Yotkaew and Saejung *Fixed Point Theory and Applications* (2015) 2015:235
DOI 10.1186/s13663-015-0488-x

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH — Open Access

Strong convergence theorems for multivalued mappings in a geodesic space with curvature bounded above

Pongsakorn Yotkaew[1] and Satit Saejung[1,2*]

*Correspondence: saejung@kku.ac.th
[1] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen, 40002, Thailand
[2] The Centre of Excellence in Mathematics, Commission on Higher Education (CHE), Sri Ayudthaya Road, Bangkok, 10400, Thailand

Abstract
We prove two strong convergence results of the Ishikawa iteration for a multivalued quasi-nonexpansive mapping in a complete geodesic space with curvature bounded above. Our results improve significantly the recent results of Panyanak (Fixed Point Theory Appl. 2014:1, 2014) in the sense that many restrictions in his results are weakened. In particular, we can conclude that the convergence result of the Mann iteration which cannot be deduced from Panyanak's results.

Keywords: CAT(κ) space; fixed point; Ishikawa iteration; Mann iteration; multivalued mapping; strong convergence

1 Introduction

For a metric space X, a point $p \in X$ is said to be a *fixed point* of a multivalued mapping $T : X \to 2^X$ if $p \in Tp$. The set of all fixed points of T is denoted by $\mathrm{F}(T)$. The purpose of this paper is to discuss an iterative scheme to approximate a fixed point of a multivalued mapping.

For a geodesic space X, there have been many algorithms for approximating fixed points of the multivalued mapping T. We are interested in an iterative sequence $\{x_n\}$ generated by the *Ishikawa algorithm* [2], that is,

$$\begin{cases} x_1 \in X \text{ is arbitrarily chosen;} \\ y_n = \beta_n v_n \oplus (1-\beta_n)x_n \text{ where } v_n \in Tx_n; \\ x_{n+1} = \alpha_n u_n \oplus (1-\alpha_n)x_n \text{ where } u_n \in Ty_n; \end{cases} \tag{1.1}$$

for all $n \in \mathbb{N}$. If $\beta_n \equiv 0$, then such a sequence $\{x_n\}$ is said to be generated by the *Mann algorithm* [3].

In 2009, Shahzad and Zegeye [4] proved the following result.

Theorem 1.1 ([4], Theorems 2.3 and 2.5) *Let K be a closed convex subset of a uniformly convex Banach space, $T : K \to \mathrm{CB}(K)$ be a quasi-nonexpansive multivalued mapping with the endpoint condition, and let $\{x_n\}$ be defined by* (1.1). *Assume that one of the following holds*:

(A′) *T satisfies condition* (I) *with respect to K, and $\{\alpha_n\}, \{\beta_n\} \subset [a, b] \subset (0, 1)$;*

(A*) *T is hemicompact and continuous, and $\{\alpha_n\}, \{\beta_n\} \subset [0,1)$, $\lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$.*

Then $\{x_n\}$ converges to a fixed point of T.

Recall that the mapping T satisfies the *endpoint condition* if $\mathrm{F}(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in \mathrm{F}(T)$.

Recently, Panyanak [1] presented the analog result of Shahzad and Zegeye in the framework of a CAT(1) space whose diameter is less than $\pi/2$ and whose metric is convex. (See the relevant definitions in Section 2.) We quote all his main results as follows.

Theorem 1.2 ([1], Theorems 3.2 and 3.3) *Let X be a complete* CAT(1) *space, $T : X \to \mathrm{CB}(X)$ be a quasi-nonexpansive mapping with the endpoint condition, and let $\{x_n\}$ be the Ishikawa algorithm. Assume that one of the following holds*:

(A′) *T satisfies condition* (I) *with respect to X, and $\{\alpha_n\}, \{\beta_n\} \subset [a,b] \subset (0,1)$;*
(A*) *T is hemicompact and continuous, and $\{\alpha_n\}, \{\beta_n\} \subset [0,1)$, $\lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$.*

If the metric is convex and $\mathrm{diam}(X) < \pi/2$, *then $\{x_n\}$ converges to a fixed point of T.*

In this paper, we improve both Theorems 3.2 and 3.3 of Panyanak (see Theorem 1.2 above) in the following aspects. (1) We do not assume that the metric is convex. (2) The restrictions on the mapping T and the parameters $\{\alpha_n\}$, $\{\beta_n\}$ are relaxed. (3) The condition $\mathrm{diam}(X) < \pi/2$ is replaced by a more general one and this condition is sharp. Note that the Mann algorithm is nothing but the Ishikawa algorithm (1.1) with $\beta_n \equiv 0$. We also show that our result can conclude the convergence for the Mann algorithm while this is not the case in Theorem 1.2.

