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Self-adaptive algorithms for proximal split feasibility problems and strong convergence analysis

Yonghong Yao[1], Zhangsong Yao[2], Afrah AN Abdou[3] and Yeol Je Cho[3,4*]

*Correspondence: yjcho@gnu.ac.kr
[3]Department of Mathematics, King Abdulaziz University, Jeddah, 21589, Saudi Arabia
[4]Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju, 660-701, Korea
Full list of author information is available at the end of the article

Abstract

The purpose of the paper is to study the proximal split feasibility problems. For solving the problems, we present new self-adaptive algorithms with the regularization technique. By using these algorithms, we give some strong convergence theorems for the proximal split feasibility problems.

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1 Introduction

The split feasibility problem has received much attention due to its applications in signal processing and image reconstruction [1] with particular progress in intensity modulated therapy [2]. Recently, the split feasibility problem (1.3) has been studied extensively by many authors (see, for instance, [3–16]).

Our purpose of the present manuscript is to study the more general case of the proximal split minimization problems by introducing new algorithms with the regularization technique.

In the sequel, we assume that H_1 and H_2 are two real Hilbert spaces, $f : H_1 \to \mathcal{R} \cup \{+\infty\}$ and $g : H_2 \to \mathcal{R} \cup \{+\infty\}$ are two proper and lower semi-continuous convex functions and $A : H_1 \to H_2$ is a bounded linear operator.

Now, we focus on the following minimization problem:

$$\min_{x^\dagger \in H_1} \{ f(x^\dagger) + g_\lambda(Ax^\dagger) \}, \tag{1.1}$$

where g_λ stands for the Moreau-Yosida approximate of the function g of parameter λ, that is,

$$g_\lambda(x) = \min_{y \in H_2} \left\{ g(y) + \frac{1}{2\lambda} \|x - y\|^2 \right\}.$$

Remark 1.1 (1) The problem (1.1) includes the split feasibility problem as a special case. In fact, we choose f and g as the indicator functions of two nonempty closed convex sets

$C \subset H_1$ and $Q \in H_2$, that is,

$$f(x^\dagger) = \delta_C(x^\dagger) = \begin{cases} 0, & \text{if } x^\dagger \in C, \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$g(x^\dagger) = \delta_Q(x^\dagger) = \begin{cases} 0, & \text{if } x^\dagger \in Q, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then the problem (1.1) reduces to

$$\min_{x^\dagger \in H_1} \{\delta_C(x^\dagger) + (\delta_Q)_\lambda(Ax^\dagger)\},$$

which equals

$$\min_{x^\dagger \in C} \left\{ \frac{1}{2\lambda} \|(I - \mathrm{proj}_Q)(Ax^\dagger)\|^2 \right\}. \tag{1.2}$$

(2) Now, we know that to solve (1.2) is exactly to solve the split feasibility problem of finding $x^\ddagger$ such that

$$x^\ddagger \in C \quad \text{and} \quad Ax^\ddagger \in Q \tag{1.3}$$

provided $C \cap A^{-1}(Q) \neq \emptyset$.

In order to solve (1.3), one of key ideas is to use fixed point technique, that is, $x^\dagger$ solves (1.3) if and only if

$$x^\dagger = \mathrm{proj}_C\big(I - \gamma A^*(I - \mathrm{proj}_Q)A\big)x^\dagger,$$

where $\gamma > 0$ is a constant and proj_C and proj_Q stand for the orthogonal projections on the closed convex sets C and Q, respectively.

According to the above fixed point equation, a popular algorithm to solve the split feasibility problems is the CQ method ([4]):

$$x_{n+1} = \mathrm{proj}_C\big(x_n - \tau_n A^*(I - \mathrm{proj}_Q)Ax_n\big),$$

where the step size $\tau_n \in (0, 2/\|A\|^2)$.

However, the determination of the step size τ_n depends on the operator norm $\|A\|$ (or the largest eigenvalue of A^*A) which is in general not an easy work in practice. To overcome the above difficulty, the so-called self-adaptive method which permits step size τ_n being selected self-adaptively was developed.

Self-adaptive algorithm ([17]) Let $x_0 \in H_1$ be an initial arbitrarily point. Assume that a sequence $\{x_n\}$ in C has been constructed with $\nabla\bar{h}(x_n) \neq 0$ as follows: Compute x_{n+1} via the rule

$$x_{n+1} = \operatorname{proj}_C\big(x_n - \tau_n A^*(I - \operatorname{proj}_Q)Ax_n\big), \tag{1.4}$$

where $\tau_n = \rho_n \frac{\bar{h}(x_n)}{\|\nabla \bar{h}(x_n)\|^2}$ with $0 < \rho_n < 4$ and $\bar{h}(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$.

