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A new technique for convergence theorem of fixed point problem of quasi-nonexpansive mapping

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Abstract

For the purpose of this paper, we use the method different from the relaxed extragradient method for finding a common element of the set of fixed points of a quasi-nonexpansive mapping, the set of solutions of equilibrium problems, and the set of solutions of a modified system of variational inequalities without demiclosed condition of W and $W_\omega := (1-\omega)I + \omega W$, where W is a quasi-nonexpansive mapping and $\omega \in (0, \frac{1}{2})$ in the framework of Hilbert space. By using our main result, we obtain a strong convergence theorem involving a finite family of nonspreading mappings and another corollary. Moreover, we give a numerical example to encourage our main theorem.

Keywords: quasi-nonexpansive mapping; equilibrium problem; variational inequality problem; fixed point problem

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that the mapping $W : C \to C$ is called *quasi-nonexpansive* if

$$\|Wp - q\| \le \|p - q\|,$$

for all $p \in C$ and $q \in F(W)$. We denote by $F(W)$ the set of fixed points of W. Fixed point problems have been widely studied and developed in the literature.

Let Ψ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The *equilibrium problem* for $\Psi : C \times C \to \mathbb{R}$ is to find $p \in C$ such that

$$\Psi(p, \zeta) \ge 0, \quad \forall \zeta \in C. \tag{1.1}$$

We denote the set of solutions of (1.1) by $\mathrm{EP}(\Psi)$. Equilibrium problems were introduced by Blum and Oettli [1] in 1994 and included many well-known problems such as the variational inequality problem, the optimization problem, and the nonexpansive mapping and fixed point problem.

A mapping $D_1 : C \to H$ is called d_1-*inverse strongly monotone* if there exists a positive real number $d_1 > 0$ such that

$$\langle D_1 p - D_1\zeta, p - \zeta\rangle \geq d_1 \|D_1 p - D_1 \zeta\|^2,$$

for all $p, \zeta \in C$.

Let $B : C \to H$. The *variational inequality* is to find a point $\phi \in C$ such that

$$\langle B\phi, \psi - \phi\rangle \geq 0, \tag{1.2}$$

for all $\psi \in C$. The set of solutions of (1.2) is denoted by $\mathrm{VIP}(C,B)$. The variational inequalities were initially studied and introduced by Lions and Stampacchia [2].

The concept of quasi-nonexpansive mapping was investigated by Diaz and Metcalf [3]. In 2007, Su *et al.* [4] introduced strong convergence theorems for quasi-nonexpansive mappings, the monotone hybrid iteration method used to approximate the fixed point of quasi-nonexpansive mappings. In 2011, Tian and Jin [5] introduced an iterative method of a quasi-nonexpansive mapping in the framework of Hilbert space. They proved the strong convergence theorem of iterative scheme $\{p_n\}$ generated by (1.3) as follows.

Theorem 1.1 *Let H be a real Hilbert space, let F be a κ-Lipschitzian and η-strongly monotone operator on H with $\kappa > 0, \eta > 0$ and let W be a quasi-nonexpansive mapping on H, and f is a L-Lipschitzian mapping with coefficient $L > 0$ for all $p, \zeta \in H$. Assume the set $F(W)$ of fixed points of W is nonempty closed and convex. Let $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \gamma < \mu(\eta - \frac{\mu\kappa^2}{2})/L = \tau/L$ and start with an arbitrary chosen $p_0 \in H$, let the sequence $\{p_n\}$ be generated by*

$$p_{n+1} = \alpha_n \gamma f(p_n) + (I - \alpha_n \mu F)W_\omega p_n, \tag{1.3}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Also $\omega \in (0, \frac{1}{2})$, $W_\omega := (1-\omega)I + \omega W$ with two conditions on W:

1. *$\|Wp - q\| \leq \|p - q\|$ for any $p \in H$, and $q \in F(W)$; this means that W is a quasi-nonexpansive mapping*;
2. *W is demiclosed on H; that is, if $\{\zeta_k\} \subset H$, $\zeta_k \rightharpoonup \xi$, and $(I - W)\zeta_k \to 0$, then $\xi \in F(W)$.*

Then $\{p_n\}$ converges strongly to the $p^* \in F(W)$ which is the unique solution of the VIP:

$$\big\langle (\mu F - \gamma f)p^*, p - p^*\big\rangle \leq 0, \quad \forall p \in F(W).$$

Many strong convergence theorems of quasi-nonexpansive mapping W were proved by assuming the following conditions:

1. $W_\omega := (1-\omega)I + \omega W$ for all $\omega \in (0, \frac{1}{2})$,
2. W is demiclosed on H.

In 2012, Dong *et al.* [6] proved strong convergence theorem by using a relaxed extragradient method as follows.

Theorem 1.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mappings $D_1, D_2 : C \to H$ be d_1-inverse strongly monotone and d_2-inverse strongly monotone, respectively. Let Ψ be a bifunction from $C \times C \to \mathbb{R}$ satisfying* (J1)-(J4) *and let $\{W_n\}_{n=1}^{\infty} : C \to C$ be a countable family of nonexpansive mappings such that $\Omega :=$*

$\bigcap_{n=1}^{\infty} F(W_n) \cap \mathrm{EP}(\Psi) \cap F(G) \neq \emptyset$. *Let* $f : C \to C$ *be a contraction with coefficient* $\rho \in (0, 1/2)$. *Set* $\beta_0 = 1$. *For given* $p_1 \in C$ *arbitrarily, let the sequences* $\{p_n\}$, $\{\zeta_n\}$, $\{\xi_n\}$, *and* $\{\phi_n\}$ *be generated by*

$$\begin{cases} \Psi(\phi_n, \zeta) + \frac{1}{g_n}\langle \zeta - \phi_n, \phi_n - p_n\rangle \geq 0, \quad \forall \zeta \in C, \\ \xi_n = P_C(\phi_n - \lambda_B D_2 \phi_n), \\ \zeta_n = \alpha_n f(p_n) + (1-\alpha_n) P_C(\xi_n - \lambda_A D_1 \xi_n), \\ p_{n+1} = \beta_n p_n + \sigma_n \sum_{i=1}^{\infty} (\beta_{i-1} - \beta_i) W_i \zeta_n \\ \qquad + (1-\beta_n)(1-\sigma_n) P_C(\xi_n - \lambda_A D_1 \xi_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\lambda_A \in (0, 2d_1)$, $\lambda_B \in (0, 2d_2)$, *and the sequences* $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset [0,1]$, $\{\sigma_n\} \subset [0,1]$, *and* $\{g_n\} \subset (r, \infty)$, $r > 0$, *are such that*

(i) $\{\beta_n\}$ *is strictly decreasing*,
(ii) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$,
(iii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iv) $\sigma_n > 1/2(1-\rho)$, $\sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < \infty$,
(v) $\sum_{n=1}^{\infty} |g_n - g_{n-1}| < \infty$.

Then the sequence $\{p_n\}$ *generated by* (1.4) *converges strongly to* $p^* = P_\Omega \cdot f(p^*)$, *and* (p^*, ζ^*) *is a solution of the general system of variational inequalities* (1.5) *where* $\zeta^* = P_C(p^* - \lambda_B D_2 p^*)$.

Many authors used the extragradient method to prove fixed point theorem of nonlinear mappings.

Let $D_1, D_2 : C \to H$ be two mappings. In 2008, Ceng *et al.* [7] introduced a relaxed extragradient method for finding solutions of problem $(p^*, \xi^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda_A D_1 \xi^* + p^* - \xi^*, p - p^*\rangle \geq 0, \quad \forall p \in C, \\ \langle \lambda_B D_2 p^* + \xi^* - p^*, p - \xi^*\rangle \geq 0, \quad \forall p \in C, \end{cases} \tag{1.5}$$

which is called a system of variational inequalities where $\lambda_A, \lambda_B > 0$.

