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Metric fixed point theory: a brief retrospective

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Abstract

These remarks are based on a talk the writer gave at the 11th International Conference in Fixed Point Theory and Applications, held at Galatasaray University in Istanbul, Turkey, July 20-24, 2015. They represent selected thoughts on a career in research, largely devoted to metric fixed point theory, that has spanned over 50 years.

1 Introduction

This is not intended to be a review of 'metric fixed point theory' from its inception, but rather an overview of the emergence of the theory as I experienced it over the past 50 years.

As many know, in 1965 I published a paper that had a clear impact on the development of 'metric' fixed point theory. In these remarks I will discuss how this paper emerged, as well as the early days of my research and selected topics that I have studied throughout the years. I was trained as a geometer. However, my very earliest research had a fixed point theory component. I then ventured into functional analysis, and later became interested in the logical underpinnings of certain aspects of the theory. Later my research largely came full circle back to the study of fixed point theory in geodesic spaces.

Throughout this discussion '*Goebel-Kirk*', '*Khamsi-Kirk*', and '*Kirk-Shahzad*' will refer, respectively, to the following books:

- K Goebel, WA Kirk, *Topics in metric fixed point theory*. Cambridge Studies in Advanced Mathematics, 28. Cambridge University Press, Cambridge, 1990. viii+244 pp.
- MA Khamsi, WA Kirk, *An introduction to metric spaces and fixed point theory*. Pure and Applied Mathematics (New York). Wiley-Interscience, New York, 2001. x+302 pp.
- WA Kirk, N Shahzad, *Fixed point theory in distance spaces*. Springer, Cham, 2014. xii+173 pp.

The present discussion will involve only some aspects of metric fixed point theory as it has developed over the years. The questions of what has been accomplished is clear; what remains to be done is less clear. It is perhaps noteworthy that this year (2015) marks the 25th anniversary of the appearance of the *Goebel-Kirk* book and the 50th anniversary of the appearance of my 1965 fixed point theorem on nonexpansive mappings (discussed below).

I first turn to some of the history of fixed point theory conferences.

2 Conferences

The Istanbul conference was the 11th in a series of conferences entitled 'Fixed Point Theory and its Applications'. This series is devoted mainly to metric and functional analytic aspects of the theory. It is noteworthy that the inaugural conference in this series was 26 years ago. Since then, conferences in this series have been held every 2-4 years throughout the world.

The metric/functional analytic series:

France (Marseille, 1989) [Jean-Bernard Baillon, Michel Théra],
Canada (Halifax, 1991) [S Swaminathan, K-K Tan],
Spain (Seville, 1995) [Universidad de Sevilla, José María Ayerbe, *et al.*],
Poland (Kazimierz Dolny, 1997) [K Goebel, S Prus, T Sękowski, A Stachura],
Israel (Haifa, 2001) [Simeon Reich],
Spain (Valencia, 2003) [Jesús Garcia Falset, Enrique Llorens Fuster],
Mexico (Guanajuato, 2005) [Helga Fetter Nathansky, Berta Gamboa de Buen],
Thailand (Chiang Mai, 2007) [S Dhompongsa, Suthep Suantai, Bancha Panyanak],
Taiwan (Changhua, 2009) [Lai Jiu Lin],
Romania (Cluj-Napoca, 2012) [Adrian Petruşel, *et al.*],
Turkey (Istanbul, 2015) [Erdal Karapinar].

Fortunately I have been able to attend (and give talks) at each of the above conferences. As far as I know only two others, Kazimierz Goebel and Brailey Sims, can say this as well.

It will be apparent from the remarks below that in the early days more well established topologists promoted fixed point theory as an independent discipline and expanded the scope of their conferences accordingly. My first introduction to a conference devoted *exclusively* to fixed point theory, in fact probably the very first such conference, was 40 years ago. This was a seminar on 'Fixed Point Theory and its Applications' held in Halifax, Nova Scotia, June 9-12, 1975. This seminar was sponsored jointly by the Canadian Mathematical Congress and Dalhousie University. Geometrical aspects of fixed point theory were emphasized, although some topological and analytical aspects were also discussed. In 1975 there were so few people working in fixed point theory that at that time the theory had not separated into two branches - 'metric' and 'topological-order theoretic'. It is not clear to me just how or when this separation occurred, but I think over time so many people began working fixed point theory a single conference devoted to all aspects of the theory had become unmanageable.

Throughout the discussion I will mention the names of several mathematicians, some likely unknown to younger mathematicians today. However in many ways these people had an impact on development metric fixed point theory.

There were 38 participants in the 1975 Halifax conference, among them:

Felix Browder (Chicago),
Kim-Peu Chew (Malayasia),
MM Day (Illinois),
Michael Edelstein (Dalhousie),
Andrzej Granas (Montreal),
LA Karlovitz (Maryland),
WA Kirk (Iowa),
V Lakshmikantham (Arlington, TX),
Anthony To-Ming Lau (Alberta),

Heinz-Otto Peitgen (Bonn),
WV Petryshyn (Rutgers, NJ),
J Reinermann (Aachen, Germany),
R Schöneberg (Aachen, Germany),
S Swaminathan (Dalhousie) (the conference organizer),
Kok Keong Tan (Dalhousie),
Chi Song Wong (Windsor).

Among the main speakers at this conference were Felix Browder,[a] MM Day, A Granas, and V Lakshmikantham.

