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C^*-Algebra-valued b-metric spaces and related fixed point theorems

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Abstract

Based on the concept and properties of C^*-algebras, the paper introduces a concept of C^*-algebra-valued b-metric spaces which generalizes the concept of C^*-algebra-valued metric spaces and gives some basic fixed point theorems for self-map with contractive condition on such spaces. As applications, existence and uniqueness results for a type of operator equation and an integral equation are given.

MSC: 47H10; 46L07

Keywords: C^*-algebra; C^*-algebra-valued b-metric; contractive mapping; fixed point theorem

1 Preliminary

As is well known, the Banach contraction mapping principle is a very useful, simple and classical tool in modern analysis, and it has many applications in applied mathematics. In particular, it is an important tool for solving existence problems in many branches of mathematics and physics.

In 1993 another axiom for semimetric spaces, which is weaker than the triangle inequality, was put forth by Czerwik [1] with a view of generalizing the Banach contraction mapping theorem. Since then, several papers have dealt with fixed point theory in such spaces [2–5]. This same relaxation of the triangle inequality is also discussed in Fagin and Stockmeyer [6], who call this new distance measure nonlinear elastic matching (NEM). The authors of that paper remark that this measure has been used, for example, in [7] for trademark shapes and in [8] to measure ice floes. Later, Xia [9] used this semimetric distance to study the optimal transport path between probability measures. Xia has chosen to call these spaces b-metric space (or quasimetric space). For details of b-metric space, one can see [10].

In [11], the authors introduced the concept of C^*-algebra-valued metric spaces. The main idea consists in using the set of all positive elements of a unital C^*-algebra instead of the set of real numbers. Obviously such spaces generalize the concept of metric spaces. In this paper, as generalization of b-metric spaces and operator-valued metric spaces [12], we introduce a new type of metric spaces, namely, C^*-algebra-valued b-metric spaces, and give some fixed point theorems for self-map with contractive condition on such spaces.

To begin with, we collect some definitions and basic facts on the theory of C^*-algebras, which will be needed in the sequel. Suppose that $\mathbb{A}$ is an unital algebra with the unit I. An involution on $\mathbb{A}$ is a conjugate-linear map $a \mapsto a^*$ on $\mathbb{A}$ such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathbb{A}$. The pair $(\mathbb{A}, *)$ is called a $*$-algebra [13, 14]. A Banach $*$-algebra is a $*$-algebra $\mathbb{A}$ together with a complete submultiplicative norm such that $\|a^*\| = \|a\|$ $(\forall a \in \mathbb{A})$. A C^*-algebra is a Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$ $(\forall a \in \mathbb{A})$.

Notice that the seeming mild requirement on a C^*-algebra above is in fact very strong. It is clear that under the norm topology, $L(H)$, the set of all bounded linear operators on a Hilbert space H, is a C^*-algebra. Furthermore, given a C^*-algebra $\mathbb{A}$, there exists a Hilbert space H and a faithfully $*$-representation (π, H) of $\mathbb{A}$ such that $\pi(\mathbb{A})$ can be made a closed C^*-subalgebra of $L(H)$ [13].

2 Main results

Throughout this paper, by $\mathbb{A}$ we always denote an unital C^*-algebra with a unit I. Set $\mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$. We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$, if $x \in \mathbb{A}_h$ and $\sigma(x) \subset [0, \infty)$, where $\sigma(x)$ is the spectrum of x. Using positive elements, one can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows: $x \preceq y$ if and only if $y - x \succeq \theta$. From now on, by $\mathbb{A}_+$ we denote the set $\{x \in \mathbb{A} : x \succeq \theta\}$ and $|x| = (x^*x)^{\frac{1}{2}}$.

Lemma 2.1 [13, 15] *Suppose that $\mathbb{A}$ is a unital C^*-algebra with a unit I.*

(1) *For any $x \in \mathbb{A}_+$ we have $x \preceq I \Leftrightarrow \|x\| \leq 1$.*
(2) *If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $I - a$ is invertible and $\|a(I - a)^{-1}\| < 1$.*
(3) *Suppose that $a, b \in \mathbb{A}$ with $a, b \succeq \theta$ and $ab = ba$, then $ab \succeq \theta$.*
(4) *By $\mathbb{A}'$ we denote the set $\{a \in \mathbb{A} : ab = ba, \forall b \in \mathbb{A}\}$. Let $a \in \mathbb{A}'$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq \theta$, and $I - a \in \mathbb{A}'_+$ is an invertible operator, then*

$$(I - a)^{-1}b \succeq (I - a)^{-1}c.$$

Notice that in a C^*-algebra, if $\theta \preceq a, b$, one cannot conclude that $\theta \preceq ab$. For example, consider the C^*-algebra $\mathbb{M}_2(\mathbb{C})$ and set $a = \left(\begin{smallmatrix} 3 & 2 \\ 2 & 3 \end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix} 1 & -2 \\ -2 & 4 \end{smallmatrix}\right)$, then $ab = \left(\begin{smallmatrix} -1 & 2 \\ -4 & 8 \end{smallmatrix}\right)$. Clearly $a, b \in \mathbb{M}_2(\mathbb{C})_+$, while ab is not.