If $\nabla \bar{h}(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of the problem (1.3) and the iterative process stops. Otherwise, we set $n := n + 1$ and go to the sequence (1.4).

In the present manuscript, our main purpose is to solve the problem (1.1) by using the fixed point technique and the self-adaptive methods. First, by the differentiability of the Yosida approximate g_λ, we have

$$\begin{aligned} \partial\big(f(x^\dagger) + g_\lambda(Ax^\dagger)\big) &= \partial f(x^\dagger) + A^*\nabla g_\lambda(Ax^\dagger) \\ &= \partial f(x^\dagger) + A^*\left(\frac{I - \operatorname{prox}_{\lambda g}}{\lambda}\right)(Ax^\dagger), \end{aligned} \tag{1.5}$$

where $\partial f(x^\dagger)$ denotes the subdifferential of f at $x^\dagger$ and $\operatorname{prox}_{\lambda g}(x^\dagger)$ is the proximal mapping of g, that is,

$$\partial f(x^\dagger) = \{x^* \in H_1 : f(x^\ddagger) \geq f(x^\dagger) + \langle x^*, x^\ddagger - x^\dagger\rangle, \forall x^\ddagger \in H_1\}$$

and

$$\operatorname{prox}_{\lambda g}(x^\dagger) = \arg\min_{x^\ddagger \in H_2}\left\{g(x^\ddagger) + \frac{1}{2\lambda}\|x^\ddagger - x^\dagger\|^2\right\}.$$

Note that the optimality condition of (1.5) is as follows:

$$0 \in \partial f(x^\dagger) + A^*\left(\frac{I - \operatorname{prox}_{\lambda g}}{\lambda}\right)(Ax^\dagger),$$

which can be rewritten as

$$0 \in \mu\lambda\partial f(x^\dagger) + \mu A^*(I - \operatorname{prox}_{\lambda g})(Ax^\dagger), \tag{1.6}$$

which is equivalent to the fixed point equation:

$$x^\dagger = \operatorname{prox}_{\mu\lambda f}\big(x^\dagger - \mu A^*(I - \operatorname{prox}_{\lambda g})(Ax^\dagger)\big) \tag{1.7}$$

for all $\mu > 0$.

If $\arg\min f \cap A^{-1}(\arg\min g) \neq \emptyset$, then (1.1) is reduced to the following proximal split feasibility problem.

Find $x^\dagger$ such that

$$x^\dagger \in \arg\min f \quad \text{and} \quad Ax^\dagger \in \arg\min g, \tag{1.8}$$

where

$$\arg\min f = \{x^* \in H_1 : f(x^*) \leq f(x^\dagger), \forall x^\dagger \in H_1\}$$

and

$$\arg\min g = \{x^\dagger \in H_2 : g(x^\dagger) \leq g(x), \forall x \in H_2\}.$$

In the sequel, we use Γ to denote the solution set of the problem (1.8).

Recently, in order to solve the problem (1.8), Moudafi and Thakur [18] presented the following split proximal algorithm with a way of selecting the step sizes such that its implementation does not need any prior information as regards the operator norm.

Self-adaptive split proximal algorithm For an initialization $x_0 \in H_1$, assume that a sequence $\{x_n\}$ in H has been constructed and $\theta(x_n) \neq \emptyset$ as follows: Compute x_{n+1} via

$$x_{n+1} = \operatorname{prox}_{\mu_n \lambda f}\big(x_n - \mu_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n\big) \tag{1.9}$$

for all $n \geq 0$, where the step size $\mu_n = \rho_n \frac{h(x_n)+l(x_n)}{\theta^2(x_n)}$ in which $0 < \rho_n < 4$,

$$h(x_n) = \frac{1}{2}\|(I - \operatorname{prox}_{\lambda g})Ax_n\|^2, \qquad l(x_n) = \frac{1}{2}\|(I - \operatorname{prox}_{\mu_n \lambda f})x_n\|^2$$

and

$$\theta(x_n) = \sqrt{\|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2}.$$

If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of the problem (1.8) and the iterative process stops. Otherwise, we set $n := n + 1$ and go to the sequence (1.9).

Consequently, they demonstrated the following weak convergence of the above split proximal algorithm.