My talk at the Halifax conference was entitled: 'Caristi's Fixed Point Theorem and the Theory of Normal Solvability'. One of Felix Browder's talks at this conference was entitled: 'On a theorem of Caristi and Kirk'. The point of Browder's talk was to give a proof of Caristi's theorem which, in contrast to Caristi's original proof, 'avoids the use of transfinite induction completely'. The theory of normal solvability has been long forgotten, but certainly the study of Caristi's theorem and its extensions lives on. We discuss Caristi's theorem in more detail below. The proceedings of the Dalhousie conference were published in:

Fixed point theory and its applications. Proceedings of the seminar held at Dalhousie University, Halifax, N. S., June 9-12, 1975. Edited by Srinivasa Swaminathan. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1976. xiii+216 pp.

There were two other noteworthy conferences devoted exclusively to fixed point theory prior to the current 'metric' series. The first was a *Seminar on Fixed Point Theory* held at the Mathematics Research Institute at Oberwolfach, Germany, September, 1977. Again participants included both people working in functional analysis and topology. The organizers were topologists: Albrecht Dold and Edward Fadell. Many of the Halifax participants were there, as well as many additional participants from Europe. Here is a complete list of participants, taken from the official 'Teilnehmer' of the conference:

J Alexander (College Park),
JC Becker (West Lafayette),
H Bell (Cincinnati),
DG Bourgin (Houston),
RF Brown (Los Angeles),
FR Cohen (De Kalb),
G de Cecco (Lecce),
K Delinic (Heidelberg),
A Dold (Heidelberg),
J Dugundgi (Los Angeles),
W End (Heidelberg),
H Engl (Linz),
D Erle (Dortmund),
E Fadell (Madison),
CC Fenske (Giessen),
M Feshbach (Evanston),
G Fournier (Sherbrooke),
FB Fuller (Pasedena),
H Glover (Columbus),

K Goebel (Lublin),
DH Gottleib (West Lafayette),
B Halpern (Bloomington),[b]
KA Hardie (Rondebosch),
SY Husseini (Madison),
J Jawoworski (Bloomington),[c]
RP Jerrard (Coventry),
WA Kirk (Iowa City),
RJ Knill (New Orleans),
J Matkowski (Bielsko-Biata),
M Nakaoka (Osaka),
J Pak (Detroit),
H-O Peitgen (Bonn),
WV Petryshyn (New Brunswick),
C Prieto (Heidelberg),
M Prieto (Heidelberg),
D Puppe (Heidelberg),
J Reinermann (Aachen),
H Schirmer (Ottawa),
R Schöneberg (Aachen),
W Singhof (Köln),
V Stallbohm (Aachen),
H Steinlein (Müchen),
H Ulrich (Heidelberg),
E Vogt (Heidelberg),
F Wille (Kassel),
TJ Wilmore (Durham).

In his Oberwalfach Abstract, quoted below, Goebel called attention to a surprising anomaly in the stability of the fixed point property of nonexpansive mappings.

> 'Irregular convex sets with the fixed point property for nonexpansive mappings'
>
> Let X be a Banach space and let $C \subset X$ be a convex closed and bounded set. It is known that the fixed point property for nonexpansive mappings depends strongly on some 'nice' geometrical properties of the set C (such as, *e.g.*, uniform convexity, normal structure). The aim of this lecture is to show examples of singularities occurring in this field. The examples of sets having f.p.p. which do not satisfy all known and commonly used regularity conditions are given. One of the constructions shows a decreasing sequence C_n of closed convex subsets of ℓ^1 such that C_{2n+1} has and C_{2n} does not have the fixed point property. The intersection of all the C_n's may or may not have the fixed point property up to our choice.

These results were subsequently published in the paper:

K Goebel, T Kuczumow, Irregular convex sets with fixed-point property for nonexpansive mappings. Colloq. Math. **40** (1978/79), no. 2, 259-264.

My Oberwolfach Abstract highlighted a subtle connection between fixed point theory for nonexpansive mappings and the seemingly unrelated 'accretive' mappings.

> 'Fixed point theorems for nonexpansive mappings in Banach spaces'
>
> Let E denote a real Banach space and $D \subset E$. A mapping $T : D \to E$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, $x, y \in D$. It has been known since 1967 that a firm link exists between the fixed point theory for nonexpansive mappings and mapping theory for accretive mappings. (A mapping $f : D \to E$ is called *accretive* if $\forall u, v \in D$, $\langle f(u) - f(v), j\rangle \geq 0$ for some $j \in J(u - v)$ where $J : E \to 2^{E^*}$ denotes the normalized duality mapping defined by $j \in J(x) \Leftrightarrow \langle x, j\rangle = \|x\|^2 = \|j\|^2$, $x \in E$.) A technique due to Felix Browder combined with a result of RH Martin, Jr. directly implies the following. *Theorem*. *Suppose E has the property*: (i) *bounded closed convex subsets of E have the common fixed point property with respect to commuting families of nonexpansive mappings. Then every continuous accretive mapping* $f : E \to E$ *which satisfies* $\|f(x)\| \to \infty$ *as* $\|x\| \to \infty$ *is surjective*. R Schöneberg and the speaker have recently shown that condition (i) may be weakened to (ii): Closed balls in E have the fixed point property with respect to nonexpansive self-mappings. Recent results of Karlovitz and Goebel show that (ii) may hold while (i) fails.

A paper based on my abstract subsequently appeared in:

WA Kirk, R Schöneberg, Zeros of m-accretive operators in Banach spaces. Israel J. Math. **35** (1980), no. 1-2, 1-8.