With the help of the positive elements in $\mathbb{A}$, one can give the definition of a C^*-algebra-valued b-metric space.

Definition 2.1 Let X be a nonempty set, and $A \in \mathbb{A}'$ such that $A \succeq I$. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta \Leftrightarrow x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \preceq A[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then d is called a C^*-algebra-valued b-metric on X and $(X, \mathbb{A}, d)$ is a C^*-algebra-valued b-metric space.

Definition 2.2 Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space. Suppose that $\{x_n\} \subset X$ and $x \in X$. If for any $\varepsilon > 0$ there is N such that for all $n > N$, $\|d(x_n, x)\| \leq \varepsilon$, then $\{x_n\}$ is said to be converge with respect to $\mathbb{A}$, and $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote it by $\lim_{n\to\infty} x_n = x$.

If for any $\varepsilon > 0$ there is N such that for all $n, m > N$, $\|d(x_n, x_m)\| \leq \varepsilon$, then $\{x_n\}$ is called a Cauchy sequence with respect to $\mathbb{A}$.

We say $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued b-metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

Remark 2.1 If $A = I$, then the ordinary triangle inequality condition in a C^*-algebra-valued metric space is satisfied. Thus a C^*-algebra-valued b-metric space is an ordinary C^*-algebra-valued metric space. In particular, if $\mathbb{A} = \mathbf{C}$ and $A = 1$, the C^*-algebra-valued b-metric spaces are just the ordinary metric spaces. The following example illustrates that, in general, a C^*-algebra-valued metric space is not necessary a C^*-algebra-valued b-metric space. For details of C^*-algebra-valued metric spaces, one can see [11].

Example 2.1 Let $X = \mathbb{R}$ and $\mathbb{A} = M_n(\mathbb{R})$. Define $d(x,y) = \operatorname{diag}(c_1|x-y|^p, c_2|x-y|^p, \ldots, c_n|x-y|^p)$ with 'diag' denotes a diagonal matrix, and $x,y \in \mathbb{R}$, $c_i > 0$ $(i = 1,2,\ldots,n)$ are constants and $p > 1$. It is easy to verify that $d(\cdot,\cdot)$ is a complete C^*-algebra-valued b-metric, for proving (3) of Definition 2.1 we only need to use the following inequality:

$$|x-y|^p \leq 2^p\bigl(|x-z|^p + |z-y|^p\bigr),$$

which implies that $d(x,y) \preceq A[d(x,z) + d(z,y)]$ for all $x,y,z \in X$, where $A = 2^p I \in \mathbb{A}'$ and $A \succ I$ by $2^p > 1$. But $|x-y|^p \leq |x-z|^p + |z-y|^p$ is impossible for all $x > z > y$ [16]. Thus $(X, M_n(\mathbb{R}), d)$ is not a C^*-algebra-valued metric space.

Definition 2.3 Suppose that $(X,\mathbb{A},d)$ is a C^*-algebra-valued b-metric space. We call a mapping $T : X \to X$ a C^*-algebra-valued contractive mapping on X, if there exists a $B \in \mathbb{A}$ with $\|B\| < 1$ such that

$$d(Tx, Ty) \preceq B^* d(x,y)B, \quad \forall x,y \in X.$$

Theorem 2.1 *If $(X,\mathbb{A},d)$ is a complete C^*-algebra-valued b-metric space and $T : X \to X$ is a contractive mapping, there exists a unique fixed point in X.*

Proof It is clear that if $B = \theta$, T maps the X into a single point. Thus without loss of generality, one can suppose that $B \neq \theta$.

Choose an $x_0 \in X$ and set $x_{n+1} = Tx_n = \cdots = T^{n+1}x_0$, $n = 1,2,\ldots$. For convenience, by B_0 we denote the element $d(x_1,x_0)$ in $\mathbb{A}$.