2 Preliminaries

For a real number κ, we denote by M_κ the following: (i) M_κ is the spherical space $(1/\sqrt{\kappa})\mathbb{S}^2$ if $\kappa > 0$; (ii) M_κ is the Euclidean space $\mathbb{R}^2$ if $\kappa = 0$; (iii) M_κ is the hyperbolic space $(1/\sqrt{-\kappa})\mathbb{H}^2$ if $\kappa < 0$. In particular, $D_\kappa := \mathrm{diam}(M_\kappa) = \infty$ if $\kappa \leq 0$ and $D_\kappa = \pi/\sqrt{\kappa}$ if $\kappa > 0$.

A metric space X is a *geodesic space* if for every $u, v \in X$, there exists a *geodesic part* from u to v, that is, $g : [0,l] \to X$ is an isometry such that $g(0) = u$ and $g(l) = v$. A *geodesic triangle* $\triangle(u,v,w)$ consists of three points $u, v, w \in X$ and all the images of each geodesic part joining two of them. We say that a geodesic triangle $\triangle(u,v,w)$, where $u, v, w \in X$ and $d(u,v) + d(v,w) + d(w,u) < 2D_\kappa$, satisfies *the* CAT($\kappa$) *inequality* if there exist three corresponding points $\overline{u}, \overline{v}, \overline{w} \in M_\kappa$ such that

- $d(u,v) = d_{M_\kappa}(\overline{u},\overline{v})$, $d(v,w) = d_{M_\kappa}(\overline{v},\overline{w})$, and $d(u,w) = d_{M_\kappa}(\overline{u},\overline{w})$;
- $d(p,q) \leq d_{M_\kappa}(\overline{p},\overline{q})$ for all $p, q \in \triangle(u,v,w)$ and $\overline{p}, \overline{q} \in \triangle(\overline{u},\overline{v},\overline{w})$.

A geodesic space X is a CAT(κ) *space* if every geodesic triangle in X with perimeter less than $2D_\kappa$ satisfies the CAT(κ) inequality. If X is a CAT(κ) space, then it is D_κ-uniquely geodesic, that is, there exists a unique geodesic part from u to v for all $u, v \in X$ with $d(u,v) < D_\kappa$. In this case, for $\alpha \in [0,1]$ and $u, v \in X$ with $d(u,v) < D_\kappa$, we denote by $w = \alpha u \oplus (1-\alpha)v$ the point on the image of the geodesic part from u to v such that $d(u,w) = (1-\alpha)d(u,v)$ and $d(v,w) = \alpha d(u,v)$. Moreover, if CAT(κ) space with $\mathrm{diam}(X) < D_\kappa/2$, then the metric d

is *convex* (see [5]), that is,

$$d\big(p, \alpha u \oplus (1-\alpha)v\big) \le \alpha d(p,u) + (1-\alpha) d(p,v)$$

for all $u, v, p \in X$ and $\alpha \in [0,1]$. But the converse is not true in general. In the recent work of Panyanak [1], it was assumed that X is a CAT(1) space whose metric is convex and $\operatorname{diam}(X) < \pi/2$. The convexity of its metric is actually a consequence of the condition $\operatorname{diam}(X) < \pi/2$.

The following lemma, proved by Kimura and Satô [6], gives a very important property of CAT(1) spaces.