Theorem 1.2 *Suppose that* $\Gamma \neq \emptyset$. *Assume that the parameters satisfy the condition*:

$$\epsilon \leq \rho_n \leq \frac{4h(x_n)}{h(x_n) + l(x_n)} - \epsilon$$

for some $\epsilon > 0$ *small enough. Then the sequence* $\{x_n\}$ *generated by* (1.9) *weakly converges to a solution of the problem* (1.8).

Note that Theorem 1.2 has only the weak convergence. So, a natural problem arises:

Could we design a new algorithm such that the strong convergence is obtained?

In this paper, our main purpose is to adapt the algorithm (1.9) by using the regularization means such that the strong convergence is guaranteed.

2 Preliminaries

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, respectively and C be a nonempty closed convex subset of H.

Recall that a mapping $T : C \to C$ is said to be:

(1) *L-Lipschitz* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|$$

for all $x, y \in C$. If $L \in (0,1)$, then we call T the *L-contraction*. If $L = 1$, we call T a *nonexpansive mapping*.

(2) *Firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$$

for all $x, y \in C$, where I denotes the identity, which is equivalent to

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$$

for all $x, y \in C$. Also, the mapping $I - T$ is firmly nonexpansive.

(3) *Strongly positive* if there exists a constant $\zeta > 0$ such that

$$\langle Tx, x\rangle \geq \zeta\|x\|^2$$

for all $x \in C$.

Note that the proximal mapping of g is firmly nonexpansive, namely,

$$\langle \operatorname{prox}_{\lambda g} x - \operatorname{prox}_{\lambda g} y, x - y\rangle \geq \|\operatorname{prox}_{\lambda g} x - \operatorname{prox}_{\lambda g} y\|^2$$

for all $x, y \in H_2$ and it is also the case for the complement $I - \operatorname{prox}_{\lambda g}$. Thus $A^*(I - \operatorname{prox}_{\lambda g})A$ is cocoercive with coefficient $\frac{1}{\|A\|^2}$, where we recall that a mapping $B : H_1 \to H_1$ is *cocoercive* if there exists $\alpha > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2$$

for all $x, y \in H_1$. If $\mu \in (0, \frac{1}{\|A\|^2})$, then $I - \mu A^*(I - \operatorname{prox}_{\lambda g})A$ is nonexpansive.

Let C be a nonempty closed convex subset of H. For all $x \in H$, there exists a unique nearest point in C, denoted by $\operatorname{proj}_C x$, such that

$$\|x - \operatorname{proj}_C x\| \leq \|x - y\|$$

for all $y \in C$. The mapping proj_C is called the *metric projection* of H onto C. It is well known that proj_C is a nonexpansive mapping and is characterized by the following property:

$$\langle x - \operatorname{proj}_C x, y - \operatorname{proj}_C x\rangle \leq 0 \qquad (2.1)$$

for all $x \in H$ and $y \in C$.

Now, we introduce two lemmas for our main results in this paper.

Lemma 2.1 ([19]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \delta_n$$

for all $n \geq 0$, *where*

(a) $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $\limsup_{n\to\infty} \sigma_n \leq 0$;

(c) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 ([20]) *Let* $\{\gamma_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{\gamma_{n_i}\}$ *of* $\{\gamma_n\}$ *such that* $\gamma_{n_i} < \gamma_{n_i+1}$ *for all* $i \geq 1$. *Then there exists a nondecreasing sequence* $\{m_k\}$ *of positive integers such that* $\lim_{k\to\infty} m_k = \infty$ *and the following properties are satisfied by all* (*sufficiently large*) *positive integers* k:

$$\gamma_{m_k} \leq \gamma_{m_k+1}, \qquad \gamma_k \leq \gamma_{m_k+1}.$$

In fact, m_k *is the largest number* n *in the set* $\{1,\ldots,k\}$ *such that the condition* $\gamma_n < \gamma_{n+1}$ *holds.*

3 Main results

Now, we first introduce our self-adaptive algorithm. Let H_1 and H_2 be two real Hilbert spaces. Let $f : H_1 \to \mathcal{R} \cup \{+\infty\}$ and $g : H_2 \to \mathcal{R} \cup \{+\infty\}$ be two proper and lower semicontinuous convex functions and $A : H_1 \to H_2$ be a bounded linear operator. Let $\psi : H_1 \to H_1$ be a κ-contraction and $B : H_1 \to H_1$ be a strongly positive bounded linear operator with coefficient $\zeta > \kappa$.