Notice that in a C^*-algebra, if $a,b \in \mathbb{A}_+$ and $a \preceq b$, then for any $x \in \mathbb{A}$ both x^*ax and x^*bx are positive elements and $x^*ax \preceq x^*bx$ [13]. Thus

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n, Tx_{n-1}) \preceq B^* d(x_n,x_{n-1})B \\ &\preceq \bigl(B^*\bigr)^2 d(x_{n-1},x_{n-2})B^2 \\ &\preceq \cdots \\ &\preceq \bigl(B^*\bigr)^n d(x_1,x_0)B^n \\ &= \bigl(B^n\bigr)^* B_0 B^n. \end{aligned}$$

For any $m \geq 1$ and $p \geq 1$, it follows that

$$\begin{aligned} d(x_{m+p},x_m) &\preceq A\bigl[d(x_{m+p},x_{m+p-1}) + d(x_{m+p-1},x_m)\bigr] \\ &= Ad(x_{m+p},x_{m+p-1}) + Ad(x_{m+p-1},x_m) \end{aligned}$$

$$\begin{aligned}
&\preceq A d(x_{m+p},x_{m+p-1}) + A^2\big[d(x_{m+p-1},x_{m+p-2}) + d(x_{m+p-2},x_m)\big] \\
&= A d(x_{m+p},x_{m+p-1}) + A^2 d(x_{m+p-1},x_{m+p-2}) \\
&\quad + A^2 d(x_{m+p-2},x_m) \\
&\preceq A d(x_{m+p},x_{m+p-1}) + A^2 d(x_{m+p-1},x_{m+p-2}) + \cdots \\
&\quad + A^{p-1} d(x_{m+2},x_{m+1}) + A^{p-1} d(x_{m+1},x_m) \\
&\preceq A\big(B^*\big)^{m+p-1} B_0 B^{m+p-1} + A^2\big(B^*\big)^{m+p-2} B_0 B^{m+p-2} \\
&\quad + A^3\big(B^*\big)^{m+p-3} B_0 B^{m+p-3} + \cdots + A^{p-1}\big(B^*\big)^{m+1} B_0 B^{m+1} \\
&\quad + A^{p-1}\big(B^*\big)^{m} B_0 B^{m} \\
&= \sum_{k=1}^{p-1} A^k\big(B^*\big)^{m+p-k} B_0 B^{m+p-k} + A^{p-1}\big(B^*\big)^m B_0 B^m \\
&= \sum_{k=1}^{p-1} \big(\big(B^*\big)^{m+p-k} A^{\frac{k}{2}} B_0^{\frac{1}{2}}\big)\big(B_0^{\frac{1}{2}} A^{\frac{k}{2}} B^{m+p-k}\big) \\
&\quad + \big(\big(B^*\big)^m A^{\frac{p-1}{2}} B_0^{\frac{1}{2}}\big)\big(B_0^{\frac{1}{2}} A^{\frac{p-1}{2}} B^m\big) \\
&= \sum_{k=1}^{p-1} \big(B_0^{\frac{1}{2}} A^{\frac{k}{2}} B^{m+p-k}\big)^* \big(B_0^{\frac{1}{2}} A^{\frac{k}{2}} B^{m+p-k}\big) \\
&\quad + \big(B_0^{\frac{1}{2}} A^{\frac{p-1}{2}} B^m\big)^* \big(B_0^{\frac{1}{2}} A^{\frac{p-1}{2}} B^m\big) \\
&= \sum_{k=1}^{p-1} \big|B_0^{\frac{1}{2}} A^{\frac{k}{2}} B^{m+p-k}\big|^2 + \big|B_0^{\frac{1}{2}} A^{\frac{p-1}{2}} B^m\big|^2 \\
&\preceq \sum_{k=1}^{p-1} \big\|B_0^{\frac{1}{2}} A^{\frac{k}{2}} B^{m+p-k}\big\|^2 I + \big\|B_0^{\frac{1}{2}} A^{\frac{p-1}{2}} B^m\big\|^2 I \\
&\preceq \big\|B_0^{\frac{1}{2}}\big\|^2 \sum_{k=1}^{p-1} \|B\|^{2(m+p-k)} \|A\|^k I + \big\|B_0^{\frac{1}{2}}\big\|^2 \big\|A^{\frac{p-1}{2}}\big\|^2 \big\|B^m\big\|^2 I \\
&= \|B_0\| \|B\|^{2(m+p)} \frac{\|A\|((\|A\|\|B\|^{-2})^{p-1} - 1)}{\|A\| - \|B\|^2} I \\
&\quad + \|B_0\| \big\|A^{\frac{p-1}{2}}\big\|^2 \big\|B^m\big\|^2 I \\
&\preceq \|B_0\| \frac{\|A\|^p \|B\|^{2(m+1)}}{\|A\| - \|B\|^2} I + \|B_0\| \big\|A^{\frac{p-1}{2}}\big\|^2 \|B\|^{2m} I \\
&\to \theta \quad (m \to \infty).
\end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X,\mathbb{A},d)$, there exists an $x \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = x$.