Lemma 2.1 *Let* $\alpha \in [0,1]$ *and* $\triangle(u,v,w)$ *be a geodesic triangle in a* CAT(1) *space such that* $d(u,v) + d(v,w) + d(w,u) < 2\pi$. *Then*

$$\begin{aligned}&\cos d\big(\alpha u \oplus (1-\alpha)v, w\big) \sin d(u,v)\\ &\quad \ge \cos d(u,w) \sin\big(\alpha d(u,v)\big) + \cos d(v,w) \sin\big((1-\alpha) d(u,v)\big).\end{aligned}$$

In particular, $\cos d(\alpha u \oplus (1-\alpha)v, w) \ge \alpha \cos d(u,w) + (1-\alpha)\cos d(v,w)$.

For a metric space X, we denote by $\mathrm{CB}(X)$ the family of all nonempty closed bounded subsets of X. For $u \in X$ and $\varnothing \ne A \subset X$, we write $d(u,A) := \inf\{d(u,a) : a \in A\}$. We denote by H the Hausdorff distance on $\mathrm{CB}(X)$; that is, for each $A, B \in \mathrm{CB}(X)$,

$$H(A,B) = \max\Big\{\sup_{a\in A} d(a,B), \sup_{b \in B} d(b,A)\Big\}.$$

Recall that a multivalued mapping $T : C \to 2^X$, where $\varnothing \ne C \subset X$, is said to:

- be quasi-nonexpansive if $\mathrm{F}(T) \ne \varnothing$ and $H(Tu, \{p\}) \le d(u,p)$ for all $u \in C$ and $p \in \mathrm{F}(T)$;
- be *closed at zero* if each strongly convergent sequence $\{u_n\}$ in C satisfying $d(u_n, Tu_n) \to 0$ has its limit in $\mathrm{F}(T)$;
- be *hemicompact* if each bounded sequence $\{u_n\}$ in C satisfying $d(u_n, Tu_n) \to 0$ has a strongly convergent subsequence;
- satisfy *condition* (I) with respect to C [7] if $\mathrm{F}(T) \ne \varnothing$ and there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ such that f vanishes only at zero and $f(d(u, \mathrm{F}(T))) \le d(u, Tu)$ for all $u \in C$.

Remark 2.2 (1) $\mathrm{F}(T)$ is a closed set if T is quasi-nonexpansive. (2) Every continuous mapping is closed at zero.

The following lemma is taken from the result of Senter and Dotson [7]. However, it is established for a multivalued mapping in a metric space, while the original result is for a single-valued mapping in a Banach space.

Lemma 2.3 *Let* K *be a bounded closed subset of a metric space* X, *and let* $T : K \to 2^X \setminus \{\varnothing\}$ *be a multivalued mapping with a fixed point. If* T *is hemicompact and closed at zero, then it satisfies condition* (I) *with respect to* K.

Proof Letting $a := \sup\{d(u, \mathrm{F}(T)) : u \in K\}$. The result is obvious if $a = 0$. Suppose that $a > 0$. Putting $K_\alpha := \{u \in X : d(u, \mathrm{F}(T)) \geq \alpha\}$ for all $\alpha \in (0, a)$. Notice that K_α is nonempty, bounded, and closed. Next, we define $f : [0, \infty) \to [0, \infty)$ by

$$f(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0; \\ \inf_{u \in K_\alpha} d(u, Tu) & \text{if } \alpha \in (0, a); \\ \sup_{\beta \in (0,a)} \inf_{u \in K_\beta} d(u, Tu) & \text{if } \alpha \geq a. \end{cases}$$

It is easy to see that f is a nondecreasing function and $f(d(u, \mathrm{F}(T))) \leq d(u, Tu)$ for all $u \in C$. Finally, we prove that $f(\alpha) > 0$ for all $\alpha \in (0, a)$. Assume that $f(\gamma) = 0$ for some $\gamma \in (0, a)$. Thus there exists a bounded sequence $\{v_n\}$ in K_γ such that $d(v_n, Tv_n) \to 0$. Since T is hemicompact and closed at zero, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $v_{n_k} \to v \in \mathrm{F}(T)$. By the closedness of K_t and $v_{n_k} \to v$, we get $v \in K_\gamma$. Hence $0 = d(u, \mathrm{F}(T)) \geq \gamma$ is a contradiction and the proof is finished. □

3 Main results

We first introduce the concept of Fejér monotonicity in a metric space. Let F be a nonempty subset of a metric space X and let $\gamma > 0$. A sequence $\{u_n\}$ in X is called (γ, F)-*Fejér monotone* if whenever $d(u_k, p) < \gamma$ for some $k \in \mathbb{N}$ and $p \in F$ it follows that

$$d(u_{n+1}, p) \leq d(u_n, p) \quad \text{for all } n \geq k.$$

Obviously, every subsequence of a (γ, F)-Fejér monotone sequence is also (γ, F)-Fejér monotone.