Algorithm 3.1 Set

$$h(x) = \frac{1}{2}\left\|(I - \operatorname{prox}_{\lambda g})Ax\right\|^2, \qquad l(x) = \frac{1}{2}\left\|(I - \operatorname{prox}_{\lambda f})x\right\|^2$$

and

$$\theta(x) = \sqrt{\left\|\nabla h(x)\right\|^2 + \left\|\nabla l(x)\right\|^2}$$

for all $x \in H_1$. For an initialization $x_0 \in H_1$, assume that a sequence $\{x_n\}$ has been constructed in H_1 with $\theta(x_n) \neq \emptyset$ as follows.

Compute x_{n+1} via

$$x_{n+1} = \alpha_n \psi(x_n) + (I - \alpha_n B)\operatorname{prox}_{\lambda f}\bigl(x_n - \mu_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n\bigr) \tag{3.1}$$

for all $n \geq 0$, where $\{\alpha_n\} \subset [0,1]$ is a real number sequence and μ_n is the step size satisfying $\mu_n = \rho_n \frac{h(x_n)+l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$.

If $\theta(x_n) = 0$, then x_n is a solution of the problem (1.8) and the iterative process stops. Otherwise, we set $n := n + 1$ and go to the sequence (3.1).

Theorem 3.2 *Suppose that* $\Gamma \neq \emptyset$. *Assume the parameters* $\{\alpha_n\}$ *and* $\{\rho_n\}$ *satisfy the conditions*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\epsilon \leq \rho_n \leq \frac{4h(x_n)}{h(x_n)+l(x_n)} - \epsilon$ *for some* $\epsilon > 0$ *small enough.*

Then the sequence $\{x_n\}$ *generated by* (3.1) *converges strongly to a point* $z = \mathrm{proj}_\Gamma(\psi + I - B)(z)$.

Proof From (2.1), we deduce that $z = \mathrm{proj}_\Gamma(\psi + I - B)(z)$ implies

$$\langle (\psi + I - B)(z) - z, x - z \rangle \le 0$$

for all $x \in \Gamma$, which has a unique solution. Let $x^* \in \Gamma$. Since minimizers of any function are exactly fixed points of its proximal mappings, we have $x^* = \mathrm{prox}_{\lambda f} x^*$ and $Ax^* = \mathrm{prox}_{\lambda g} Ax^*$. Since $\mathrm{prox}_{\lambda f}$ is nonexpansive, by (3.1), we can derive

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\quad = \|\alpha_n \psi(x_n) + (I - \alpha_n B)\,\mathrm{prox}_{\lambda f}\big(x_n - \mu_n A^*(I - \mathrm{prox}_{\lambda g})Ax_n\big) - x^*\| \\
&\quad = \|(I - \alpha_n B)\big(\mathrm{prox}_{\lambda f}\big(x_n - \mu_n A^*(I - \mathrm{prox}_{\lambda g})Ax_n\big) - x^*\big) + \alpha_n\big(\psi(x_n) - Bx^*\big)\| \\
&\quad = \|I - \alpha_n B\|\,\|\mathrm{prox}_{\lambda f}\big(x_n - \mu_n A^*(I - \mathrm{prox}_{\lambda g})Ax_n\big) - \mathrm{prox}_{\lambda f} x^*\| \\
&\qquad + \alpha_n \|\psi(x_n) - \psi(x^*)\| + \alpha_n \|\psi(x^*) - Bx^*\| \\
&\quad \le \alpha_n \kappa \|x_n - x^*\| + \alpha_n \|\psi(x^*) - Bx^*\| \\
&\qquad + (1 - \zeta\alpha_n)\|\mathrm{prox}_{\lambda f}\big(x_n - \mu_n A^*(I - \mathrm{prox}_{\lambda g})Ax_n\big) - \mathrm{prox}_{\lambda f} x^*\| \\
&\quad \le \alpha_n \kappa \|x_n - x^*\| + \alpha_n \|\psi(x^*) - Bx^*\| \\
&\qquad + (1 - \zeta\alpha_n)\|x_n - \mu_n A^*(I - \mathrm{prox}_{\lambda g})Ax_n - x^*\|.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \alpha_n \kappa \|x_n - x^*\|^2 + (\zeta - \kappa)\alpha_n \frac{\|\psi(x^*) - Bx^*\|^2}{(\zeta - \kappa)^2} \\
&\quad + (1 - \zeta\alpha_n)\|x_n - \mu_n A^*(I - \mathrm{prox}_{\lambda g})Ax_n - x^*\|^2.
\end{aligned} \tag{3.2}$$