Since

$$\begin{aligned}
\theta &\preceq d(Tx,x) \preceq A\big[d(Tx,Tx_n) + d(Tx_n,x)\big] \\
&\preceq AB^* d(x,x_n)B + d(x_{n+1},x) \to \theta \quad (n \to \infty),
\end{aligned}$$

hence, $Tx = x$, *i.e.*, x is a fixed point of T.

Now suppose that y ($\neq x$) is another fixed point of T. Since

$$\theta \preceq d(x,y) = d(Tx,Ty) \preceq B^* d(x,y)B,$$

we have

$$\begin{aligned} 0 &\leq \|d(x,y)\| = \|d(Tx,Ty)\| \\ &\leq \|B^* d(x,y)B\| \\ &\leq \|B^*\| \|d(x,y)\| \|B\| \\ &= \|B\|^2 \|d(x,y)\| \\ &< \|d(x,y)\|. \end{aligned}$$

It is impossible. So $d(x,y) = \theta$ and $x = y$, which implies that the fixed point is unique. □

Theorem 2.2 (Chatterjea-type [17]) *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b-metric space. Suppose that the mapping $T : X \to X$ satisfies for all $x, y \in X$*

$$d(Tx,Ty) \preceq B\big(d(Tx,y) + d(Ty,x)\big) \quad (\forall x,y \in X),$$

where $B \in \mathbb{A}'_+$ and $\|BA\| < \frac{1}{2}$. Then there exists a unique fixed point in X.

Proof Without loss of generality, one can suppose that $B \neq \theta$. Notice that $B \in \mathbb{A}'_+$, $B(d(Tx,y) + d(Ty,x))$ is also a positive element.

Choose $x_0 \in X$, set $x_{n+1} = Tx_n = T^{n+1}x_0$, $n = 1,2,\dots$, by B_0 we denote the element $d(x_1,x_0)$ in $\mathbb{A}$. Then

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1}) \\ &\preceq B\big(d(Tx_n,x_{n-1}) + d(Tx_{n-1},x_n)\big) \\ &= B\big(d(Tx_n,Tx_{n-2}) + d(Tx_{n-1},Tx_{n-1})\big) \\ &\preceq BA\big(d(Tx_n,Tx_{n-1}) + d(Tx_{n-1},Tx_{n-2})\big) \\ &= BAd(Tx_n,Tx_{n-1}) + BAd(Tx_{n-1},Tx_{n-2}) \\ &= BAd(x_{n+1},x_n) + BAd(x_n,x_{n-1}). \end{aligned}$$

Using Lemma 2.1,

$$(I - BA)d(x_{n+1},x_n) \preceq BAd(x_n,x_{n-1}).$$

Since $A, B \in \mathbb{A}'_+$ with $\|BA\| < \frac{1}{2}$ and $A \succeq I$, we have $I - BA \preceq I - B$ and furthermore $(I - BA)^{-1} \in \mathbb{A}'_+$ with $\|(I - BA)^{-1}BA\| < 1$ by Lemma 2.1. Therefore,

$$d(x_{n+1},x_n) \preceq (I - BA)^{-1}BAd(x_n,x_{n-1}) = td(x_n,x_{n-1}),$$

where $t = (I - BA)^{-1}BA$.

For any $m \geq 1$ and $p \geq 1$, it follows that

$$\begin{aligned}
d(x_{m+p}, x_m) &\preceq A\bigl[d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_m)\bigr] \\
&= Ad(x_{m+p}, x_{m+p-1}) + Ad(x_{m+p-1}, x_m) \\
&\preceq Ad(x_{m+p}, x_{m+p-1}) + A^2\bigl[d(x_{m+p-1}, x_{m+p-2}) + d(x_{m+p-2}, x_m)\bigr] \\
&= Ad(x_{m+p}, x_{m+p-1}) + A^2 d(x_{m+p-1}, x_{m+p-2}) \\
&\quad + A^2 d(x_{m+p-2}, x_m) \\
&\preceq Ad(x_{m+p}, x_{m+p-1}) + A^2 d(x_{m+p-1}, x_{m+p-2}) \\
&\quad + A^3 d(x_{m+p-2}, x_{m+p-3}) + \cdots \\
&\quad + A^{p-1} d(x_{m+2}, x_{m+1}) + A^{p-1} d(x_{m+1}, x_m) \\
&\preceq At^{m+p-1}B_0 + A^2 t^{m+p-2}B_0 + A^3 t^{m+p-3}B_0 + \cdots \\
&\quad + A^{p-1}t^{m+1}B_0 + A^{p-1}t^m B_0 \\
&= \sum_{k=1}^{p-1} A^k t^{m+p-k} B_0 + A^{p-1}t^m B_0 \\
&= \sum_{k=1}^{p-1} \bigl|B_0^{\frac{1}{2}} t^{\frac{m+p-k}{2}} A^{\frac{k}{2}}\bigr|^2 + \bigl|B_0^{\frac{1}{2}} t^{\frac{m}{2}} A^{\frac{p-1}{2}}\bigr|^2 \\
&\preceq \|B_0\| \sum_{k=1}^{p-1} \|A\|^k \|t\|^{m+p-k} I + \|A\|^{p-1}\|t\|^m \|B_0\| I \\
&\preceq \frac{\|B_0\|\|A\|^p\|t\|^{m+1}}{\|A\| - \|t\|} I + \|A\|^{p-1}\|t\|^m \|B_0\| I \\
&\to \theta \quad (m \to \infty).
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$, *i.e.*, $\lim_{n\to\infty} Tx_{n-1} = x$. Since