Lemma 3.1 *Let F be a nonempty closed subset of a complete metric space X and let $\gamma > 0$. Suppose that $\{u_n\} \subset X$ is (γ, F)-Fejér monotone. Then $\{u_n\}$ converges to an element of F if and only if $\liminf_{n\to\infty} d(u_n, F) = 0$.*

Proof The necessity is obvious. We now prove the sufficiency. Assume that

$$\liminf_{n\to\infty} d(u_n, F) = 0.$$

Let $0 < \varepsilon < \gamma$. Then there exist a subsequence $\{v_n\}$ of $\{u_n\}$ and a sequence $\{p_n\}$ in F such that $d(v_n, p_n) < \varepsilon/2^{n+1} < \gamma$ for all $n \in \mathbb{N}$. Since $\{v_n\}$ is (γ, F)-Fejér monotone, $d(v_{n+l}, p_n) \leq d(v_n, p_n)$ for all $n, l \in \mathbb{N}$. Consequently,

$$d(v_{n+l}, v_n) \leq d(v_{n+l}, p_n) + d(v_n, p_n) \leq 2d(v_n, p_n) < \varepsilon/2^n$$

for all $n, l \in \mathbb{N}$. Hence, $\{v_n\}$ is a Cauchy sequence in X. By the completeness of X, we get $v_n \to p$ for some element $p \in X$. Since $d(v_n, p_n) \to 0$ and F is closed, we have $p_n \to p \in F$. We finally show that the whole sequence $\{u_n\}$ converges to p. Since $v_n \to p$ and $\{v_n\}$ is the subsequence of $\{u_n\}$, there exists $k \in \mathbb{N}$ such that $d(u_k, p) < \gamma$. Since $\{u_n\}$ is (γ, F)-Fejér monotone, we get $d(u_{n+1}, p) \leq d(u_n, p)$ for all $n \geq k$, and hence $\lim_{n\to\infty} d(u_n, p)$ exists. In particular, $\lim_{n\to\infty} d(u_n, p) \leq \lim_{n\to\infty} d(v_n, p)$. This implies that $\lim_{n\to\infty} d(u_n, p) = 0$. □

Lemma 3.2 *Let K be a subset of a metric space X and let $\{u_n\}$ be a bounded sequence in K. Let $T : K \to 2^X \setminus \{\varnothing\}$ be a multivalued mapping with a fixed point. If T satisfies condition* (I) *with respect to $\{u_n\}$ and $\lim_{n\to\infty} d(u_n, Tu_n) = 0$, then $\lim_{n\to\infty} d(u_n, \mathrm{F}(T)) = 0$.*

Proof Let f be the nondecreasing function in the condition (I) of T with respect to $\{u_n\}$. Then $f(d(u_n, \mathrm{F}(T))) \le d(u_n, Tu_n)$ for all $n \in \mathbb{N}$. This implies that

$$\limsup_{n\to\infty} f\bigl(d\bigl(u_n, \mathrm{F}(T)\bigr)\bigr) \le \lim_{n\to\infty} d(u_n, Tu_n) = 0.$$

Since f vanishes only at zero, the conclusion follows. □

Lemma 3.3 *Let $\{u_n\}$, $\{v_n\}$ be two sequences in a* CAT(1) *space X and $p \in X$. Suppose that $\{\alpha_n\}$ is a real sequence in $[0,1]$, $\lambda \in [0,\pi/2)$, and $d(u_n, v_n) < \pi$ for all $n \in \mathbb{N}$. Suppose that the following conditions hold*:

(i) $\lim_{n\to\infty} d(\alpha_n u_n \oplus (1-\alpha_n)v_n, p) = \lambda$.

(ii) $\limsup_{n\to\infty} d(u_n, p) \le \lambda$;

(iii) $\limsup_{n\to\infty} d(v_n, p) \le \lambda$.