Another conference that had an important impact on metric fixed point theory was one entitled '*Fixed Point Theory*' and held in Sherbrooke, Quebec, Canada, June 2-21, 1980. This conference was organized by the topologists Edward Fadell and Gilles Fournier, and the following were among the listed participants (this is only a partial list). As before, note the mixture of people specializing in functional analysis and in topology. However, the increase in the number of participants working in functional analysis is apparent. Unfortunately, several of the people listed here (and elsewhere) are now deceased:

JC Alexander,
Felix Browder,
Robert F Brown,
Edward Fadell,
Gilles Fournier,
Lech Gorniewicz,
Jean Pierre Gossez,
Benjamin Halpern,
George Isac,
Jan Jaworowski,
WA Kirk,
Enrique Lami Dozo,
Mario Martelli,
Silvio Massa,
Roger D Nussbaum,[d]
Heinz-Otto Peitgen,
Walter Petryshyn,
William O Ray,
BE Rhoades,
Helga Schirmer,

Robert Sine,
Sankatha P Singh,
Heinrich Steinlein.

The proceedings of the Sherbrooke conference were published in:

Fixed point theory. Proceedings of a Conference held at the Université de Sherbrooke, Sherbrooke, Que., June 2-21, 1980. Edited by Edward Fadell and Gilles Fournier. Lecture Notes in Mathematics, 886. Springer, Berlin, 1981. xii+511 pp.

There have been a number of conferences over the years which, although not devoted exclusively to fixed point theory, have had significant fixed point theory components. I mention five that I attended. The first of these was the 31st *Summer Research Institute of the American Mathematical Society* held at the University of California, Berkeley from July 11 to July 29, 1983. The scope of this symposium, chaired by Felix Browder, was very broad, but many talks were devoted to fixed point theory. The proceedings were published in the two volumes of the AMS *Proceedings of Symposia in Pure Mathematics* (vol. 45, parts 1 and 2), 1986. The second was the *NATO Advanced Study on Nonlinear Analysis and Fixed Point Theory*, in Maratea, Italy, April 1985, organized by Sankatha P Singh (1937-2013) of Memorial University of Newfoundland. This was a major conference and, as in the past, fixed point theory was a generic term encompassing both the topological and functional analytic aspects of the theory. The main speakers were: J Alexander, H Berestycki, Felix Browder, Edward Fadell, K Geba, J Ize, JM Lasry, JM Mawhin, CA Stewart, and J Toland. The proceedings of the Maratea conference appear in:

Nonlinear functional analysis and its applications. Proceedings of the NATO Advanced Study Institute held in Maratea, April 22-May 3, 1985. Edited by SP Singh. NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, 173. D Reidel Publishing Co., Dordrecht, 1986. xii+418 pp.

Later I attended conferences sponsored by the International Federation of Nonlinear Analysts, an organization founded by V Lakshmikantham.[e] All of these conferences had sessions devoted exclusively to fixed point theory. The IFNA conferences that I attended were: The Second World Congress, Athens, Greece, July 1996; The Third World Congress, Catania, Italy, July 2000; The Fourth World Congress, Orlando, Florida, July 2004.

3 Early work - geodesic geometry

I received my PhD degree at the University of Missouri in 1962 under the supervision of Leonard M Blumenthal.[f] As a result I turned to geometry as I began my research career. Soon after accepting my first appointment at UC Riverside I became fascinated by a class of geodesic spaces introduced by the geometer Herbert Busemann.[g]

A *G-space* R in the sense of Busemann (*The Geometry of Geodesics*, Academic Press, New York, 1955) is a metric space which is (i) finitely compact (or proper, *i.e.*, bounded closed sets are compact), (ii) metrically convex, and for which (iii) prolongation is locally possible and unique.

Precisely, (iii) means that to every point $p \in R$ there corresponds a number $\rho_p > 0$ such that if $x, y \in U(p; \rho_p)$ (the open ball) with $x \neq y$ there exists a point $z \in R$, $z \neq y$, for which

$$d(x,y) + d(y,z) = d(x,z),$$

and moreover, if $d(x,y) + d(y,z_1) = d(x,z_1)$ and $d(x,y) + d(y,z_2) = d(x,z_2)$, then $d(y,z_1) = d(y,z_2) \Rightarrow z_1 = z_2$.

A mapping φ of a G-space R onto itself is called a *local isometry* if for every $p \in R$ there exists a number $\eta_p > 0$ such that ψ maps the open ball $U(p;\eta_p)$ isometrically onto the ball $U(\varphi(p);\eta_p)$. The following fact is almost immediate.

A locally isometric mapping of a G-space onto itself is a motion (*surjective isometry*) *if and only if it is one-to-one* (*bijective*).

This led Busemann to pose the following problem (see p.405, (27) of Busemann's book).

Problem *Find conditions, in particular conditions applying to an ordinary cylinder, under which a non-compact G-space has the property that every locally isometric mapping of the space onto itself is a motion.*

I soon discovered a solution to Busemann's problem which led to my *very first* published paper:

WA Kirk, On locally isometric mappings of a G-space on itself, Proc. Amer. Math. Soc. **15** (1964), 584-586.

In this paper I first proved the following 'fixed point' result.

Theorem 1 *A locally isometric mapping of a G-space onto itself which has a fixed point is a motion.*

The group of motions of a G-space is said to be *transitive* if given any two points of the space there is a motion of the space that maps one into the other. Busemann had noted that among two-dimensional G-spaces, it is known that the cylinder (and torus) with a Minkowskian metric has a transitive group of motions.

Therefore the following simple application of Theorem 1 provides an affirmative answer to Busemann's problem.

Theorem 2 *If a G-space R has a transitive group of motions then every locally isometric mapping of R onto itself is a motion.*

Proof Let ϕ be a locally isometric mapping of R onto itself, let $p \in R$, and let ψ be a motion of R such that $\psi \circ \phi(p) = p$. Then by Theorem 1 $\psi \circ \phi$ is a motion and hence one-to-one. This trivially implies ϕ is one-to-one; hence a motion. □

The proceedings paper was submitted in March, 1963 and it appeared in print in August, 1964. (At that time such a lapse between submission and publication was typical.)