$$\begin{aligned}
d(Tx, x) &\preceq A\bigl[d(Tx, Tx_n) + d(Tx_n, x)\bigr] \\
&\preceq A\bigl[B\bigl(d(Tx, x_n) + d(Tx_n, x)\bigr) + d(x_{n+1}, x)\bigr] \\
&\preceq ABA\bigl(d(Tx, x) + d(x, x_n)\bigr) + ABd(x_{n+1}, x) + Ad(x_{n+1}, x).
\end{aligned}$$

This is equivalent to

$$\bigl(I - A^2B\bigr)d(Tx, x) \preceq A^2Bd(x, x_n) + (AB + A)d(x_{n+1}, x).$$

Then

$$\begin{aligned}
\bigl\|d(Tx, x)\bigr\| &\leq \bigl\|\bigl(I - A^2B\bigr)^{-1}A^2B\bigr\| \bigl\|d(x, x_n)\bigr\| \\
&\quad + \bigl\|\bigl(I - A^2B\bigr)^{-1}(AB + A)\bigr\| \bigl\|d(x_{n+1}, x)\bigr\| \\
&\to 0 \quad (n \to \infty).
\end{aligned}$$

This implies that $Tx = x$, *i.e.*, x is a fixed point of T. Now if y ($\neq x$) is another fixed point of T, then

$$\begin{aligned}\theta \preceq d(x,y) &= d(Tx,Ty)\\ &\preceq B\bigl(d(Tx,y)+d(Ty,x)\bigr)\\ &= B\bigl(d(x,y)+d(y,x)\bigr),\end{aligned}$$

i.e.,

$$d(x,y) \preceq (I-B)^{-1}Bd(x,y).$$

Since $\|B(I-B)^{-1}\| < 1$,

$$\begin{aligned}0 \leq \bigl\|d(x,y)\bigr\| &= \bigl\|d(Tx,Ty)\bigr\|\\ &\leq \bigl\|(I-B)^{-1}Bd(x,y)\bigr\|\\ &\leq \bigl\|(I-B)^{-1}B\bigr\|\bigl\|d(x,y)\bigr\|\\ &< \bigl\|d(x,y)\bigr\|.\end{aligned}$$

This means that

$$d(x,y) = \theta \quad \Leftrightarrow \quad x = y.$$

Therefore the fixed point is unique and the proof is complete. □

Theorem 2.3 (Kannan-type [18]) *Let $(X,\mathbb{A},d)$ be a complete C^* algebra-valued b-metric space. Suppose that the mapping $T: X \to X$ satisfies*

$$d(Tx,Ty) \preceq B\bigl(d(Tx,x)+d(Ty,y)\bigr) \quad (\forall x,y \in X),$$

where $B \in \mathbb{A}'_+$ and $\|B\| < \frac{1}{2}$. Then there exists a unique fixed point in X.

Proof Without loss of generality, one can suppose that $B \neq \theta$. Notice that $B \in \mathbb{A}'_+$, $B(d(Tx,x)+d(Ty,y))$ is also a positive element.