In 2013, Kangtunyakarn [8] modified (1.5) for finding $(p^*, \xi^*) \in C \times C$ such that

$$\begin{cases} \langle p^* - (I - \lambda_A D_1)(ap^* + (1-a)\xi^*), p - p^*\rangle \geq 0, \quad \forall p \in C, \\ \langle \xi^* - (I - \lambda_B D_2) p^*, p - \xi^*\rangle \geq 0, \quad \forall p \in C, \end{cases} \tag{1.6}$$

which is called a modification of system of variational inequalities, for every $\lambda_A, \lambda_B > 0$ and $a \in [0,1]$. If $a = 0$, (1.6) reduces to (1.5). He introduced the relation between solutions of (1.6) and fixed point of the mapping G as follows.

Lemma 1.3 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and let* $D_1, D_2 : C \to H$ *be mappings. For every* $\lambda_A, \lambda_B > 0$ *and* $a \in [0,1]$, *the following statements are equivalent*:

1. $(p^*, \xi^*) \in C \times C$ *is a solution of problem* (1.6),
2. p^* *is a fixed point of the mapping* $G : C \to C$, *i.e.*, $p^* \in F(G)$, *defined by*

$$G(p) = P_C(I - \lambda_A D_1)\bigl(ap + (1-a) P_C(I - \lambda_B D_2) p\bigr),$$

where $\xi^* = P_C(I - \lambda_B D_2) p^*$.

After we investigated Theorem 1.1, Theorem 1.2 and researchers in the same direction, we have the questions as follows:

(1) Can we prove strong convergence theorem without demiclosed condition and $W_\omega := (1-\omega)I + \omega W$, where W is a quasi-nonexpansive mapping and $\omega \in (0, \frac{1}{2})$ in the framework of Hilbert space?

(2) Can we prove strong convergence theorem without relaxed extragradient method?

In this paper, we give the answer for the mentioned questions and introduce the method of iterative scheme $\{p_n\}$ for finding a common element of the set of fixed points of a quasi-nonexpansive mapping, the set of solutions of equilibrium problems and the set of solutions of a modified system of variational inequalities. Applying our main result, we prove strong convergence theorem involving a finite family of nonspreading mappings and another corollary. Moreover, We also give a numerical example to support our main theorem.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. In this paper, we use the symbol of weak and strong convergence by '$\rightharpoonup$' and '$\to$', respectively. For every $p \in H$, there exists a unique nearest point $P_C p$ in C such that $\|p - P_C p\| \leq \|p - \zeta\|$ for all $\zeta \in C$. P_C is called the *metric projection* of H onto C.

Remark 2.1 It is well known that metric projection P_C has the following properties:

1. P_C is firmly nonexpansive, *i.e.*,

$$\|P_C p - P_C \zeta\|^2 \leq \langle P_C p - P_C \zeta, p - \zeta\rangle, \quad \forall p, \zeta \in H.$$

2. For each $p \in H$,

$$\xi = P_C(p) \quad \Leftrightarrow \quad \langle p - \xi, \xi - \zeta\rangle \geq 0, \quad \forall \zeta \in C.$$

Recall that H satisfies *Opial's condition* [9], *i.e.*, for any sequence $\{p_n\}$ with $p_n \rightharpoonup p$, the inequality

$$\liminf_{n\to\infty} \|p_n - p\| < \liminf_{n\to\infty} \|p_n - \zeta\|$$

holds for every $\zeta \in H$ with $\zeta \neq p$.

Lemma 2.2 *Let H be a real Hilbert space. Then we have the following well-known results*:

1. $\|p \pm \zeta\|^2 = \|p\|^2 \pm 2\langle p, \zeta\rangle + \|\zeta\|^2$,
2. $\|p + \zeta\|^2 \leq \|p\|^2 + 2\langle \zeta, p + \zeta\rangle$,

for all $p, \zeta \in H$.

Lemma 2.3 ([10]) *Let* $(E, \langle\cdot,\cdot\rangle)$ *be an inner product space. Then, for all* $p, \zeta, \xi \in E$ *and* $\alpha_1, \alpha_2, \alpha_3 \in [0,1]$ *with* $\alpha_1 + \alpha_2 + \alpha_3 = 1$, *we have*

$$\begin{aligned}\|\alpha_1 p + \alpha_2 \zeta + \alpha_3 \xi\|^2 &= \alpha_1\|p\|^2 + \alpha_2\|\zeta\|^2 + \alpha_3\|\xi\|^2 - \alpha_1\alpha_2\|p - \zeta\|^2 \\ &\quad - \alpha_1\alpha_3\|p - \xi\|^2 - \alpha_2\alpha_3\|\zeta - \xi\|^2.\end{aligned}$$

For solving the equilibrium problem, we assume that the bifunction $\Psi : C \times C \to \mathbb{R}$ satisfies the following conditions:

(J1) $\Psi(p,p) = 0$ for all $p \in C$;

(J2) Ψ is monotone, *i.e.*, $\Psi(p,\zeta) + \Psi(\zeta,p) \leq 0$ for all $p,\zeta \in C$;

(J3) for each $p,\zeta,\xi \in C$,

$$\lim_{t\downarrow 0} \Psi\bigl(t\xi + (1-t)p,\zeta\bigr) \leq \Psi(p,\zeta);$$

(J4) for each $p \in C$, $\zeta \mapsto \Psi(p,\zeta)$ is convex and lower semicontinuous.

Lemma 2.4 ([1]) *Let C be a nonempty closed convex subset of H and let Ψ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (J1)-(J4). *Let $r > 0$ and $p \in H$. Then there exists $\xi \in C$ such that*

$$\Psi(\xi,\zeta) + \frac{1}{r}\langle \zeta - \xi, \xi - p\rangle \geq 0, \quad \forall \zeta \in C.$$

Lemma 2.5 ([11]) *Assume that $\Psi : C \times C \to \mathbb{R}$ satisfies* (J1)-(J4). *For $r > 0$, define a mapping $W_r : H \to C$ as follows*:

$$W_r(p) = \left\{\xi \in C : \Psi(\xi,\zeta) + \frac{1}{r}\langle \zeta - \xi, \xi - p\rangle \geq 0, \forall \zeta \in C\right\},$$

for all $p \in H$. Then the following hold:

(1) *W_r is single-valued*;

(2) *W_r is firmly nonexpansive, i.e., for any $p,\zeta \in H$*,

$$\bigl\|W_r(p) - W_r(\zeta)\bigr\|^2 \leq \bigl\langle W_r(p) - W_r(\zeta), p - \zeta\bigr\rangle;$$

(3) $F(W_r) = \mathrm{EP}(\Psi)$;

(4) $\mathrm{EP}(\Psi)$ *is closed and convex*.

Lemma 2.6 ([12]) *Let $\{h_n\}$ be a sequence of nonnegative real numbers satisfying*

$$h_{n+1} \leq (1-\alpha_n)h_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} h_n = 0$.

Lemma 2.7 ([13]) *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let D_1 be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$*,

$$u = P_C(I - \lambda D_1)u \quad \Leftrightarrow \quad u \in \mathrm{VIP}(C, D_1),$$

where P_C is the metric projection of H onto C.

Lemma 2.8 ([14]) *Let C be a nonempty closed convex subset of a real Hilbert space H and let $W: C \to C$ be a quasi-nonexpansive mapping with $F(W) \neq \emptyset$. Then* $\mathrm{VIP}(C, I - W) = F(W)$.

Remark 2.9 From Lemmas 2.7 and 2.8, we have

$$F(W) = \mathrm{VIP}(C, I - W) = F\big(P_C\big(I - \lambda(I - W)\big)\big),$$

for all $\lambda > 0$.