Since $\mathrm{prox}_{\lambda g}$ is firmly nonexpansive, we deduce that $I - \mathrm{prox}_{\lambda g}$ is also firmly nonexpansive. Hence we have

$$\begin{aligned}
&\langle A^*(I - \mathrm{prox}_{\lambda g})Ax_n, x_n - x^* \rangle \\
&\quad = \langle (I - \mathrm{prox}_{\lambda g})Ax_n, Ax_n - Ax^* \rangle \\
&\quad = \langle (I - \mathrm{prox}_{\lambda g})Ax_n - (I - \mathrm{prox}_{\lambda g})Ax^*, Ax_n - Ax^* \rangle \\
&\quad \ge \|(I - \mathrm{prox}_{\lambda g})Ax_n\|^2 \\
&\quad = 2h(x_n).
\end{aligned} \tag{3.3}$$

Note that $\nabla h(x_n) = A^*(I - \mathrm{prox}_{\lambda g})Ax_n$ and $\nabla l(x_n) = (I - \mathrm{prox}_{\lambda f})x_n$. Thus it follows from (3.3) that

$$\begin{aligned}
&\|x_n - \mu_n A^*(I - \mathrm{prox}_{\lambda g})Ax_n - x^*\|^2 \\
&\quad = \|x_n - x^*\|^2 + \mu_n^2 \|A^*(I - \mathrm{prox}_{\lambda g})Ax_n\|^2 - 2\mu_n \langle A^*(I - \mathrm{prox}_{\lambda g})Ax_n, x_n - x^* \rangle
\end{aligned}$$

$$\begin{aligned}
&= \|x_n - x^*\|^2 + \mu_n^2 \|\nabla h(x_n)\|^2 - 2\mu_n \langle \nabla h(x_n), x_n - x^* \rangle \\
&\le \|x_n - x^*\|^2 + \mu_n^2 \|\nabla h(x_n)\|^2 - 4\mu_n h(x_n) \\
&= \|x_n - x^*\|^2 + \rho_n^2 \frac{(h(x_n) + l(x_n))^2}{\theta^4(x_n)} \|\nabla h(x_n)\|^2 - 4\rho_n \frac{h(x_n) + l(x_n)}{\theta^2(x_n)} h(x_n) \\
&\le \|x_n - x^*\|^2 + \rho_n^2 \frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)} - 4\rho_n \frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)} \frac{h(x_n)}{h(x_n) + l(x_n)} \\
&= \|x_n - x^*\|^2 - \rho_n \left(\frac{4h(x_n)}{h(x_n) + l(x_n)} - \rho_n \right) \frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)}. \qquad (3.4)
\end{aligned}$$

By the condition (C3), without loss of generality, we can assume that

$$\frac{4h(x_n)}{h(x_n) + l(x_n)} - \frac{\rho_n}{1 - \alpha_n} \ge 0$$

for all $n \ge 0$. Thus, from (3.2) and (3.4), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \le \alpha_n \kappa \|x_n - x^*\|^2 + (\zeta - \kappa)\alpha_n \frac{\|\psi(x^*) - Bx^*\|^2}{(\zeta - \kappa)^2} \\
&\qquad + (1 - \zeta\alpha_n) \left[\|x_n - x^*\|^2 - \rho_n \left(\frac{4h(x_n)}{h(x_n) + l(x_n)} - \rho_n \right) \frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)} \right] \\
&\quad = (\zeta - \kappa)\alpha_n \frac{\|\psi(x^*) - Bx^*\|^2}{(\zeta - \kappa)^2} + \left[1 - (\zeta - \kappa)\alpha_n\right] \|x_n - x^*\|^2 \\
&\qquad - (1 - \zeta\alpha_n)\rho_n \left(\frac{4h(x_n)}{h(x_n) + l(x_n)} - \rho_n \right) \frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)} \\
&\quad \le (\zeta - \kappa)\alpha_n \frac{\|\psi(x^*) - Bx^*\|^2}{(\zeta - \kappa)^2} + \left[1 - (\zeta - \kappa)\alpha_n\right] \|x_n - x^*\|^2 \\
&\quad \le \max\left\{ \frac{\|\psi(x^*) - Bx^*\|^2}{(\zeta - \kappa)^2}, \|x_n - x^*\|^2 \right\}. \qquad (3.5)
\end{aligned}$$

Hence $\{x_n\}$ is bounded.