Choose $x_0 \in X$, set $x_{n+1} = Tx_n = T^{n+1}x_0$, $n = 1,2,\ldots$, by B_0 we denote the element $d(x_1,x_0)$ in $\mathbb{A}$. Then

$$\begin{aligned}d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1})\\ &\preceq B\bigl(d(Tx_n,x_n)+d(Tx_{n-1},x_{n-1})\bigr)\\ &= B\bigl(d(x_{n+1},x_n)+d(x_n,x_{n-1})\bigr).\end{aligned}$$

Thus

$$d(x_{n+1},x_n) \preceq (I-B)^{-1}Bd(x_n,x_{n-1}) = td(x_n,x_{n-1}),$$

where $t = (I - B)^{-1}B$. For any $m \geq 1$, $p \geq 1$,

$$\begin{aligned}
d(x_{m+p}, x_m) &\preceq A\big[d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_m)\big] \\
&= Ad(x_{m+p}, x_{m+p-1}) + Ad(x_{m+p-1}, x_m) \\
&\preceq Ad(x_{m+p}, x_{m+p-1}) + A^2\big[d(x_{m+p-1}, x_{m+p-2}) + d(x_{m+p-2}, x_m)\big] \\
&= Ad(x_{m+p}, x_{m+p-1}) + A^2 d(x_{m+p-1}, x_{m+p-2}) \\
&\quad + A^2 d(x_{m+p-2}, x_m) \\
&\preceq Ad(x_{m+p}, x_{m+p-1}) + A^2 d(x_{m+p-1}, x_{m+p-2}) \\
&\quad + A^3 d(x_{m+p-2}, x_{m+p-3}) + \cdots \\
&\quad + A^{p-1} d(x_{m+2}, x_{m+1}) + A^{p-1} d(x_{m+1}, x_m) \\
&\preceq At^{m+p-1}B_0 + A^2 t^{m+p-2}B_0 + A^3 t^{m+p-3}B_0 + \cdots \\
&\quad + A^{p-1}t^{m+1}B_0 + A^{p-1}t^m B_0 \\
&= \sum_{k=1}^{p-1} A^k t^{m+p-k} B_0 + A^{p-1}t^m B_0 \\
&= \sum_{k=1}^{p-1} \big|B_0^{\frac{1}{2}} t^{\frac{m+p-k}{2}} A^{\frac{k}{2}}\big|^2 + \big|B_0^{\frac{1}{2}} A^{\frac{p-1}{2}} t^{\frac{m}{2}}\big|^2 \\
&\preceq \|B_0\| \sum_{k=1}^{p-1} \|A\|^k \|t\|^{m+p-k} I + \|A\|^{p-1}\|t\|^m \|B_0\| I \\
&\preceq \frac{\|B_0\| \|A\|^p \|t\|^{m+1}}{\|A\| - \|t\|} I + \|A\|^{p-1}\|t\|^m \|B_0\| I \\
&\to \theta \quad (m \to \infty).
\end{aligned}$$

This implies $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$, *i.e.*, $\lim_{n\to\infty} Tx_{n-1} = x$. Since

$$\begin{aligned}
d(Tx, x) &\preceq A\big[d(Tx, Tx_n) + d(Tx_n, x)\big] \\
&\preceq A\big[B\big(d(Tx, x) + d(Tx_n, x_n)\big) + d(Tx_n, x)\big] \\
&= AB\big(d(Tx, x) + d(Tx_n, x_n)\big) + Ad(Tx_n, x).
\end{aligned}$$

This is equivalent to

$$d(Tx, x) \preceq (I - AB)^{-1}ABd(Tx_n, Tx_{n-1}) + (I - AB)^{-1}Ad(Tx_n, x).$$

Then

$$\begin{aligned}
\big\|d(Tx, x)\big\| &\leq \big\|(I - AB)^{-1}AB\big\| \big\|d(Tx_n, x_n)\big\| \\
&\quad + \big\|(I - AB)^{-1}A\big\| \big\|d(Tx_n, x)\big\| \\
&\to 0 \quad (n \to \infty).
\end{aligned}$$

This implies that $Tx = x$, *i.e.*, x is a fixed point of T. Now if y ($\neq x$) is another fixed point of T, then

$$\theta \preceq d(x,y) = d(Tx,Ty) \preceq B\big(d(Tx,x) + d(Ty,y)\big) = \theta.$$

Hence, $x = y$. Therefore the fixed point is unique and the proof is complete. □

3 Applications

As applications of contractive mapping theorem on complete C^*-algebra-valued b-metric spaces, existence and uniqueness results for a type of operator equation and integral equation are given.

Example 3.1 Suppose that H is a Hilbert space, $L(H)$ is the set of linear bounded operators on H. Let $A_1, A_2, \ldots, A_n, \ldots \in L(H)$ which satisfy $\sum_{n=1}^{\infty} \|A_n\| < 1$ and $Q \in L(H)_+$. Then the operator equation

$$X - \sum_{n=1}^{\infty} A_n^* X A_n = Q$$

has a unique solution in $L(H)$.

Proof Set $\alpha = (\sum_{n=1}^{\infty} \|A_n\|)^p$ with $p \geq 1$, then $\|\alpha\| < 1$. Without loss of generality, one can suppose that $\alpha > 0$.