3 Main result

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H, let $\Psi_1, \Psi_2 : C \times C \to \mathbb{R}$ be bifunctions satisfying* (J1)-(J4) *and let $W: C \to C$ be a quasi-nonexpansive mapping. Let $D_1, D_2 : C \to H$ be d_1, d_2-inverse strongly monotone mappings, respectively. Define the mapping $G: C \to C$ by $G(p) = P_C(I - \lambda_A D_1)(ap + (1-a)P_C(I - \lambda_B D_2)p)$ for all $p \in C$ and $a \in [0,1]$. Assume $\mathcal{F} = \mathrm{EP}(\Psi_1) \cap \mathrm{EP}(\Psi_2) \cap F(G) \cap F(W) \neq \emptyset$. Suppose that $p_1, u \in C$ and let $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ be sequences generated by*

$$\begin{cases} \Psi_1(\phi_n, \zeta) + \frac{1}{g_n}\langle \zeta - \phi_n, \phi_n - p_n\rangle \geq 0, & \forall \zeta \in C, \\ \Psi_2(\psi_n, \zeta) + \frac{1}{h_n}\langle \zeta - \psi_n, \psi_n - p_n\rangle \geq 0, & \forall \zeta \in C, \\ p_{n+1} = \alpha_n u + \beta_n p_n + \gamma_n P_C(I - \lambda_n(I - W))\phi_n + \delta_n G(\psi_n), & \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where the sequences $\lambda_A \in (0, 2d_1)$, $\lambda_B \in (0, 2d_2)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \in \mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$ *for some* $c, d > 0$ *and for all* $n \geq 1$,

(iii) $0 < e \leq g_n, h_n \leq f$ *for some* $e, f > 0$ *and for all* $n \geq 1$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |g_{n+1} - g_n| < \infty$, $\sum_{n=1}^{\infty} |h_{n+1} - h_n| < \infty$.

Then $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ converge strongly to $p_0 = P_{\mathcal{F}}u$ and (p_0, ξ_0) is a solution of (1.6) *where $\xi_0 = P_C(I - \lambda_B D_2)p_0$.*

Proof First, we show that G is a nonexpansive mapping. Let $p, \zeta \in C$. Since D_1, D_2 are d_1, d_2-inverse strongly monotone, $\lambda_A \in (0, 2d_1)$, and $\lambda_B \in (0, 2d_2)$, we have

$$\begin{aligned} &\big\|(I - \lambda_A D_1)p - (I - \lambda_A D_1)\zeta\big\|^2 \\ &\quad = \|p - \zeta\|^2 - 2\lambda_A\langle p - \zeta, D_1 p - D_1\zeta\rangle + \lambda_A^2\|D_1 p - D_1\zeta\|^2 \\ &\quad \leq \|p - \zeta\|^2 - 2d_1\lambda_A\|D_1 p - D_1\zeta\|^2 + \lambda_A^2\|D_1 p - D_1\zeta\|^2 \\ &\quad = \|p - \zeta\|^2 + \lambda_A(\lambda_A - 2d_1)\|D_1 p - D_1\zeta\|^2 \\ &\quad \leq \|p - \zeta\|^2. \end{aligned}$$

Then $I - \lambda_A D_1$ is a nonexpansive mapping. Similarly $I - \lambda_B D_2$ is a nonexpansive mapping. Then G is a nonexpansive mapping.

Next, we show $\{p_n\}$ is bounded. Let $\xi \in \mathcal{F}$, then $\phi_n = W_{g_n}p_n$ and $\psi_n = W_{h_n}p_n$. It is clear that $\|\phi_n - \xi\| \leq \|p_n - \xi\|$ and $\|\psi_n - \xi\| \leq \|p_n - \xi\|$. By Remark 2.9, we have

$$\xi \in F\big(P_C\big(I - \lambda_n(I - W)\big)\big). \tag{3.2}$$

Observe that

$$\begin{aligned} \|W\phi_n - \xi\|^2 &= \big\|(\phi_n - \xi) - (I - W)\phi_n\big\|^2 \\ &= \|\phi_n - \xi\|^2 - 2\big\langle\phi_n - \xi, (I - W)\phi_n\big\rangle + \big\|(I - W)\phi_n\big\|^2 \\ &\leq \|\phi_n - \xi\|^2. \end{aligned}$$

It implies that

$$\big\|(I - W)\phi_n\big\|^2 \leq 2\big\langle\phi_n - \xi, (I - W)\phi_n\big\rangle. \tag{3.3}$$

From (3.2) and (3.3), we have

$$\begin{aligned} \big\|P_C\big(I - \lambda_n(I - W)\big)\phi_n - \xi\big\|^2 &= \big\|P_C\big(I - \lambda_n(I - W)\big)\phi_n - P_C\big(I - \lambda_n(I - W)\big)\xi\big\|^2 \\ &\leq \big\|(\phi_n - \xi) - \lambda_n\big((I - W)\phi_n - (I - W)\xi\big)\big\|^2 \\ &= \|\phi_n - \xi\|^2 - 2\lambda_n\big\langle\phi_n - \xi, (I - W)\phi_n\big\rangle \\ &\quad + \lambda_n^2\big\|(I - W)\phi_n\big\|^2 \\ &\leq \|\phi_n - \xi\|^2 + \lambda_n(\lambda_n - 1)\big\|(I - W)\phi_n\big\|^2 \\ &\leq \|\phi_n - \xi\|^2. \end{aligned} \tag{3.4}$$

From the definition of p_n and (3.4), we have

$$\begin{aligned} \|p_{n+1} - \xi\| &= \big\|\alpha_n(u - \xi) + \beta_n(p_n - \xi) + \gamma_n\big(P_C\big(I - \lambda_n(I - W)\big)\phi_n - \xi\big) \\ &\quad + \delta_n\big(G(\psi_n) - \xi\big)\big\| \\ &\leq \alpha_n\|u - \xi\| + \beta_n\|p_n - \xi\| + \gamma_n\big\|P_C\big(I - \lambda_n(I - W)\big)\phi_n - \xi\big\| \\ &\quad + \delta_n\big\|G(\psi_n) - \xi\big\| \\ &\leq \alpha_n\|u - \xi\| + \beta_n\|p_n - \xi\| + \gamma_n\|\phi_n - \xi\| + \delta_n\|\psi_n - \xi\| \\ &\leq \alpha_n\|u - \xi\| + \beta_n\|p_n - \xi\| + \gamma_n\|p_n - \xi\| + \delta_n\|p_n - \xi\| \\ &= \alpha_n\|u - \xi\| + (1 - \alpha_n)\|p_n - \xi\|. \end{aligned}$$

By induction, we can conclude that

$$\|p_n - \xi\| \leq \max\big\{\|u - \xi\|, \|p_1 - \xi\|\big\},$$

for all $n \geq 1$. This implies that the sequence $\{p_n\}$ is bounded and so are $\{\phi_n\}$, $\{\psi_n\}$, $\{(I - W)\phi_n\}$, and $\{P_C(I - \lambda_n(I - W))\phi_n\}$.

Then we show that $\lim_{n\to\infty}\|p_{n+1} - p_n\| = 0$.

From the definition of p_n and nonexpansiveness of G, we have

$$\begin{aligned}
\|p_{n+1}-p_n\| &= \big\|(\alpha_n-\alpha_{n-1})u+\beta_n(p_n-p_{n-1})+(\beta_n-\beta_{n-1})p_{n-1} \\
&\quad +\gamma_n\big(P_C\big(I-\lambda_n(I-W)\big)\phi_n-P_C\big(I-\lambda_{n-1}(I-W)\big)\phi_{n-1}\big) \\
&\quad +(\gamma_n-\gamma_{n-1})P_C\big(I-\lambda_{n-1}(I-W)\big)\phi_{n-1} \\
&\quad +\delta_n\big(G(\psi_n)-G(\psi_{n-1})\big)+(\delta_n-\delta_{n-1})G(\psi_{n-1})\big\| \\
&\le |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|p_n-p_{n-1}\|+|\beta_n-\beta_{n-1}|\|p_{n-1}\| \\
&\quad +\gamma_n\big\|P_C\big(I-\lambda_n(I-W)\big)\phi_n-P_C\big(I-\lambda_{n-1}(I-W)\big)\phi_{n-1}\big\| \\
&\quad +|\gamma_n-\gamma_{n-1}|\big\|P_C\big(I-\lambda_{n-1}(I-W)\big)\phi_{n-1}\big\| \\
&\quad +\delta_n\big\|G(\psi_n)-G(\psi_{n-1})\big\|+|\delta_n-\delta_{n-1}|\big\|G(\psi_{n-1})\big\| \\
&\le |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|p_n-p_{n-1}\|+|\beta_n-\beta_{n-1}|\|p_{n-1}\| \\
&\quad +\gamma_n\big\|(\phi_n-\phi_{n-1})-\big(\lambda_n(I-W)\phi_n-\lambda_n(I-W)\phi_{n-1}\big) \\
&\quad -\big(\lambda_n(I-W)\phi_{n-1}-\lambda_{n-1}(I-W)\phi_{n-1}\big)\big\| \\
&\quad +|\gamma_n-\gamma_{n-1}|\big\|P_C\big(I-\lambda_{n-1}(I-W)\big)\phi_{n-1}\big\|+\delta_n\|\psi_n-\psi_{n-1}\| \\
&\quad +|\delta_n-\delta_{n-1}|\big\|G(\psi_{n-1})\big\| \\
&\le |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|p_n-p_{n-1}\|+|\beta_n-\beta_{n-1}|\|p_{n-1}\| \\
&\quad +\gamma_n\|\phi_n-\phi_{n-1}\|+\lambda_n\big\|(I-W)\phi_n-(I-W)\phi_{n-1}\big\| \\
&\quad +|\lambda_n-\lambda_{n-1}|\big\|(I-W)\phi_{n-1}\big\| \\
&\quad +|\gamma_n-\gamma_{n-1}|\big\|P_C\big(I-\lambda_{n-1}(I-W)\big)\phi_{n-1}\big\|+\delta_n\|\psi_n-\psi_{n-1}\| \\
&\quad +|\delta_n-\delta_{n-1}|\big\|G(\psi_{n-1})\big\|. \qquad (3.5)
\end{aligned}$$