Let $z = P_\Gamma(\psi + I - B)z$. From (3.5), we deduce

$$\begin{aligned}
0 &\le (1 - \zeta\alpha_n)\rho_n \left(\frac{4h(x_n)}{h(x_n) + l(x_n)} - \rho_n \right) \frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)} \\
&\le (\zeta - \kappa)\alpha_n \frac{\|\psi(z) - Bz\|^2}{(\zeta - \kappa)^2} + \left[1 - (\zeta - \kappa)\alpha_n\right] \|x_n - z\|^2 - \|x_{n+1} - z\|^2. \qquad (3.6)
\end{aligned}$$

We consider the following two cases.

Case 1. $\|x_{n+1} - z\| \le \|x_n - z\|$ for all $n \ge n_0$ large enough.

In this case, $\lim_{n\to\infty} \|x_n - z\|$ exists and is finite, and hence

$$\lim_{n\to\infty} \left(\|x_{n+1} - z\| - \|x_n - z\| \right) = 0.$$

This together with (3.6) implies that

$$\rho_n \left(\frac{4h(x_n)}{h(x_n) + l(x_n)} - \rho_n \right) \frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)} \to 0.$$

Since $\rho_n(\frac{4h(x_n)}{h(x_n)+l(x_n)} - \rho_n) \geq \epsilon^2$ by the condition (C3), we have

$$\frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)} \to 0.$$

Noting that $\theta^2(x_n) = \|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2$ is bounded, we deduce immediately that

$$\lim_{n\to\infty} \bigl(h(x_n) + l(x_n)\bigr) = 0.$$

Therefore, we have

$$\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} l(x_n) = 0. \tag{3.7}$$

Next, we prove that

$$\limsup_{n\to\infty} \bigl\langle (\psi + I - B)z - z, x_n - z \bigr\rangle \leq 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z^\dagger$ and

$$\limsup_{n\to\infty} \bigl\langle (\psi + I - B)z - z, x_n - z \bigr\rangle = \lim_{i\to\infty} \bigl\langle (\psi + I - B)z - z, x_{n_i} - z \bigr\rangle.$$

By the lower semi-continuity of h, we have

$$0 \leq h\bigl(z^\dagger\bigr) \leq \liminf_{i\to\infty} h(x_{n_i}) = \lim_{n\to\infty} h(x_n) = 0.$$

So, we have

$$h\bigl(z^\dagger\bigr) = \frac{1}{2}\bigl\|(I - \mathrm{prox}_{\lambda g})Az^\dagger\bigr\| = 0,$$

that is, $Az^\dagger$ is a fixed point of the proximal mapping of g or, equivalently, $0 \in \partial g(Az^\dagger)$. In other words, $Az^\dagger$ is a minimizer of g.

Similarly, from the lower semi-continuity of l, we have

$$0 \leq l\bigl(z^\dagger\bigr) \leq \liminf_{i\to\infty} l(x_{n_i}) = \lim_{n\to\infty} l(x_n) = 0.$$

Therefore, we have

$$l\bigl(z^\dagger\bigr) = \frac{1}{2}\bigl\|(I - \mathrm{prox}_{\lambda f})z^\dagger\bigr\| = 0,$$

that is, $z^\dagger$ is a fixed point of the proximal mapping of f or, equivalently, $0 \in \partial f(z^\dagger)$. In other words, $z^\dagger$ is a minimizer of f. Hence $z^\dagger \in \Gamma$. Therefore, we have

$$\begin{aligned} \limsup_{n\to\infty} \bigl\langle (\psi + I - B)z - z, x_n - z \bigr\rangle &= \lim_{i\to\infty} \bigl\langle (\psi + I - B)z - z, x_{n_i} - z \bigr\rangle \\ &= \bigl\langle (\psi + I - B)z - z, z^\dagger - z \bigr\rangle \leq 0. \end{aligned} \tag{3.8}$$

By (3.4), we have

$$\big\|\mathrm{prox}_{\lambda f}\big[x_n-\mu_n A^*(I-\mathrm{prox}_{\lambda g})Ax_n\big]-z\big\|\le\|x_n-z\|.$$