Choose a positive operator $T \in L(H)$. For $X, Y \in L(H)$ and $p \geq 1$, set

$$d(X,Y) = \|X - Y\|^p T.$$

Then $d(X,Y)$ is a C^*-algebra-valued b-metric and $(L(H), d)$ is complete since $L(H)$ is a Banach space. Indeed, it suffices to check the third condition of Definition 2.1 as follows.

Suppose that $X, Y, Z \in L(H)$ and set $U = X - Z$, $V = Z - Y$. Using the well-known inequality

$$(a+b)^p \leq \big(2\max\{a,b\}\big)^p \leq 2^p\big(a^p + b^p\big) \quad \text{for all } a, b \geq 0,$$

we have

$$\begin{aligned} \|X - Y\|^p &= \|U + V\|^p \leq \big(\|U\| + \|V\|\big)^p \\ &\leq 2^p\big(\|U\|^p + \|V\|^p\big) \\ &= 2^p\big(\|X - Z\|^p + \|Z - Y\|^p\big), \end{aligned}$$

which implies that

$$d(X,Y) \leq A\big[d(X,Z) + d(Z,Y)\big],$$

where $A = 2^p I$. Consider the map $F : L(H) \to L(H)$ defined by

$$F(X) = \sum_{n=1}^{\infty} A_n^* X A_n + Q.$$

Then

$$\begin{aligned}
d\big(F(X),F(Y)\big) &= \big\|F(X)-F(Y)\big\|^p T \\
&= \left\|\sum_{n=1}^{\infty} A_n^*(X-Y)A_n\right\|^p T \\
&\preceq \sum_{n=1}^{\infty} \|A_n\|^{2p}\|X-Y\|^p T \\
&\preceq \alpha^2 d(X,Y) \\
&= (\alpha I)^* d(X,Y)(\alpha I).
\end{aligned}$$

Using Theorem 2.1, there exists a unique fixed point X in $L(H)$. Furthermore, since $\sum_{n=1}^{\infty} A_n^* X A_n + Q$ is a positive operator, the solution is a Hermitian operator. □

As a special case of Example 3.1, one can consider the following matrix equation, which can also be found in [19]:

$$X - A_1^* X A_1 - \cdots - A_m^* X A_m = Q,$$

where Q is a positive definite matrix and $A_1,\ldots,A_m$ are arbitrary $n \times n$ matrices with $\sum_{n=1}^{m} \|A_n\| < 1$ with $p \geq 1$. Using Example 3.1, there exists a unique Hermitian matrix solution.

Example 3.2 Consider the integral equation

$$x(t) = \int_E F\big(t,x(s)\big)\,\mathrm{d}s + g(t), \quad t \in E,$$

where E is a Lebesgue measurable set. Suppose that

(1) $F : E \times \mathbb{R} \to \mathbb{R}$ and $g \in L^\infty(E)$;

(2) there exists a continuous function $\varphi : E \times E \to \mathbb{R}$ and $k \in (0,1)$ such that

$$\big|F\big(t,u(s)\big) - F\big(t,v(s)\big)\big| \leq k\big|\varphi(t,s)\big|\big|u(s)-v(s)\big|$$

for $t \in E$ and $u, v \in L^\infty(E)$;

(3) $\sup_{t\in E} \int_E |\varphi(t,s)|\,\mathrm{d}s \leq 1$.

Then the integral equation has a unique solution x^* in $L^\infty(E)$.

Proof Let $X = L^\infty(E)$ and $H = L^2(E)$. For $f, g \in X$ and $p > 1$, we set $d : X \times X \to L(H)$ by

$$d(f,g) = \pi_{|f-g|^p},$$

where $\pi_h : H \to H$ is the multiplication operator defined by

$$\pi_\psi(\varphi) = h \cdot \psi$$

for $\psi \in H$. Then d is a C^*-algebra-valued b-metric by Example 2.1 and $(X, L(H), d)$ is a complete C^*-algebra-valued b-metric space by Example 2.1 in [11].