On the other hand, from $\phi_n = W_{g_n}p_n$ and $\phi_{n+1} = W_{g_{n+1}}p_{n+1}$, we have

$$\Psi_1(\phi_n,\zeta)+\frac{1}{g_n}\langle\zeta-\phi_n,\phi_n-p_n\rangle\ge 0,\quad \forall\zeta\in C \qquad (3.6)$$

and

$$\Psi_1(\phi_{n+1},\zeta)+\frac{1}{g_{n+1}}\langle\zeta-\phi_{n+1},\phi_{n+1}-p_{n+1}\rangle\ge 0,\quad \forall\zeta\in C. \qquad (3.7)$$

Putting $\zeta = \phi_{n+1}$ in (3.6) and $\zeta = \phi_n$ in (3.7), we have

$$\Psi_1(\phi_n,\phi_{n+1})+\frac{1}{g_n}\langle\phi_{n+1}-\phi_n,\phi_n-p_n\rangle\ge 0$$

and

$$\Psi_1(\phi_{n+1},\phi_n)+\frac{1}{g_{n+1}}\langle\phi_n-\phi_{n+1},\phi_{n+1}-p_{n+1}\rangle\ge 0.$$

From (J2), we have

$$\left\langle \phi_{n+1} - \phi_n, \frac{\phi_n - p_n}{g_n} - \frac{\phi_{n+1} - p_{n+1}}{g_{n+1}} \right\rangle \geq 0.$$

So

$$\left\langle \phi_{n+1} - \phi_n, \phi_n - \phi_{n+1} + \phi_{n+1} - p_n - \frac{g_n}{g_{n+1}}(\phi_{n+1} - p_{n+1}) \right\rangle \geq 0.$$

Then

$$\begin{aligned} \|\phi_{n+1} - \phi_n\|^2 &\leq \left\langle \phi_{n+1} - \phi_n, p_{n+1} - p_n + \phi_{n+1} - p_{n+1} - \frac{g_n}{g_{n+1}}(\phi_{n+1} - p_{n+1}) \right\rangle \\ &= \left\langle \phi_{n+1} - \phi_n, p_{n+1} - p_n + \left(1 - \frac{g_n}{g_{n+1}}\right)(\phi_{n+1} - p_{n+1}) \right\rangle \\ &\leq \|\phi_{n+1} - \phi_n\| \left(\|p_{n+1} - p_n\| + \left|1 - \frac{g_n}{g_{n+1}}\right| \|\phi_{n+1} - p_{n+1}\| \right), \end{aligned}$$

and hence

$$\begin{aligned} \|\phi_{n+1} - \phi_n\| &\leq \|p_{n+1} - p_n\| + \frac{1}{g_{n+1}}|g_{n+1} - g_n| \|\phi_{n+1} - p_{n+1}\| \\ &\leq \|p_{n+1} - p_n\| + \frac{1}{e}|g_{n+1} - g_n| \|\phi_{n+1} - p_{n+1}\|. \end{aligned} \tag{3.8}$$

We use $\psi_n = W_{h_n} p_n$ and $\psi_{n+1} = W_{h_{n+1}} p_{n+1}$. By using the same method as (3.8), we have

$$\|\psi_{n+1} - \psi_n\| \leq \|p_{n+1} - p_n\| + \frac{1}{e}|h_{n+1} - h_n| \|\psi_{n+1} - p_{n+1}\|. \tag{3.9}$$

From (3.5), (3.8), and (3.9), we have

$$\begin{aligned} \|p_{n+1} - p_n\| \leq{}& |\alpha_n - \alpha_{n-1}| \|u\| + \beta_n \|p_n - p_{n-1}\| + |\beta_n - \beta_{n-1}| \|p_{n-1}\| \\ &+ \gamma_n \left(\|p_{n+1} - p_n\| + \frac{1}{e}|g_{n+1} - g_n| \|\phi_{n+1} - p_{n+1}\| \right) \\ &+ \lambda_n \left\| (I - W)\phi_n - (I - W)\phi_{n-1} \right\| + |\lambda_n - \lambda_{n-1}| \left\| (I - W)\phi_{n-1} \right\| \\ &+ |\gamma_n - \gamma_{n-1}| \left\| P_C\big(I - \lambda_{n-1}(I - W)\big)\phi_{n-1} \right\| \\ &+ \delta_n \left(\|p_{n+1} - p_n\| + \frac{1}{e}|h_{n+1} - h_n| \|\psi_{n+1} - p_{n+1}\| \right) \\ &+ |\delta_n - \delta_{n-1}| \left\| G(\psi_{n-1}) \right\| \\ \leq{}& |\alpha_n - \alpha_{n-1}| \|u\| + \beta_n \|p_n - p_{n-1}\| + |\beta_n - \beta_{n-1}| \|p_{n-1}\| \\ &+ \gamma_n \|p_{n+1} - p_n\| + \frac{1}{e}|g_{n+1} - g_n| \|\phi_{n+1} - p_{n+1}\| \\ &+ \lambda_n \left\| (I - W)\phi_n - (I - W)\phi_{n-1} \right\| + |\lambda_n - \lambda_{n-1}| \left\| (I - W)\phi_{n-1} \right\| \\ &+ |\gamma_n - \gamma_{n-1}| \left\| P_C\big(I - \lambda_{n-1}(I - W)\big)\phi_{n-1} \right\| + \delta_n \|p_{n+1} - p_n\| \\ &+ \frac{1}{e}|h_{n+1} - h_n| \|\psi_{n+1} - p_{n+1}\| + |\delta_n - \delta_{n-1}| \left\| G(\psi_{n-1}) \right\| \end{aligned}$$

$$\begin{aligned}&\le (1-\alpha_n)\|p_n-p_{n-1}\| + |\alpha_n-\alpha_{n-1}|M + |\beta_n-\beta_{n-1}|M \\ &\quad + |\gamma_n-\gamma_{n-1}|M + |\delta_n-\delta_{n-1}|M + |\lambda_n-\lambda_{n-1}|M + \lambda_n M \\ &\quad + \frac{1}{e}|g_{n+1}-g_n|M + \frac{1}{e}|h_{n+1}-h_n|M,\end{aligned}$$

where

$$\begin{aligned}M := \max_{n\in\mathbb{N}}\bigl\{&\|u\|, \|p_n\|, \bigl\|P_C\bigl(I-\lambda_n(I-W)\bigr)\phi_n\bigr\|, \bigl\|G(\psi_n)\bigr\|, \bigl\|(I-W)\phi_n\bigr\|, \\ &\bigl\|(I-W)\phi_{n+1}-(I-W)\phi_n\bigr\|, \|\phi_n-p_n\|, \|\psi_n-p_n\|\bigr\}.\end{aligned}$$