4 Nonexpansive mappings in Banach spaces

A closed convex subset K of a Banach space is said to have *normal structure* [1] if, given any convex subset H of K consisting of more than one point, there exists a nondiametral point in H, that is, there exist a point $p \in H$ and a positive number $r < \operatorname{diam}(H)$ such that

$$H \subseteq B(p;r).$$

A mapping $T : K \to K$ is said to be *nonexpansive* if for each $x, y \in K$, $\|T(x) - T(y)\| \leq \|x - y\|$.

Here is a statement of my 1965 fixed point result.

Theorem 3 *Let K be a bounded closed convex subset of a reflexive Banach space, and suppose that K has normal structure. Then every nonexpansive mapping $T : K \to K$ has at least one fixed point.*

There is a little history associated with this theorem. While at UC Riverside (where I proved Theorem 3) a colleague of mine, Hajimu Ogawa, with whom I had discussed my theorem, attended a lecture given by Felix Browder at UC Berkeley. When Ogawa returned he told me that Browder had announced a theorem that sounded very similar to the one I had told him about. I sent Browder a preprint of my paper and he immediately replied that we had indeed hit upon the same idea. He said he would refer to my paper in a footnote to his paper (which he did), and that his paper was scheduled to appear soon in the *Proceedings of the National Academy of Sciences*. Browder's theorem was the same as mine except that he assumed that the space was uniformly convex and therefore he was able to drop the normal structure assumption.

My paper had already been accepted by the *American Mathematical Monthly*, but it had languished for some time awaiting publication. I showed Browder's letter to my Department Chair, F Burton Jones.[h] Later I found out that Jones called the Managing Editor of the *Monthly* (FA Ficken) and that Ficken apparently had arranged to expedite publication of my paper. It turned out that my paper and Browder's appeared simultaneously in 1965 avoiding any question of priority. Subsequently, Browder and I discovered that Dietrich Göhde of Leipzig (East Germany at that time) had also proved that uniformly convex Banach spaces have the fixed point property for nonexpansive mappings. Remarkably, Göhde's paper also appeared in 1965. Regarding the three proofs, my proof and Browder's were quite similar, both relying on an application of Zorn's lemma, while Göhde's was somewhat more constructive.

The relevant papers are:

FE Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Nat. Acad. Sci. U.S.A. **54** (1965), 1040-1044.

D Göhde, Zum Prinzip der kontraktiven Abbildung. (German) Math. Nachr. **30** (1965), 251-258.

WA Kirk, A fixed point theorem for mappings which do not increase distances. Amer. Math. Monthly **72** (1965), 1004-1006.

I have been more adept at identifying open problems than solving them. In the early days interest in metric fixed point theory was tied exclusively to the theory of Banach spaces. More abstract theories had yet to emerge.

The first problem I identified involved the necessity of 'normal structure' in the fundamental existence theorem for nonexpansive mappings in weakly compact sets. It was shown early on that the answer is negative. A second problem that arose early (see [2]) was whether a commutative family of nonexpansive mappings under the assumption of Theorem 3 always has a common fixed point. This was answered in the affirmative a few years later, independently, by Teck-Cheong Lim[i] and RE Bruck[j] [3, 4].

5 The 1995 Seville workshop

In my talk at the Seville conference [5], the third in the present metric series, I mentioned a number of open problems in Banach space fixed point theory, many of which remain open to this day.

It is gratifying to see that some people are still working on these problems despite the somewhat understandable tendency of people to shy away from such problems out of concern that they might be too difficult. However, over the years there has been progress, some of which I summarize below. We say that a Banach space satisfies FPP (resp. weak FPP) if every bounded closed convex (resp. weakly compact convex) subset of X has the fixed point property for nonexpansive self-mappings. We only mention progress on problems discussed in [5] subsequent to the survey of *Goebel-Kirk* [6].

Question (IV) Can either ℓ_1 or c_0 be renormed so that the resulting space has the FPP?

Question (VI) Does either of the following implications for a Banach space X hold?

$$X \text{ is reflexive} \quad \Leftrightarrow \quad X \text{ has an equivalent norm relative to which } X \text{ has the FPP.}$$

In [7] T Domínguez Benavides proved that any reflexive space can be so renormed. (See also [8].)

Question (XIV) Suppose X is a Banach space which has the property that a closed convex subset K of X has the fixed point property for nonexpansive mappings $\Leftrightarrow$ K is bounded. Is X a Hilbert space?

This question has been answered. In [9] it is shown that every unbounded closed convex subset of c_0 fails to satisfy the FPP. Earlier it had been shown [10] that a bounded closed convex subset K of c_0 has the FPP $\Leftrightarrow$ K is weakly compact. The fact that c_0 has the weak FPP is of course a very deep and fundamental result of B Maurey [11].

However, as far as I know, the following question remains open to this day.

Question (XV) Does there exist an unbounded closed convex subset of a Banach space which has the fixed point property for nonexpansive mappings?

Other fundamental questions about fixed point theory for nonexpansive mappings were also discussed in [5]. Let X and Y be two Banach spaces and $p \in [1, \infty]$. Let $X \oplus_p Y$ denote the product space $X \oplus Y$ equipped with the norm:

$$\|(x,y)\| := \left(\|x\|^p + \|y\|^p\right)^{1/p} \quad \text{if } p \in [0, \infty);$$
$$\|(x,y)\| := \max\left\{\|x\|, \|y\|\right\} \quad \text{if } p = \infty.$$

Question (XVI) If both X and Y have the FPP, does $X \oplus_p Y$ have the FPP?

Question (XVIII) When is the fixed point set of a nonexpansive mapping in $X \oplus_p Y$ a nonexpansive retract of $X \oplus_p Y$?

