, set $y_t = ty + (1-t)x^*$. Then $\tau(y_t, x^*) \le 0$ since $y_t \in C$. By condition (A_1) and (A_4), we have

$$0 = \tau(y_t, y_t) \le t\tau(y_t, y) + (1-t)\tau\big(y_t, x^*\big) \le t\tau(y_t, y).$$

Dividing both sides of the above equation by t, we have $\tau(y_t, y) \ge 0$, $\forall y \in C$. Letting $t \downarrow 0$, from condition (A_3), we have $\tau(x^*, y) \ge 0$, $\forall y \in C$, *i.e.*, $x^* \in \Omega$ and so $x^* \in G$.

(VI) $x^* = \Pi_G x_1$, and so $x_n \to \Pi_G x_1$ as $n \to \infty$.

Put $u = \Pi_G x_1$. Since $u \in G \subset C_n$ and $x_n = \Pi_{C_n} x_1$, we have $\phi(x_n, x_1) \le \phi(u, x_1)$, $\forall n \ge 1$. Then

$$\phi\big(x^*, x_1\big) = \lim_{n \to \infty} \phi(x_n, x_1) \le \phi(u, x_1), \tag{3.19}$$

which implies that $x^* = u$ since $u = \Pi_{Gx_1}$, and hence $x_n \to x^* = \Pi_F x_1$ as $n \to \infty$. This completes the proof. □

A numerical result is given as follows.

Example 3.2 Let $E = \mathbb{R}^1$ with the standard norm $\|\cdot\| = |\cdot|$ and $C = [0,1]$. Let $\{T_i\}_{i=1}^{\infty} : C \to 2^C$ be a sequence of multi-valued nonlinear mappings defined by

$$T_i x = \left\{ \frac{(\lambda x)^{i+1}}{i+1} : \lambda \in [0,1] \right\}.$$

Consider the following iteration sequence generated by

$$\begin{cases} x_1 \in C; \quad C_1 = C, \\ y_n = J^{-1}[\alpha_n Jx_n + (1-\alpha_n)Jz_n], \\ z_n = J^{-1}[\beta_n Jx_n + (1-\beta_n)Jw_{i_n}], \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \ge 1, \end{cases} \tag{3.20}$$

where $w_i := \frac{x^{i+1}}{i+1} \in T_i x$, $\{\alpha_n\} = \{\frac{2}{3} - \frac{1}{4n}\}$, $\{\beta_n\} = \{\frac{4}{5} - \frac{1}{2n}\}$, and $\Pi_{C_{n+1}}(x) := \arg\inf_{y \in C_{n+1}} |y - x|$. Note that $J = I$ and $\phi(x, y) = |x - y|^2$ for all $x, y \in E$ since E is a Hilbert space. Moreover, it is not difficult to obtain $C_{n+1} = [0, \frac{x_n + y_n}{2}]$ for all $n \ge 1$. Then (3.20) is reduced to

$$\begin{cases} x_1 \in C; \quad C_1 = C, \\ y_n = (\frac{2}{3} - \frac{1}{4n})x_n + (\frac{1}{3} + \frac{1}{4n})z_n, \\ z_n = (\frac{4}{5} - \frac{1}{2n})x_n + (\frac{1}{5} + \frac{1}{2n})w_{i_n}, \\ C_{n+1} = \{v \in C_n : |v - y_n| \le |v - x_n|\}, \\ x_{n+1} = \frac{x_n + y_n}{2}, \quad \forall n \ge 1, \end{cases} \tag{3.21}$$

where i_n is the solution to the positive integer equation: $n = i_n + \frac{(m_n - 1)m_n}{2}$ ($m_n \ge i_n$, $n = 1, 2, \ldots$). It is clear that $\{T_i\}$ is a sequence of closed and totally quasi-ϕ-asymptotically nonexpansive multi-valued mappings with a common fixed point zero. It then can be shown by similar way of Theorem 3.1 that $\{x_n\}$ converges strongly to zero. The numerical experiment outcome obtained by using MATLAB 7.10.0.499 shows that, as $x_1 = 1$, the computations of x_{100}, x_{200}, x_{300}, and x_{400} are 0.023899039, 0.00074538945, 0.000024001481, and 0.00000078318587, respectively. This example illustrates the effectiveness of the introduced algorithm for countable families of totally quasi-ϕ-asymptotically nonexpansive multi-valued mappings.

Competing interests
The author declares that they have no competing interests.

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