Thus it follows from (3.1) that

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
&\quad=\alpha_n\big\langle\psi(x_n)-\psi(z),x_{n+1}-z\big\rangle+\alpha_n\big\langle\psi(z)-Bz,x_{n+1}-z\big\rangle\\
&\qquad+(I-\alpha_n B)\big\langle\mathrm{prox}_{\lambda f}\big(x_n-\mu_n A^*(I-\mathrm{prox}_{\lambda g})Ax_n\big)-z,x_{n+1}-z\big\rangle\\
&\quad\le\alpha_n\big\|\psi(x_n)-\psi(z)\big\|\|x_{n+1}-z\|+\alpha_n\big\langle\psi(z)-Bz,x_{n+1}-z\big\rangle\\
&\qquad+\|I-\alpha_n B\|\big\|\mathrm{prox}_{\lambda f}\big(x_n-\mu_n A^*(I-\mathrm{prox}_{\lambda g})Ax_n\big)-z\big\|\|x_{n+1}-z\|\\
&\quad\le\alpha_n\kappa\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\big\langle\psi(z)-Bz,x_{n+1}-z\big\rangle\\
&\qquad+(1-\zeta\alpha_n)\|x_n-z\|\|x_{n+1}-z\|\\
&\quad\le\frac{1-(\zeta-\kappa)\alpha_n}{2}\|x_n-z\|^2+\frac{1}{2}\|x_{n+1}-z\|^2+\alpha_n\big\langle\psi(z)-Bz,x_{n+1}-z\big\rangle.
\end{aligned}$$

Thus it follows that

$$\|x_{n+1}-z\|^2\le\big[1-(\zeta-\kappa)\alpha_n\big]\|x_n-z\|^2+2\alpha_n\big\langle\psi(z)-Bz,x_{n+1}-z\big\rangle. \tag{3.9}$$

From Lemma 2.1, (3.8) and (3.9) we deduce that $x_n\to z$.

Case 2. There exists a subsequence $\{\|x_{n_j}-z\|\}$ of $\{\|x_n-z\|\}$ such that

$$\|x_{n_j}-z\|<\|x_{n_j+1}-z\|$$

for all $j\ge1$. By Lemma 2.2, there exists a strictly increasing sequence $\{m_k\}$ of positive integers such that $\lim_{k\to\infty}m_k=+\infty$ and the following properties are satisfied: for all $k\in\mathbb{N}$,

$$\|x_{m_k}-z\|\le\|x_{m_{k+1}}-z\|,\qquad\|x_k-z\|\le\|x_{m_{k+1}}-z\|. \tag{3.10}$$

Consequently, we have

$$\begin{aligned}
0&\le\lim_{k\to\infty}\big(\|x_{m_{k+1}}-z\|-\|x_{m_k}-z\|\big)\\
&\le\limsup_{n\to\infty}\big(\|x_{n+1}-z\|-\|x_n-z\|\big)\\
&\le\limsup_{n\to\infty}\big(\alpha_n\|u-z\|+(1-\alpha_n)\|x_n-z\|-\|x_n-z\|\big)\\
&=\limsup_{n\to\infty}\alpha_n\big(\|u-z\|-\|x_n-z\|\big)=0
\end{aligned}$$

and so

$$\lim_{k\to\infty}\big(\|x_{m_{k+1}}-z\|-\|x_{m_k}-z\|\big)=0. \tag{3.11}$$

By a similar argument to Case 1, we can prove that

$$\limsup_{k\to\infty}\langle(\psi+I-B)z-z,x_{m_k}-z\rangle\leq 0$$

and

$$\|x_{m_k+1}-z\|^2\leq\left[1-(\zeta-\kappa)\alpha_n\right]\|x_{m_k}-z\|^2+\alpha_{m_k}\sigma_{m_k},$$

where $\sigma_{m_k}=2\langle\psi(z)-Bz,x_{m_k+1}-z\rangle$. In particular, we have

$$\begin{aligned}(\zeta-\kappa)\alpha_{m_k}\|x_{m_k}-z\|^2&\leq\|x_{m_k}-z\|^2-\|x_{m_k+1}-z\|+\alpha_{m_k}\sigma_{m_k}\\&\leq\alpha_{m_k}\sigma_{m_k}.\end{aligned}$$

Then we have

$$\limsup_{k\to\infty}\|x_{m_k}-z\|^2\leq\limsup_{k\to\infty}\sigma_{m_k}\leq 0.$$

Thus it follows from (3.10) and (3.11) that

$$\limsup_{k\to\infty}\|x_k-z\|\leq\limsup_{k\to\infty}\|x_{m_k+1}-z\|=0,$$

which implies that $x_n\to z$. This completes the proof. □

Algorithm 3.3 For an initialization $x_0\in H_1$. Assume that a sequence $\{x_n\}$ has been constructed as follows: Set

$$h(x_n)=\frac{1}{2}\left\|(I-\operatorname{prox}_{\lambda g})Ax_n\right\|^2,\qquad l(x_n)=\frac{1}{2}\left\|(I-\operatorname{prox}_{\lambda f})x_n\right\|^2$$

and

$$\theta(x_n)=\sqrt{\left\|\nabla h(x_n)\right\|^2+\left\|\nabla l(x_n)\right\|^2}$$

for all $n\in\mathbb{N}$.