From the conditions (i), (iv), (v), and Lemma 2.6, we have

$$\lim_{n\to\infty}\|p_{n+1}-p_n\| = 0. \tag{3.10}$$

Since W_{g_n} is a firmly nonexpansive mapping, we obtain

$$\begin{aligned}\|\phi_n-\xi\|^2 &= \|W_{g_n}p_n - W_{g_n}\xi\|^2 \\ &\le \langle W_{g_n}p_n - W_{g_n}\xi, p_n-\xi\rangle \\ &\le \langle \phi_n-\xi, p_n-\xi\rangle \\ &= \frac{1}{2}\bigl(\|\phi_n-\xi\|^2 + \|p_n-\xi\|^2 - \|\phi_n-p_n\|^2\bigr).\end{aligned}$$

It implies that

$$\|\phi_n-\xi\|^2 \le \|p_n-\xi\|^2 - \|\phi_n-p_n\|^2. \tag{3.11}$$

By using the same method as (3.11), we have

$$\|\psi_n-\xi\|^2 \le \|p_n-\xi\|^2 - \|\psi_n-p_n\|^2. \tag{3.12}$$

From the definition of p_n, (3.4), (3.11), and (3.12), we have

$$\begin{aligned}\|p_{n+1}-\xi\|^2 &= \bigl\|\alpha_n(u-\xi) + \beta_n(p_n-\xi) + \gamma_n\bigl(P_C\bigl(I-\lambda_n(I-W)\bigr)\phi_n-\xi\bigr) \\ &\quad + \delta_n\bigl(G(\psi_n)-\xi\bigr)\bigr\|^2 \\ &\le \alpha_n\|u-\xi\|^2 + \beta_n\|p_n-\xi\|^2 + \gamma_n\bigl\|P_C\bigl(I-\lambda_n(I-W)\bigr)\phi_n-\xi\bigr\|^2 \\ &\quad + \delta_n\bigl\|G(\psi_n)-\xi\bigr\|^2 - \beta_n\gamma_n\bigl\|P_C\bigl(I-\lambda_n(I-W)\bigr)\phi_n-p_n\bigr\|^2 \\ &\quad - \beta_n\delta_n\bigl\|G(\psi_n)-p_n\bigr\|^2 \\ &\le \alpha_n\|u-\xi\|^2 + \beta_n\|p_n-\xi\|^2 + \gamma_n\|\phi_n-\xi\|^2 + \delta_n\|\psi_n-\xi\|^2 \\ &\quad - \beta_n\gamma_n\bigl\|P_C\bigl(I-\lambda_n(I-W)\bigr)\phi_n-p_n\bigr\|^2 - \beta_n\delta_n\bigl\|G(\psi_n)-p_n\bigr\|^2 \\ &\le \alpha_n\|u-\xi\|^2 + \beta_n\|p_n-\xi\|^2 + \gamma_n\bigl(\|p_n-\xi\|^2 - \|\phi_n-p_n\|^2\bigr) \\ &\quad + \delta_n\bigl(\|p_n-\xi\|^2 - \|\psi_n-p_n\|^2\bigr) - \beta_n\delta_n\bigl\|G(\psi_n)-p_n\bigr\|^2\end{aligned}$$

$$
\begin{aligned}
&- \beta_n\gamma_n \big\| P_C\big(I-\lambda_n(I-W)\big)\phi_n - p_n\big\|^2 \\
&= \alpha_n\|u-\xi\|^2 + (1-\alpha_n)\|p_n-\xi\|^2 - \gamma_n\|\phi_n - p_n\|^2 \\
&\quad - \delta_n\|\psi_n - p_n\|^2 - \beta_n\gamma_n\big\|P_C\big(I-\lambda_n(I-W)\big)\phi_n - p_n\big\|^2 \\
&\quad - \beta_n\delta_n\big\|G(\psi_n) - p_n\big\|^2 \\
&\le \alpha_n\|u-\xi\|^2 + \|p_n-\xi\|^2 - \gamma_n\|\phi_n-p_n\|^2 - \delta_n\|\psi_n-p_n\|^2 \\
&\quad - \beta_n\gamma_n\big\|P_C\big(I-\lambda_n(I-W)\big)\phi_n - p_n\big\|^2 - \beta_n\delta_n\big\|G(\psi_n)-p_n\big\|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\gamma_n\|\phi_n - p_n\|^2 &\le \alpha_n\|u-\xi\|^2 + \|p_n-\xi\|^2 - \|p_{n+1}-\xi\|^2 \\
&\le \alpha_n\|u-\xi\|^2 + \|p_n - p_{n+1}\|\big(\|p_n-\xi\| + \|p_{n+1}-\xi\|\big).
\end{aligned}
$$

From the conditions (i), (ii), and (3.10), we have

$$
\lim_{n\to\infty}\|\phi_n - p_n\| = 0. \tag{3.13}
$$

By using the same method as (3.13), we can imply that

$$
\lim_{n\to\infty}\|\psi_n - p_n\| = \lim_{n\to\infty}\big\|P_C\big(I-\lambda_n(I-W)\big)\phi_n - p_n\big\| = \lim_{n\to\infty}\big\|G(\psi_n)-p_n\big\| = 0. \tag{3.14}
$$

From (3.13) and (3.14), we have

$$
\lim_{n\to\infty}\|\phi_n - \psi_n\| = 0. \tag{3.15}
$$

Afterwards, we show that $\limsup_{n\to\infty}\langle u-p_0, p_n-p_0\rangle \le 0$, where $p_0 = P_{\mathcal{F}}u$. To show this inequality, take a subsequence $\{p_{n_j}\}$ of $\{p_n\}$ such that

$$
\limsup_{n\to\infty}\langle u-p_0, p_n-p_0\rangle = \lim_{j\to\infty}\langle u-p_0, p_{n_j}-p_0\rangle.
$$

Without loss of generality, we may assume that $u_{n_j} \rightharpoonup \omega$ as $j\to\infty$. From (3.15), we have $v_{n_j} \rightharpoonup \omega$ as $j\to\infty$. By using the same method as [15] in Theorem 3.2, we have

$$
\omega \in \mathrm{EP}(\Psi_1) \tag{3.16}
$$

and

$$
\omega \in \mathrm{EP}(\Psi_2). \tag{3.17}
$$

Furthermore, we show that $\omega \in F(W)$. From Remark 2.9, we have $F(W) = F(P_C(I-\lambda_{n_j}(I-W)))$. Assume that $\omega \notin F(W)$, we have $\omega \neq P_C(I-\lambda_{n_j}(I-W))\omega$. From (3.13), we have $p_{n_j} \rightharpoonup \omega$ as $j\to\infty$. By (3.13), (3.14), the condition (iv), and Opial's property, we have

$$
\begin{aligned}
\liminf_{j\to\infty}\|p_{n_j} - \omega\| &< \liminf_{j\to\infty}\big\|p_{n_j} - P_C\big(I-\lambda_{n_j}(I-W)\big)\omega\big\| \\
&\le \liminf_{j\to\infty}\big(\big\|p_{n_j} - P_C\big(I-\lambda_{n_j}(I-W)\big)u_{n_j}\big\|
\end{aligned}
$$

$$\begin{aligned}
&+ \left\|P_C\big(I - \lambda_{n_j}(I - W)\big)u_{n_j} - P_C\big(I - \lambda_{n_j}(I - W)\big)p_{n_j}\right\| \\
&+ \left\|P_C\big(I - \lambda_{n_j}(I - W)\big)p_{n_j} - P_C\big(I - \lambda_{n_j}(I - W)\big)\omega\right\|\Big) \\
&\leq \liminf_{j\to\infty}\big(\|u_{n_j} - p_{n_j}\| + \lambda_{n_j}\left\|(I - W)u_{n_j} - (I - W)p_{n_j}\right\| \\
&+ \|p_{n_j} - \omega\| + \lambda_{n_j}\left\|(I - W)p_{n_j} - (I - W)\omega\right\|\big) \\
&= \liminf_{j\to\infty}\|p_{n_j} - \omega\|.
\end{aligned}$$