The origin of these questions go back to 1968 [12]. Since then there have been a number of partial results, but the basic problems seem to remain open. For more recent discussions of these questions, including their fundamental differences and some positive results, we refer to [13] and [14].

6 Logical foundations

My interest in foundations, and specifically the Axiom of Choice, was stimulated by two events. One was the fact that B Fuchssteiner [15] had recently proved that my 1965 theorem on fixed points of nonexpansive mappings, which made explicit use of the Axiom of Choice via Zorn's lemma, did in fact have a proof wholly within the basic axioms of Zermelo-Fraenkel. The other was a brief conversation I had with the Polish mathematician Roman Mańka. This conversation involved the relationship between a well-known variational principle due to Ivar Ekeland and Caristi's theorem. In the discussion below $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^+ = (0,\infty)$. Recall that if X is a metric space, a mapping $\varphi : X \to \mathbb{R}^+$ is said to be (sequentially) *lower semicontinuous* (l.s.c.) if given any sequence $\{x_n\}$ in X, the conditions $x_n \to x$ and $\varphi(x_n) \to r$ imply $\varphi(x) \le r$.

Theorem 4 (E) (Ekeland, 1974 [16]) *Let (X,d) be a complete metric space and $\varphi : X \to \mathbb{R}^+$ l.s.c. Define a partial order $\le$ on X as follows*:

$$x \le y \quad \Leftrightarrow \quad d(x,y) \le \varphi(x) - \varphi(y), \quad x,y \in X. \tag{1}$$

Then $(X,\le)$ has a maximal element.

Theorem 5 (C) (Caristi, 1976 [17]) *Let X and φ be as above. Suppose $f : X \to X$ satisfies*

$$d\bigl(x,f(x)\bigr) \le \varphi(x) - \varphi\bigl(f(x)\bigr), \quad x \in X. \tag{2}$$

Then f has a fixed point.

It is easy to see that (E) ⇔ (C) in the usual sense. (See *Kirk-Shahzad*, Chapter 9, for the details.) However, these two results are not logically equivalent. In particular the implication (C) ⇒ (E) invokes the Axiom of Choice (AC). In fact, N Brunner [18] has shown that *any* proof of (E) requires at least the basic axioms of Zermelo-Fraenkel (ZF) plus a form of the Axiom of Choice called the Axiom of Dependent Choices (DC), whereas R Mańka has shown in [19] that (C) holds wholly within (ZF). So from a purely logical point of view the two theorems are not equivalent. (DC) is strictly weaker than (AC) but somewhat stronger than the Axiom of Countable Choice. The Axiom of Dependent Choices (DC), sometimes called the 'Axiom of inductive definition of sequences',[k] appears to be essential for the development of the foundations of functional analysis at least in the separable case (see, for example, the discussions in [20, 21]).

In [22] Hiam Brézis and Felix Browder derived Ekeland's theorem from an order principle which requires only ZFDC. They then obtained Caristi's theorem as in the implication (E) ⇒ (C) described in *Kirk-Shahzad*. Hence Choice is invoked at this step. However, in *Goebel-Kirk* it is shown that Caristi's theorem can be derived directly from the order principle of Brézis and Browder without recourse to Ekeland's theorem.

In the chart below we list the authors of some of the *very* early proofs of Caristi's theorem, the methods, and the axioms used. For explicit citations, see *Kirk-Shahzad*.

Author	Axioms
Caristi (1976)	ZFAC
CS Wong (1976)	ZFAC
Kirk (1976)	ZFAC
Brøndsted (1976)	ZFAC
Browder(1976)	ZFDC
Brézis-Browder (1976)	ZFDC
Penot (1976)	ZFDC
Siegel (1977)	ZFDC
Pasicki (1978)	ZFAC
Mańka (1988)	ZF
Goebel-Kirk (1990)	ZFDC

It is interesting that to this day Caristi's theorem continues to be 'extended and/or generalized'. Indeed, according to MathSciNet, Caristi's name appears in the *titles* of over 100 papers. For a very recent extension, see Du [23]. Also it would be a huge undertaking to see how many of the literally dozens of generalizations and/or extensions of Caristi's theorem can be obtained without at least assuming (DC). At the same time many 'extensions' of Caristi's theorem turn out to be rather immediate consequences of Caristi's theorem.[l] There is another fact that is of some interest. It has been known from the outset that the validity of Caristi's theorem *characterizes* metric completeness (see [24]). The same is not true of Banach's contraction mapping theorem. For an elaborate exposition on this topic, see [25].

Some interest in foundations continues. For example, in Ackerman [26] one finds the statement:

> In this paper we will work in a fixed background model of Zermelo-Fraenkel set theory. In general we will not use the axiom of choice unless necessary. If a result does use the axiom of choice we will mark it by (∗).

On the other hand, nonstandard approaches are extremely powerful and are frequently employed to this day. See for example, the recent paper of Avigad and Iovino [27].

The general Axiom of Choice: Suppose $\mathcal{F} = \{A_\alpha\}_{\alpha\in I}$ is a collection of sets with no restriction placed on the index set I. Thus I may be very large, especially uncountable. The *Axiom of Choice* states that there is a function $f : \mathcal{F} \to \bigcup_{\alpha\in I} A_\alpha$ such that $f(A_\alpha) \in A_\alpha$ for each $\alpha \in I$. Such a function f is called a *choice function*. Knowing that such a function exists, it is now possible to define the Cartesian product $\prod_{\alpha\in I} A_\alpha$ to be the collection of all such choice functions. (For a more detailed discussion, see Appendix A3 in *Khamsi-Kirk*; also Moore [28].)