If $\theta(x_n)\neq\emptyset$, then compute x_{n+1} via

$$x_{n+1}=\alpha_n\psi(x_n)+(1-\alpha_n)\operatorname{prox}_{\lambda f}\left(x_n-\mu_nA^*(I-\operatorname{prox}_{\lambda g})Ax_n\right)\tag{3.12}$$

for all $n\geq 0$, where $\{\alpha_n\}_{n\in\mathbb{N}}\subset[0,1]$ is a real number sequence and μ_n is the step size satisfying

$$\mu_n=\rho_n\frac{h(x_n)+l(x_n)}{\theta^2(x_n)}$$

with $0<\rho_n<4$.

If $\theta(x_n)=0$, then x_n is a solution of the problem (1.8) and the iterative process stops. Otherwise, we set $n:=n+1$ and go to (3.12).

From Theorem 3.2, we have the following corollary.

Corollary 3.4 *Suppose that* $\Gamma \neq \emptyset$. *Assume the parameters* $\{\alpha_n\}$ *and* $\{\rho_n\}$ *satisfy the conditions*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\epsilon \leq \rho_n \leq \frac{4h(x_n)}{h(x_n)+l(x_n)} - \epsilon$ *for some* $\epsilon > 0$ *small enough.*

Then the sequence $\{x_n\}$ *generated by* (3.12) *converges strongly to a point* $z = \operatorname{proj}_\Gamma(\psi)(z)$.

Algorithm 3.5 For an initialization $x_0 \in H_1$. Assume that a sequence $\{x_n\}$ has been constructed as follows: Set

$$h(x_n) = \frac{1}{2}\big\|(I - \operatorname{prox}_{\lambda g})Ax_n\big\|^2, \qquad l(x_n) = \frac{1}{2}\big\|(I - \operatorname{prox}_{\lambda f})x_n\big\|^2$$

and

$$\theta(x_n) = \sqrt{\big\|\nabla h(x_n)\big\|^2 + \big\|\nabla l(x_n)\big\|^2}$$

for all $n \in \mathbb{N}$.

If $\theta(x_n) \neq \emptyset$, then compute x_{n+1} via

$$x_{n+1} = (1-\alpha_n)\operatorname{prox}_{\lambda f}\big(x_n - \mu_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n\big) \tag{3.13}$$

for all $n \geq 0$, where $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0,1]$ is a real number sequence and μ_n is the step size satisfying

$$\mu_n = \rho_n \frac{h(x_n) + l(x_n)}{\theta^2(x_n)}$$

with $0 < \rho_n < 4$.

If $\theta(x_n) = 0$, then x_n is a solution of the problem (1.8) and the iterative process stops. Otherwise, we set $n := n + 1$ and go to (3.13).

Corollary 3.6 *Suppose that* $\Gamma \neq \emptyset$. *Assume the parameters* $\{\alpha_n\}$ *and* $\{\rho_n\}$ *satisfy the following conditions*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\epsilon \leq \rho_n \leq \frac{4h(x_n)}{h(x_n)+l(x_n)} - \epsilon$ *for some* $\epsilon > 0$ *small enough.*

Then the sequence $\{x_n\}$ *generated by* (3.13) *converges strongly to a point* $z = \operatorname{proj}_\Gamma(0)$, *which is the minimum norm element in* Γ.

Remark 3.7 Where the bounded linear operator A is the identity operator, the problem (1.8) is nothing else than the problem of finding a common minimizer of f and g and (3.1) reduces to the following relaxed split proximal algorithm:

$$x_{n+1} = \alpha_n \psi(x_n) + (I - \alpha_n B)\operatorname{prox}_{\lambda f}\big((1-\mu_n)x_n + \mu_n \operatorname{prox}_{\lambda g} x_n\big)$$

for all $n \geq 0$.

Competing interests
The authors declare that they have no competing interest.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Tianjin Polytechnic University, Tianjin, 300387, China. [2]School of Information Engineering, Nanjing Xiaozhuang University, Nanjing, 211171, China. [3]Department of Mathematics, King Abdulaziz University, Jeddah, 21589, Saudi Arabia. [4]Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju, 660-701, Korea.

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