It is a contradiction. So we have

$$\omega \in F(W). \tag{3.18}$$

After that, we show that $\omega \in F(G)$. Assume that $\omega \notin F(G)$, that is, $\omega \neq G(\omega)$. Since $p_{n_j} \rightharpoonup \omega$ as $j \to \infty$, (3.14), the condition (iv), and Opial's property, we have

$$\begin{aligned}
\liminf_{j\to\infty}\|p_{n_j} - \omega\| &< \liminf_{j\to\infty}\left\|p_{n_j} - G(\omega)\right\| \\
&\leq \liminf_{j\to\infty}\big(\left\|p_{n_j} - G(\psi_{n_j})\right\| + \left\|G(\psi_{n_j}) - G(p_{n_j})\right\| \\
&\quad + \left\|G(p_{n_j}) - G(\omega)\right\|\big) \\
&\leq \liminf_{j\to\infty}\big(\|\psi_{n_j} - p_{n_j}\| + \|p_{n_j} - \omega\|\big) \\
&= \liminf_{j\to\infty}\|p_{n_j} - \omega\|.
\end{aligned}$$

It is a contradiction. So we have

$$\omega \in F(G). \tag{3.19}$$

Therefore $\omega \in \mathcal{F}$. Since $p_{n_j} \rightharpoonup \omega$ as $j \to \infty$, we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle u - p_0, p_n - p_0\rangle &= \lim_{j\to\infty}\langle u - p_0, p_{n_j} - p_0\rangle \\
&= \langle u - p_0, \omega - p_0\rangle \leq 0.
\end{aligned} \tag{3.20}$$

Finally, we show that the sequences $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ converge strongly to $p_0 = P_{\mathcal{F}}u$. From the definition of p_n, (3.4), and $p_0 = P_{\mathcal{F}}u$, we have

$$\begin{aligned}
\|p_{n+1} - p_0\|^2 &= \big\|\alpha_n(u - p_0) + \beta_n(p_n - p_0) + \gamma_n\big(P_C\big(I - \lambda_n(I - W)\big)\phi_n - p_0\big) \\
&\quad + \delta_n\big(G(\psi_n) - p_0\big)\big\|^2 \\
&\leq \big\|\beta_n(p_n - p_0) + \gamma_n\big(P_C\big(I - \lambda_n(I - W)\big)\phi_n - p_0\big) \\
&\quad + \delta_n\big(G(\psi_n) - p_0\big)\big\|^2 + 2\alpha_n\langle u - p_0, p_{n+1} - p_0\rangle \\
&\leq (1 - \alpha_n)\|p_n - p_0\|^2 + 2\alpha_n\langle u - p_0, p_{n+1} - p_0\rangle.
\end{aligned}$$

From the condition (i), (3.20), and Lemma 2.6, we can conclude that the sequence $\{p_n\}$ converges strongly to $p_0 = P_{\mathcal{F}}u$. Consequently, we see that $\{\phi_n\}$ and $\{\psi_n\}$ also converge strongly to $p_0 = P_{\mathcal{F}}u$. This completes the proof. □

From our main result, if we take $a = 0$, we have the following corollary.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H, let $\Psi_1, \Psi_2 : C \times C \to \mathbb{R}$ be bifunctions satisfying* (J1)-(J4) *and let $W : C \to C$ be a quasi-nonexpansive mapping. Let $D_1, D_2 : C \to H$ be d_1, d_2-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(p) = P_C(I - \lambda_A D_1)(P_C(I - \lambda_B D_2)p)$ for all $p \in C$. Assume $\mathcal{F} = \mathrm{EP}(\Psi_1) \cap \mathrm{EP}(\Psi_2) \cap F(G) \cap F(W) \neq \emptyset$. Suppose that $p_1, u \in C$ and let $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ be sequences generated by*

$$\begin{cases} \Psi_1(\phi_n, \zeta) + \frac{1}{g_n}\langle \zeta - \phi_n, \phi_n - p_n\rangle \geq 0, & \forall \zeta \in C, \\ \Psi_2(\psi_n, \zeta) + \frac{1}{h_n}\langle \zeta - \psi_n, \psi_n - p_n\rangle \geq 0, & \forall \zeta \in C, \\ p_{n+1} = \alpha_n u + \beta_n p_n + \gamma_n P_C(I - \lambda_n(I - W))\phi_n + \delta_n G(\psi_n), & \forall n \in \mathbb{N}, \end{cases} \tag{3.21}$$

where the sequences $\lambda_A \in (0, 2d_1)$, $\lambda_B \in (0, 2d_2)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \in \mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$ *for some* $c, d > 0$ *and for all* $n \geq 1$,

(iii) $0 < e \leq g_n, h_n \leq f$ *for some* $e,f > 0$ *and for all* $n \geq 1$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |g_{n+1} - g_n| < \infty$, $\sum_{n=1}^{\infty} |h_{n+1} - h_n| < \infty$.

Then $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ converge strongly to $p_0 = P_{\mathcal{F}}u$ and (p_0, ξ_0) is a solution of (1.5) *where $\xi_0 = P_C(I - \lambda_B D_2)p_0$.*

4 Application

In this section, we use our main result to obtain Theorem 4.7 and Theorem 4.8. Before we prove these theorems, we need the following definition and lemma. A mapping $W : C \to C$ is said to be nonspreading if

$$2\|Wp - W\zeta\|^2 \leq \|Wp - \zeta\|^2 + \|W\zeta - p\|^2, \quad \forall p, \zeta \in C. \tag{4.1}$$

Such a mapping is defined by Kohsaka and Takahashi [16].

In 2009, Iemoto and Takahashi [17] proved that (4.1) is equivalent to

$$\|Wp - W\zeta\|^2 \leq \|p - \zeta\|^2 + 2\langle p - Wp, \zeta - W\zeta\rangle, \quad \forall p, \zeta \in C. \tag{4.2}$$

Remark 4.1 A nonspreading mapping W with $F(W) \neq \emptyset$ is quasi-nonexpansive mapping.

Example 4.2 Let $W : [-5, \infty) \to [-5, \infty)$ be defined by

$$Wp = \frac{p - 5}{2}, \quad \forall p \in [-5, \infty).$$

Since W is a nonspreading mapping and $F(W) = \{-5\}$, we have W is a quasi-nonexpansive mapping.

The following lemmas and definition are used to prove the results in this section.

Lemma 4.3 ([8]) *Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2 : C \to H$ be d_1, d_2-inverse strongly monotone mappings, respectively, with $\mathrm{VIP}(C, D_1) \cap \mathrm{VIP}(C, D_2) \neq \emptyset$. Define a mapping $G : C \to C$ by*

$$G(p) = P_C(I - \lambda_A D_1)\big(ap + (1-a)P_C(I - \lambda_B D_2)p\big),$$

for every $\lambda_A \in (0, 2d_1)$, $\lambda_B \in (0, 2d_2)$ and $a \in (0,1)$. Then $F(G) = \mathrm{VIP}(C, D_1) \cap \mathrm{VIP}(C, D_2)$.

Lemma 4.4 ([16]) *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let W be a nonspreading mapping of C into itself. Then $F(W)$ is closed and convex.*

In 2009, Kangtunyakarn and Suantai [18] introduced the S-mapping generated by $W_1, W_2, W_3, \ldots, W_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ as follows.

Definition 4.5 Let C be a nonempty convex subset of a real Banach space. Let $\{W_i\}_{i=1}^N$ be a finite family of (nonexpansive) mappings of C into itself. For each $j = 1, 2, \ldots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows:

$$\begin{aligned}
&U_0 = I,\\
&U_1 = \alpha_1^1 W_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I,\\
&U_2 = \alpha_1^2 W_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I,\\
&U_3 = \alpha_1^3 W_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I,\\
&\ldots,\\
&U_{N-1} = \alpha_1^{N-1} W_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I,\\
&S = U_N = \alpha_1^N W_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called an *S-mapping* generated by $W_1, W_2, \ldots, W_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.

For every $i = 1, 2, \ldots N$, put $\alpha_3^i = 0$ in Definition 4.5, then the S-mapping is reduced to the K-mapping generated by $\alpha_1^1, \alpha_1^2, \ldots, \alpha_1^N$ where the K-mapping is defined by Kangtunyakarn and Suantai [19] as follows.

Lemma 4.6 ([20]) *Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{W_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(W_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, \ldots, N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0,1)$ for all $j = 1, 2, \ldots, N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_3^N \in [0,1)$, $\alpha_2^j \in [0,1)$ for all $j = 1, 2, \ldots, N$. Let S be the mapping generated by $W_1, W_2, \ldots, W_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(W_i)$ and S is a quasi-nonexpansive mapping.*

By using these results, we obtain the following theorems.