The recent paper of Gregoriades [29] is also of interest in this regard. Versions of some of the ideas found in [29] can also be found in a paper I published with my student Tekamül Büber[m] over 20 years ago.

Several minimization principles are proved in Büber-Kirk [30] that do not require the full Axiom of Choice. In particular, Lemma 1 of [30] implies the following minimization principle, called the Brouwer Reduction Theorem. (This is stated as Corollary 2 in [30].)

Theorem 6 *Let X be a topological space which has a countable base, let Γ be a family of nonempty closed subsets of X, and suppose every descending sequence in Γ is bounded below by a member of Γ. Then Γ has a minimal element.*

The central purpose of [30] was to prove that a fundamental theorem of MA Khamsi [31] about commutative families of nonexpansive mappings requires (in the separable case) only (DC) rather than the full Axiom of Choice. However, another consequence of the above theorem might be of independent interest. To this aim, we need some more definitions. If (M,d) is a metric space, a family of closed subsets Σ of M which contains $\emptyset$ and M and is closed under intersections is called a *convexity structure*. A convexity structure is said to be *countably compact* if every countable family in Σ has nonempty intersection when each of its finite subfamilies has that property.

For $x \in M$ and $D \in \Sigma$, denote

$$\operatorname{dist}(x,D) = \inf\{d(x,y) : y \in D\}$$

and let

$$P(x,D) = \{z \in D : d(x,z) = \operatorname{dist}(x,D)\}.$$

The sets $\{P(x,D) : x \in M \text{ and } D \in \Sigma\}$ are called the *proximinal sets* in Σ.

The following is an application of Theorem 6.

Theorem 7 *Suppose (M,d) is a bounded metric space which possesses a countably compact convexity structure Σ which contains the closed balls of M, and suppose the proximinal sets in Σ relative to some fixed point $p \in M$ are separable. Let Γ be any family of nonempty subsets of Σ which is closed under nonempty intersections. Then Γ has a minimal element.*

7 Some bizarre thoughts - transfinite iterations

Despite the constructive approach of the preceding section, non-construcive approaches can also be quite interesting - yielding even bizarre results. For a survey of early results in this direction we refer to Kirk [32].

We begin with the notion of an ultranet (or universal net). The definition is simple. A net $\{x_\alpha\}$ in a set S is an *ultranet* if given any subset G of S, either $\{x_\alpha\}$ is eventually in G or eventually in $S\backslash G$. Two facts are of paramount importance regarding ultranets. (1) An ultranet in a compact topological space always converges, and (2) every net has a subnet that is an ultranet. We refer to *Kirk-Shahzad*, Section 9.5 for further discussion and citations.

The following transfinite iteration process is described in [33]; see also [34]. Let Γ be the set of all countable ordinals. Think of Γ as a collection of nets. It is possible to associate with each limit ordinal $\alpha = \{\gamma \in \Gamma : \gamma < \alpha\}$ a *fixed* subnet $\{\beta_{\mu(\alpha)} : \mu(\alpha) \in M_\alpha\}$ of α which is an ultranet.

Specifically, M_α is a directed set with

$$\varphi : M_\alpha \to \{\beta \in \Gamma : \beta < \alpha\}$$

isotone and cofinal. (Denote $\varphi_\alpha(\mu_\alpha) = \beta_{\mu(\alpha)}$.) Now let K be a weakly compact convex subset of a Banach space and $T : K \to K$. Fix $x \in K$ and make the inductive assumption $\{x_\beta = T^\beta(x) : \beta < \alpha\}$ has been defined. Now define $T^\alpha(x)$ as follows:

(a) If $\alpha = \beta + 1$ set $T^{\alpha}(x) = T^{\beta+1}(x) = T \circ T^{\beta}(x)$.
(b) If α is a limit ordinal define $T^{\alpha}(x) = \text{weak-lim}\, T^{\alpha_\xi}(x)$. (Observe that $\{T^{\alpha_\xi}(x)\}$ is an ultranet in X. Define $T : \alpha_\xi \to X$ by defining $T(\alpha_\xi) = T^{\alpha_\xi}(x)$. It follows that $\{T^{\alpha_\xi}(x)\}$ is also an ultranet.)

There is curious anomaly related to the transfinite iteration procedure described above. This result is described in [34]; also see [33].

Theorem 8 *Suppose K is a weakly compact subset of a Banach space and suppose $T : K \to K$ is contractive in the sense that $\|T(x) - T(y)\| < \|x - y\|$ if $x, y \in K$, $x \neq y$. For $\alpha \in \Gamma$ define the mappings T^{α} as in* (a) *and* (b)*. Then there exists $z \in K$ such that for each $x \in K$, $T^{\Gamma}(x) = z$.*

Since the norm in X is lower semicontinuous relative to the weak topology, if $x, y \in K$ and if $\alpha \in \Gamma$ is a limit ordinal,

$$\left\| T^{\alpha}(x) - T^{\alpha}(y) \right\| \leq \|x - y\|.$$

Thus the chain $\{\|T^{\alpha}(x) - T^{\alpha}(y)\|\}$ is nonincreasing and, if $T^{\alpha}(x) \neq T^{\alpha}(y)$, then $\|T^{\alpha}(x) - T^{\alpha}(y)\| < \|x - y\|$. Since Γ is uncountable, it follows that $T^{\alpha}(x) = T^{\alpha}(y)$ for all α sufficiently large, from which the conclusion follows.

The following remark appears to be of particular interest.

Remark 9 The above result does not assert that z is the fixed point of T. Indeed, it remain an open question to this day whether a contractive mapping of a weakly compact convex subset of a Banach space has a fixed point. *However, if T does have a fixed point, then it must be the point z whose existence is assured by Theorem* 8.