Let $T : L^\infty(E) \to L^\infty(E)$ be

$$Tx(t) = \int_E F\bigl(t, x(s)\bigr)\,\mathrm{d}s + g(t), \quad t \in E.$$

Set $B = kI$, then $B \in L(H)_+$ and $\|B\| = k < 1$. For any $h \in H$,

$$\begin{aligned}
\bigl\|d(Tx, Ty)\bigr\| &= \sup_{\|h\|=1} (\pi_{|Tx-Ty|^p} h, h) \\
&= \sup_{\|h\|=1} \int_E \biggl[\biggl|\int_E \bigl(F\bigl(t, x(s)\bigr) - F\bigl(t, y(s)\bigr)\bigr)\,\mathrm{d}s\biggr|^p\biggr] h(t)\overline{h(t)}\,\mathrm{d}t \\
&\leq \sup_{\|h\|=1} \int_E \biggl[\int_E \bigl|F\bigl(t, x(s)\bigr) - F\bigl(t, y(s)\bigr)\bigr|\,\mathrm{d}s\biggr]^p \bigl|h(t)\bigr|^2\,\mathrm{d}t \\
&\leq \sup_{\|h\|=1} \int_E \biggl[\int_E \bigl|k\varphi(t,s)\bigl(x(s) - y(s)\bigr)\bigr|\,\mathrm{d}s\biggr]^p \bigl|h(t)\bigr|^2\,\mathrm{d}t \\
&\leq k^p \sup_{\|h\|=1} \int_E \biggl[\int_E \bigl|\varphi(t,s)\bigr|\,\mathrm{d}s\biggr]^p \bigl|h(t)\bigr|^2\,\mathrm{d}t \cdot \|x - y\|_\infty^p \\
&\leq k \sup_{t \in E} \int_E \bigl|\varphi(t,s)\bigr|\,\mathrm{d}s \cdot \sup_{\|h\|=1} \int_E \bigl|h(t)\bigr|^2\,\mathrm{d}t \cdot \|x - y\|_\infty^p \\
&\leq k\|x - y\|_\infty^p \\
&= \|B\|\bigl\|d(x, y)\bigr\|.
\end{aligned}$$

Since $\|B\| < 1$, the integral equation has a unique solution x^* in $L^\infty(E)$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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References

1. Czerwik, S: Contraction mappings in *b*-metric spaces. Acta Math. Inform. Univ. Ostrav. **1**, 5-11 (1993)
2. Bakhtin, IA: The contraction mapping principle in almost metric spaces. Funct. Anal., Gos. Ped. Inst. Ulianowsk **30**, 26-37 (1989)
3. Boriceanu, M, Bota, M, Petrusel, A: Mutivalued fractals in *b*-metric spaces. Cent. Eur. J. Math. **8**(2), 367-377 (2010)
4. Czerwik, S: Nonlinear set-valued contraction mappings in *b*-metric spaces. Atti Semin. Mat. Fis. Univ. Modena **46**, 263-276 (1998)
5. Hussian, N, Shah, MH: KKM mappings in cone *b*-metric spaces. Comput. Math. Appl. **62**, 1677-1684 (2011)
6. Fagin, R, Stockmeyer, L: Relaxing the triangle inequality in pattern matching. Int. J. Comput. Vis. **30**(3), 219-231 (1998)
7. Cortelazzo, G, Milan, G, Vezzi, G, Zamoeroni, P: Trademark shapes description by string matching techniques. Pattern Recognit. **27**(8), 1005-1018 (1994)
8. Mcconnell, R, Kwok, R, Curlander, J, Kober, W, Pang, S: ψ-s Correlation and dynamic time warping: two methods for tracking ice floes. IEEE Trans. Geosci. Remote Sens. **29**(6), 1004-1012 (1991)

9. Xia, Q: The geodesic problem in quasimetric spaces. J. Geom. Anal. **19**(2), 452-479 (2009)
10. Kirk, W, Shahzad, N: Fixed Point Theory in Distance Spaces. Springer, Cham (2014)
11. Ma, ZH, Jiang, LN, Sun, HK: C^*-Algebras-valued metric spaces and related fixed point theorems. Fixed Point Theory Appl. **2014**, 206 (2014)
12. Ma, ZH, Jiang, LN, Xin, QL: Fixed point theorems on operator-valued metric space. Trans. Beijing Inst. Techol. **34**(10), 1078-1080 (2014)
13. Murphy, GJ: C^*-Algebras and Operator Theory. Academic Press, London (1990)
14. Xu, QH, Bieke, TED, Chen, ZQ: Introduction to Operator Algebras and Noncommutative L^p Spaces. Science Press, Beijing (2010) (in Chinese)
15. Douglas, RG: Banach Algebra Techniques in Operator Theory. Springer, Berlin (1998)
16. Huang, HP, Xu, SY: Fixed point theorems of contractive mappings in cone b-metric spaces and applications. Fixed Point Theory Appl. **2013**, 112 (2013)
17. Chatterjea, SK: Fixed-point theorems. C. R. Acad. Bulgare Sci. **25**, 727-730 (1972)
18. Kannan, R: Some results on fixed point II. Am. Math. Mon. **76**, 405-408 (1969)
19. Ran, ACM, Reurings, MCB: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**(5), 1435-1443 (2004)