Theorem 4.7 *Let C be a nonempty closed convex subset of a real Hilbert space H, let $\Psi_1, \Psi_2 : C \times C \to \mathbb{R}$ be bifunctions satisfying* (J1)-(J4) *and let $W : C \to C$ be a quasi-nonexpansive mapping. Let $D_1, D_2 : C \to H$ be d_1, d_2-inverse strongly monotone mappings,* *respectively. Assume* $\mathcal{F} = \mathrm{EP}(\Psi_1) \cap \mathrm{EP}(\Psi_2) \cap F(W) \cap \mathrm{VIP}(C, D_1) \cap \mathrm{VIP}(C, D_2) \neq \emptyset$. *Suppose that* $p_1, u \in C$ *and let* $\{p_n\}$, $\{\phi_n\}$, *and* $\{\psi_n\}$ *be sequences generated by*

$$\begin{cases} \Psi_1(\phi_n, \zeta) + \frac{1}{g_n}\langle \zeta - \phi_n, \phi_n - p_n\rangle \geq 0, \quad \forall \zeta \in C, \\ \Psi_2(\psi_n, \zeta) + \frac{1}{h_n}\langle \zeta - \psi_n, \psi_n - p_n\rangle \geq 0, \quad \forall \zeta \in C, \\ p_{n+1} = \alpha_n u + \beta_n p_n + \gamma_n P_C(I - \lambda_n(I - W))\phi_n \\ \qquad + \delta_n P_C(I - \lambda_A D_1)(ap_n + (1-a)P_C(I - \lambda_B D_2)p_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{4.3}$$

where the sequences $\lambda_A \in (0, 2d_1)$, $\lambda_B \in (0, 2d_2)$, *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$ *with* $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$, *for all* $n \in \mathbb{N}$, *and* $a \in (0,1)$. *Suppose the following conditions hold*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$ *for some* $c, d > 0$ *and for all* $n \geq 1$,

(iii) $0 < e \leq g_n, h_n \leq f$ *for some* $e, f > 0$ *and for all* $n \geq 1$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |g_{n+1} - g_n| < \infty$, $\sum_{n=1}^{\infty} |h_{n+1} - h_n| < \infty$.

Then $\{p_n\}$, $\{\phi_n\}$, *and* $\{\psi_n\}$ *converge strongly to* $p_0 = P_{\mathcal{F}}u$ *and* (p_0, ξ_0) *be a solution of* (1.6) *where* $\xi_0 = P_C(I - \lambda_B D_2)p_0$.

Proof By using Theorem 3.1 and Lemma 4.3, we obtain the conclusion. □

Theorem 4.8 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, *let* $\Psi_1, \Psi_2 : C \times C \to \mathbb{R}$ *be bifunctions satisfying* (J1)-(J4). *Let* $\{W_i\}_{i=1}^N$ *be a finite family of nonspreading mappings of* C *into* C *and let* $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, \ldots, N$, *where* $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0,1)$ *for all* $j = 1, 2, \ldots, N-1$ *and* $\alpha_1^N \in (0,1]$, $\alpha_3^N \in [0,1)$, $\alpha_2^j \in [0,1)$ *for all* $j = 1, 2, \ldots, N$. *Let* S *be the mapping generated by* $W_1, W_2, \ldots, W_N$, *and* $\alpha_1, \alpha_2, \ldots, \alpha_N$. *Let* $D_1, D_2 : C \to H$ *be* d_1, d_2-*inverse strongly monotone mappings, respectively. Define the mapping* $G : C \to C$ *by* $G(p) = P_C(I - \lambda_A D_1)(ap + (1-a)P_C(I - \lambda_B D_2)p)$ *for all* $p \in C$ *and* $a \in [0,1]$. *Assume* $\mathcal{F} = \mathrm{EP}(\Psi_1) \cap \mathrm{EP}(\Psi_2) \cap F(G) \cap \bigcap_{i=1}^N F(W_i) \neq \emptyset$. *Suppose that* $p_1, u \in C$ *and let* $\{p_n\}$, $\{\phi_n\}$, *and* $\{\psi_n\}$ *are sequences generated by*

$$\begin{cases} \Psi_1(\phi_n, \zeta) + \frac{1}{g_n}\langle \zeta - \phi_n, \phi_n - p_n\rangle \geq 0, \quad \forall \zeta \in C, \\ \Psi_2(\psi_n, \zeta) + \frac{1}{h_n}\langle \zeta - \psi_n, \psi_n - p_n\rangle \geq 0, \quad \forall \zeta \in C, \\ p_{n+1} = \alpha_n u + \beta_n p_n + \gamma_n P_C(I - \lambda_n(I - S))\phi_n + \delta_n G(\psi_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{4.4}$$

where the sequences $\lambda_A \in (0, 2d_1)$, $\lambda_B \in (0, 2d_2)$, *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$ *with* $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *for all* $n \in \mathbb{N}$. *Suppose the following conditions hold*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$ *for some* $c, d > 0$ *and for all* $n \geq 1$,

(iii) $0 < e \leq g_n, h_n \leq f$ *for some* $e, f > 0$ *and for all* $n \geq 1$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |g_{n+1} - g_n| < \infty$, $\sum_{n=1}^{\infty} |h_{n+1} - h_n| < \infty$.

Then $\{p_n\}$, $\{\phi_n\}$, *and* $\{\psi_n\}$ *converge strongly to* $p_0 = P_{\mathcal{F}}u$ *and* (p_0, ξ_0) *is a solution of* (1.6) *where* $\xi_0 = P_C(I - \lambda_B D_2)p_0$.

Proof By using Theorem 3.1 and Lemma 4.6, we obtain the conclusion. □

The following result is directly proven from Theorem 4.8. Therefore, we omit the proof.

Corollary 4.9 *Let C be a nonempty closed convex subset of a real Hilbert space H, let $\Psi_1, \Psi_2 : C \times C \to \mathbb{R}$ be bifunctions satisfying* (J1)-(J4). *Let W be a nonspreading mappings of C into itself with $F(W) \neq \emptyset$. Let $D_1, D_2 : C \to H$ be d_1, d_2-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(p) = P_C(I - \lambda_A D_1)(ap + (1-a)P_C(I - \lambda_B D_2)p)$ for all $p \in C$ and $a \in [0,1]$. Assume $\mathcal{F} = \mathrm{EP}(\Psi_1) \cap \mathrm{EP}(\Psi_2) \cap F(G) \cap F(W) \neq \emptyset$. Suppose that $p_1, u \in C$ and let $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ be sequences generated by*

$$\begin{cases} \Psi_1(\phi_n, \zeta) + \frac{1}{g_n}\langle \zeta - \phi_n, \phi_n - p_n \rangle \geq 0, & \forall \zeta \in C, \\ \Psi_2(\psi_n, \zeta) + \frac{1}{h_n}\langle \zeta - \psi_n, \psi_n - p_n \rangle \geq 0, & \forall \zeta \in C, \\ p_{n+1} = \alpha_n u + \beta_n p_n + \gamma_n P_C(I - \lambda_n(I - W))\phi_n + \delta_n G(\psi_n), & \forall n \in \mathbb{N}, \end{cases} \tag{4.5}$$

where the sequences $\lambda_A \in (0, 2d_1)$, $\lambda_B \in (0, 2d_2)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \in \mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$ *for some* $c, d > 0$ *and for all* $n \geq 1$,

(iii) $0 < e \leq g_n, h_n \leq f$ *for some* $e, f > 0$ *and for all* $n \geq 1$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |g_{n+1} - g_n| < \infty$, $\sum_{n=1}^{\infty} |h_{n+1} - h_n| < \infty$.

Then $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ converge strongly to $p_0 = P_{\mathcal{F}}u$ and (p_0, ξ_0) is a solution of (1.6) *where $\xi_0 = P_C(I - \lambda_B D_2)p_0$.*

5 Example and numerical results

In this section, we give an example supporting Theorem 3.1.