There is a comment on the Kirk-Massa paper in a 1996 paper by A Melentsov [35]. I have not read Melentsov's paper (it is in Russian) but I quote from the *Mathematical Reviews* summary. This seems to suggest that Melentsov may have identified conditions that imply the existence of fixed points for contractive mappings.

> Let K be a subset of a Banach space X. An operator $T{:}K \to K$ is said to be weakly contractive, if for every point $x \in K$ there is a number $C(x)$ and for any sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = x$ there exists a number N such that $\|T(x) - T(x_n)\| \leq C(x)\|x - x_n\|$ for all $n > N$. Every contractive operator is weakly contractive. The investigation of transfinite iteration processes leads to the study of the sequence space $\Sigma(X)$ with Tikhonov's topology, to the study of fundamental and uniformly fundamental nets in the space $\Sigma(X)$. In Theorem 2, by using regular operators summing divergent sequences, necessary and sufficient conditions for the existence of a fixed point of a weakly contractive operator are given. In particular, if T is a contractive operator, then by a theorem of WA Kirk and S Massa [34] $T^{\Omega}(x) = z$ for all $x \in K$, and by Theorem 2, $T(z) = z$.

8 Two problems

There are many open problems in metric fixed point theory, but two have frustrated me over the years because of their simplicity and yet seeming intractability. The first is suggested by Remark 9 in the preceding section.

Problem 10 *Does a weakly compact convex subset K of a Banach space have the fixed point property for strictly contractive mappings* (*that is*, *mappings $T : K \to K$ for which $\|T(x) - T(y)\| < \|x - y\|$, $x, y \in K$, $x \neq y$*)*?*

Another problem involves approximate fixed point sequences. It is well known that if K is a bounded convex subset of an arbitrary Banach space (or normed space) and if $T : K \to K$ is nonexpansive, then there is a sequence $\{x_n\}$ in K for which $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$. This can be seen by simply uniformly approximating T with a sequence of contraction mapping having Lipschitz constants approaching 1. Thus for each $\varepsilon > 0$,

$$F_\varepsilon(T) := \{x \in K : \|x - T(x)\| \leq \varepsilon\} \neq \emptyset.$$

Problem 11 *If K is a bounded* (*closed*) *convex subset of a Banach space and if $T, G : K \to K$ are commuting nonexpansive mappings*, *is it the case that for each $\varepsilon > 0$*,

$$F_\varepsilon(T) \cap F_\varepsilon(G) \neq \emptyset?$$

For a discussion of some of these and related problems, see [36, 37].

9 Further reflections

Throughout most of my career my research has been primarily focused on metric fixed point theory in a functional analytic setting. However, I have always had a lingering interest in geodesic spaces, and in 2003-2004 I published two articles [38, 39] on fixed point theory in the so-called CAT(0) spaces. These papers, along with my early collaboration with R Espínola, appear to have stimulated a large amount of on-going research in this setting. Some of this research merely takes advantage of the 'Hilbert space' geometry one finds in CAT(0) space thus leads to the adaptation of known Hilbert space arguments with little change. However, one of the primary obstacles in branching out to CAT(0) spaces is the absence of a well-understood 'weak' topology in such spaces. This has been at least partially remedied by the introduction of the notion of Δ-convergence in [40]. In fact, it appears that Δ-convergence may be sufficient to carry out the transfinite iteration process of Theorem 8 (see [41]). A summary of this aspect of my research is found in *Kirk-Shahzad*, Chapter 9.

In reflecting on my overall career, I must call attention to my long collaboration with Kazimierz Goebel. My first encounter with Kaz was during the academic year 1971-72, when he visited the University of Iowa under the aegis of a Kościuszko Foundation fellowship. During this visit we introduced the often cited concept of an asymptotically nonexpansive mapping to the literature. Later, on a subsequent visit by Goebel to Iowa in 1989, we completed the manuscript for our signature book: *Topics in Metric Fixed Point Theory*.

Over the years I have made several visits to UMCS in Lublin, Poland, and one of the singular highlights of my career occurred during such a visit in June, 2001. Kaz Goebel arranged for me and our Australian friend, Brailey Sims, to travel with him and his colleague, Yuri Kozitsky, to the historic city Lwów (now Lviv, in Ukraine) where Stefan Banach spent most of his life. While in Lwów we had an opportunity to visit Banach's grave, and the building that housed the famous 'Scottish Cafe'. Brailey subsequently described this journey in an eloquent article he published in the *Australian Mathematical Gazette* (see [42]).

Competing interests
The author declares that he has no competing interest.

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Endnotes

[a] Felix Browder is regarded as the founder of a branch of analysis called 'Nonlinear Functional Analysis'. In his book *Nonlinear operators and nonlinear equations of evolution in Banach spaces* (Nonlinear functional analysis (Proc. Sympos. Pure Math., Vol. XVIII, Part 2, Chicago, Ill., 1968), pp. 1-308. Amer. Math. Soc., Providence, R. I., 1976) he singles out the monotone, nonexpansive, and accretive mappings for special study.

Browder was born in 1928 in Moscow. He received his PhD degree from Princeton University at the age of 20 under the supervision of Solomon Lefschetz. He spent many years at the University of Chicago, serving as Department Chair from 1971 to 1976 and again from 1979 to 1985 before moving to Rutgers University in 1986 as vice president for research. Two of his Chicago PhD students, Ronald E Bruck and Roger D Nussbbaum, are mentioned elsewhere in this article.

For an interesting brief biography of Felix Browder's personal life, go to: http://www-history.mcs.st-and.ac.uk/Biographies/Browder_Felix.html.

[b] Benjamin Halpern is well known in metric fixed point theory for introducing the Halpern iteration scheme. This scheme was alluded to numerous times in talks at the 11th ICFPTA. Also he and George Bergman introduced the notion of 'inward' mappings to the theory. It was this very concept that led to the discovery of Caristi's theorem.

[c] Jan Jawoworski (1928-2013) received his PhD in 1955 from the Polish Academy of Sciences under the supervision of Karol Borsuk. (Borsuk's other students include Samuel Eilenberg and Andrzej Granas.) One of Jawoworski's students, Sehie Park, is also well known in fixed point theory.

[d] I call attention to a recent book: Bas Lemmens, Roger Nussbaum, *Nonlinear Perron-Frobenius theory*. Cambridge Tracts in Mathematics, 189. Cambridge University Press, Cambridge, 2012. xii+323 pp. This book contains many applications of fixed point theory for nonexpansive mappings, especially in finite dimensional vector spaces.

[e] V Lakshmikantham (1924-2012) has made many contributions to fixed point theory as well as to many other branches of nonlinear analysis. Among other things, he was Founding Editor of the journal *Nonlinear Analysis* - TMA in 1976, and he established the International Federation of Nonlinear Analysts (IFNA) in 1991. For a brief biography of his early life and career, see R Agarwal, S Leela, A brief biography and survey of collected works of V Lakshmikantham, Nonlinear Anal. **40** (2000), 1-19.

[f] Leonard M Blumenthal (1901-1984) received his PhD in 1927 from the Johns Hopkins University under the supervision of the geometer Frank Morley (1860-1937). Morley received his BA from Cambridge in 1884 and was AMS President, 1919-1920. (Some trivia: One of Frank Morley's sons was the celebrated American novelist and essayist, Christopher Morley (1890-1957). Another son, Frank Vigor Morley was also a mathematician and a student of GH Hardy. For more information, see http://www-history.mcs.st-andrews.ac.uk/Biographies/Morley.html.)

Blumenthal's interest in abstract metrics was aroused by lectures that Karl Menger gave at the Rice University in 1931. Subsequently, this interest was further stimulated and developed during a year he spent with Professor Menger at the University of Vienna. Blumenthal came to the University of Missouri in 1936.

[g] Herbert Busemann (1905-1994) was born in Berlin, and he studied at universities Munich, Paris, and Rome. He defended his dissertation in University of Göttingen in 1931, where his advisor was Richard Courant. He remained in Göttingen as an assistant until 1933, when he escaped Nazi Germany to Copenhagen. He worked at the University of Copenhagen until 1936, when he left for the United States. He held temporary positions at the Institute of Advanced Studies, the Johns Hopkins University, Illinois Institute of Technology, Smith College, and eventually he became a professor in 1947 at University of Southern California. He advanced to a distinguished professorship in 1964, and continued working at USC until his retirement in 1970.

[h] Floyd Burton Jones (1910-1999) was a 1935 PhD student of the topologist RL Moore at the University of Texas. Moore had introduced what is now called the 'Moore method' of teaching which entailed, roughly speaking, having the students discover for themselves the mainstream of the subject based only on primitive axioms. For a detailed discussion, see FB Jones, The Moore method, Amer. Math. Monthly **84** (1977), no. 4, 273-278. Moore had 50 PhD students, many of whom have had a lasting impact on mathematical research. Among them (listed chronologically): Raymond Wilder, Gordon Whyburn, RH Bing, Gail Yourg, Jr., Edwin Moise, Richard Anderson, Mary Ellen Rudin, and Eldon Dyer. On the negative side, it is rumored that Moore refused to teach black students.

[i] I have recently learned that Teck-Cheong (TC) Lim passed away the evening of Monday, October 20, 2014, at the age of 64, near his Burke, VA home. He was an LE Dickson Instructor at the University of Chicago, before joining the faculty of George Mason University. For more information, go to: http://www.fairfaxmemorialfuneralhome.com/obituary/Teck-Cheong-Lim/Burke-VA/1444328.

[j] Ronald Bruck received his PhD in 1969 at the University of Chicago, under the supervision of Felix Browder. He spent most of his career at the University of Southern California, and was Department Chair there from 1985-90.

[k] The Axiom of Dependent Choices (DC) may be formulated precisely as follows. (See, *e.g.*, p.19 of H Rubin and JE Rubin, *Equivalents of the Axiom of Choice* II, North-Holland, Amsterdam-New York-Oxford, 1985.) (DC) If $R \neq \emptyset$ is a relation on a set such that Range(R) $\subseteq$ Domain(R) then there is a function f with domain ω such that for all $n \in \omega$, $(f(n), f(n+1)) \in R$.

[l] There is perhaps a lesson here. The name Caristi is surely one of the most well known in all of metric fixed point theory. Yet Caristi published only four related papers in mathematics. This serves as evidence that numbers of papers alone do not necessarily lead to fame.

James Caristi received his PhD from the University of Iowa in 1975 and taught for many years in the Mathematics Department at Valparaiso University in Valparaiso, Indiana. He is currently Professor and Chair of the newly formed Department of Computer and Information Sciences at Valparaiso University.

[m] M Tekamül Büber was originally from Turkey. He arrived at the University of Iowa with an MSc in Electrical Engineering, BSc in Mathematics, and in BSc Physics from Rheinisch Westphälische Technicsche Hochschule Aachen, in Aachen, Germany. Upon arriving at Iowa he attended lectures in logic by the noted philosopher Gustav Bergmann (who also had a PhD in Mathematics). Büber received his PhD in Mathematics from University of Iowa in 1994. After teaching in a small college for two years, Büber went on to earn a Masters Degree in Electrical Engineering at Syracuse University. He subsequently worked for several companies in the US and Canada and, as of this writing, he is a Principal Engineer at Maury Microwave in Ontario, CA.

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