Example 5.1 Let $\mathbb{R}$ be the set of real numbers and let the mapping $D_1, D_2 : \mathbb{R} \to \mathbb{R}$ defined by $D_1 p = \frac{p-2}{3}$ and $D_2 p = \frac{p-2}{5}$, $\forall p \in \mathbb{R}$, respectively. Let the mapping $W : \mathbb{R} \to \mathbb{R}$ be defined by $Wp = \frac{p+2}{2}$, $\forall p \in \mathbb{R}$, let $\Psi_1, \Psi_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\Psi_1(p, \zeta) = -(p - \zeta)(-4 + p + \zeta), \quad \forall p, \zeta \in \mathbb{R}$$

and

$$\Psi_2(p, \zeta) = -2(p-2)^2 + (p-2)(\zeta - 2) + (\zeta - 2)^2, \quad \forall p, \zeta \in \mathbb{R}.$$

By the definition of Ψ_1, we have

$$\begin{aligned} 0 &\leq \Psi_1(\phi_n, \zeta) + \frac{1}{g_n}\langle \zeta - \phi_n, \phi_n - p_n \rangle \\ &= -(\phi_n - \zeta)(-4 + \phi_n + \zeta) + \frac{1}{g_n}(\zeta - \phi_n)(\phi_n - p_n) \\ &= -(\phi_n - \zeta)(-4 + \phi_n + \zeta) + \frac{1}{g_n}\left(\zeta \phi_n - \zeta p_n - \phi_n^2 + \phi_n p_n\right) \end{aligned}$$

$$\begin{aligned}\Leftrightarrow \quad & 0 \le -g_n(\phi_n - \zeta)(-4 + \phi_n + \zeta) + \bigl(\zeta\phi_n - \zeta p_n - \phi_n^2 + \phi_n p_n\bigr) \\ & = 4g_n\phi_n - \phi_n^2 - g_n\phi_n^2 + \phi_n p_n + (-4g_n + \phi_n - p_n)\zeta + g_n\zeta^2.\end{aligned}$$

Let $G(\zeta) = g_n\zeta^2 + (-4g_n + \phi_n - p_n)\zeta + 4g_n\phi_n - \phi_n^2 - g_n\phi_n^2 + \phi_n p_n$, which is a quadratic function of ζ with coefficient $a = g_n$, $b = -4g_n + \phi_n - p_n$, and $c = 4g_n\phi_n - \phi_n^2 - g_n\phi_n^2 + \phi_n p_n$. Determine the discriminant Δ of G as follows:

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-4g_n + \phi_n - p_n)^2 - 4g_n\bigl(4g_n\phi_n - \phi_n^2 - g_n\phi_n^2 + \phi_n p_n\bigr) \\ &= 16g_n^2 - 8g_n\phi_n - 16g_n^2\phi_n + \phi_n^2 + 4g_n\phi_n^2 + 4g_n^2\phi_n^2 + 8g_n p_n - 2\phi_n p_n - 4g_n\phi_n p_n + p_n^2 \\ &= (-4g_n + \phi_n + 2g_n\phi_n - p_n)^2.\end{aligned}$$

We know that $G(\zeta) \ge 0$, $\forall\zeta \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta \le 0$. So we obtain

$$\phi_n = \frac{4g_n + p_n}{1 + 2g_n}. \tag{5.1}$$

By using the same method as (5.1), we have

$$\psi_n = \frac{6h_n + p_n}{1 + 3h_n}. \tag{5.2}$$

Let $p_1, u \in \mathbb{R}$, and $\{p_n\}$ be generated by (3.1) as follows:

$$\begin{cases} \Psi_1(\phi_n, \zeta) + \frac{1}{g_n}\langle \zeta - \phi_n, \phi_n - p_n\rangle \ge 0, & \forall\zeta \in C, \\ \Psi_2(\psi_n, \zeta) + \frac{1}{h_n}\langle \zeta - \psi_n, \psi_n - p_n\rangle \ge 0, & \forall\zeta \in C, \\ p_{n+1} = \alpha_n u + \beta_n p_n + \gamma_n P_C(I - \lambda_n(I - W))\phi_n + \delta_n G(\psi_n), & \forall n \in \mathbb{N}, \end{cases}$$

where $a = 0.5$, $\lambda_A = 1$, $\lambda_B = 1$, $g_n = \frac{n}{3n+1}$, $h_n = \frac{n}{4n+1}$, $\alpha_n = \frac{1}{2n}$, $\beta_n = \frac{3n-1}{16n}$, $\gamma_n = \frac{10n-3}{16n}$, $\delta_n = \frac{3n-4}{16n}$, and $\lambda_n = \frac{1}{2n^2}$ for all $n \in \mathbb{N}$. By the definitions of Ψ_1, Ψ_2, G, and W, we have $\mathrm{EP}(\Psi_1) \cap \mathrm{EP}(\Psi_2) \cap F(G) \cap F(W) = \{2\}$. From Theorem 3.1, we can conclude that the sequences $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ converge strongly to 2. From (5.1) and (5.2), we can rewrite (3.1) as follows:

$$\begin{cases} \phi_n = \frac{4g_n + p_n}{1+2g_n}, \\ \psi_n = \frac{6h_n + p_n}{1+3h_n}, \\ p_{n+1} = \frac{1}{2n}u + \frac{3n-1}{16n}p_n + \frac{10n-3}{16n}(I - \frac{1}{2n^2}(I - W))\phi_n + \frac{3n-4}{16n}G(\psi_n), & \forall n \ge 1. \end{cases} \tag{5.3}$$

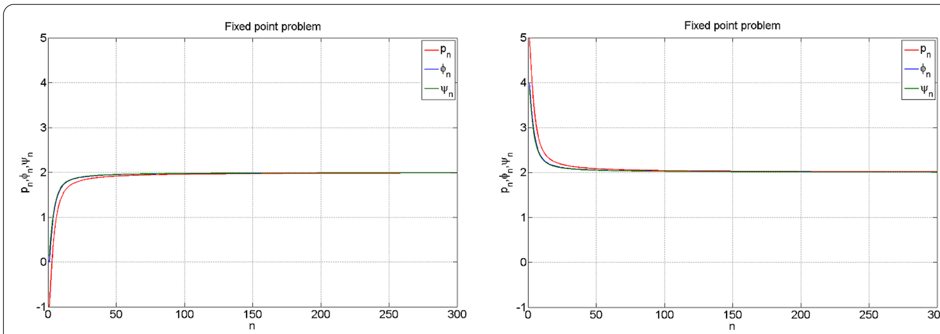
Table 1 shows the values of the sequences $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ where $u = p_1 = -1$ and $u = p_1 = 5$ and $n = 300$.

Conclusion

1. The sequences $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ in Table 1 and Figure 1 converge to 2, where $\{2\} = \mathrm{EP}(\Psi_1) \cap \mathrm{EP}(\Psi_2) \cap F(G) \cap F(W)$.
2. Theorem 3.1 ensures the convergence of $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ in Example 5.1.

Table 1 The values of $\{\phi_n\}$, $\{\psi_n\}$, and $\{p_n\}$ where $n = 300$

n	$u = p_1 = -1$			$u = p_1 = 5$		
	ϕ_n	ψ_n	p_n	ϕ_n	ψ_n	p_n
1	0.0000	0.1250	−1.0000	4.000000	3.8750	5.0000
2	0.4339	0.5234	−0.4609	3.5661	3.4766	4.4609
3	0.7688	0.8360	0.0301	3.2312	3.1640	3.9699
4	1.0254	1.0771	0.4256	2.9746	3.9229	3.5744
5	1.2188	1.2595	0.7306	2.7812	2.7405	3.2694
⋮	⋮	⋮	⋮	⋮	⋮	⋮
150	1.9837	1.9845	1.9728	2.0163	2.0155	2.0272
⋮	⋮	⋮	⋮	⋮	⋮	⋮
296	1.9918	1.9922	1.9863	2.0082	2.0078	2.0137
297	1.9918	1.9922	1.9864	2.0082	2.0078	2.0136
298	1.9918	1.9922	1.9864	2.0082	2.0078	2.0136
299	1.9919	1.9923	1.9865	2.0081	2.0077	2.0135
300	1.9919	1.9923	1.9865	2.0081	2.0077	2.0135



(a) $u = p_1 = -1$ (b) $u = p_1 = 5$

Figure 1 The convergence comparison of the sequences $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ with different initial values u and p_1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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