If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, then $\lim_{n\to\infty} d(u_n, v_n) = 0$.

Proof Let $\{n_k\}$ be an increasing sequence of positive integers such that the following limits exist: $\lim_{k\to\infty} d(u_{n_k}, v_{n_k}) = a$, $\lim_{k\to\infty} d(u_{n_k}, p) = b$, $\lim_{k\to\infty} d(v_{n_k}, p) = c$, and $\lim_{k\to\infty} \alpha_{n_k} = \alpha \in (0,1)$. We show that $a = 0$. Note that $b \le \lambda < \pi/2$ and $c \le \lambda < \pi/2$. We may assume that $d(u_{n_k}, p) < \pi/2$ and $d(v_{n_k}, p) < \pi/2$ for all $k \in \mathbb{N}$. In particular, $d(u_{n_k}, p) + d(p, v_{n_k}) + d(v_{n_k}, u_{n_k}) < 2\pi$ for all $k \in \mathbb{N}$. Using Lemma 2.1 gives

$$\begin{aligned}&\cos d\bigl(\alpha_{n_k} u_{n_k} \oplus (1-\alpha_{n_k})v_{n_k}, p\bigr) \sin d(u_{n_k}, v_{n_k}) \\ &\quad\ge \cos d(u_{n_k}, p) \sin\bigl(\alpha_{n_k} d(u_{n_k}, v_{n_k})\bigr) + \cos d(v_{n_k}, p) \sin\bigl((1-\alpha_{n_k}) d(u_{n_k}, v_{n_k})\bigr).\end{aligned}$$

Consequently,

$$\cos\lambda \sin a \ge \cos b \sin(\alpha a) + \cos c \sin\bigl((1-\alpha)a\bigr) \ge \cos\lambda\bigl(\sin(\alpha a) + \sin\bigl((1-\alpha)a\bigr)\bigr).$$

It follows that $\sin a \ge \sin(\alpha a) + \sin((1-\alpha)a)$ and hence $a = 0$.

By the double extract subsequence principle, we have $\lim_{n\to\infty} d(u_n, v_n) = 0$. □

Lemma 3.4 *Let u, v, q be three elements of a* CAT(1) *space X such that*

$$d(u,q) < \pi/2 \quad \textit{and} \quad d(v,q) < \pi/2.$$

If $w = \alpha v \oplus (1-\alpha)u$ for some $\alpha \in [0,1]$, then $d(w,q) \le \max\{d(u,q), d(v,q)\}$.

Proof Note that the element w is well defined. By Lemma 2.1, we have

$$\cos d(w,q) = \cos d\bigl(\alpha v \oplus (1-\alpha)u, q\bigr) \ge \alpha \cos d(v,q) + (1-\alpha)\cos d(u,q).$$

Since $\alpha \cos d(v,q) + (1-\alpha)\cos d(u,q) \ge \min\{\cos d(v,q), \cos d(u,q)\}$, we get

$$\cos d(w,q) \ge \min\bigl\{\cos d(v,q), \cos d(u,q)\bigr\}.$$

This implies that

$$\begin{aligned} & d(w,q) \\ & \quad \le \min\bigl\{d(w,u)+d(u,q), d(w,v)+d(v,q)\bigr\} \\ & \quad = \min\bigl\{\alpha d(v,u)+d(u,q), (1-\alpha)d(v,u)+d(v,q)\bigr\} \\ & \quad \le \min\{\alpha, 1-\alpha\}d(v,u)+\frac{\pi}{2} \\ & \quad \le \frac{1}{2}d(v,u)+\frac{\pi}{2} < \pi. \end{aligned}$$

Using elementary trigonometry gives $d(w,q) \le \max\{d(u,q), d(v,q)\}$. □

Lemma 3.5 *Let X be a* CAT(1) *space, $T : X \to \mathrm{CB}(X)$ be a quasi-nonexpansive mapping with the endpoint condition, and let $\{x_n\}$ be defined by* (1.1) *and be such that $d(x_1,p) < \pi/2$ for some $p \in \mathrm{F}(T)$. Then*

$$d(x_{n+1},p) \le d(x_n,p) \quad \textit{and} \quad d(y_n,p) \le d(x_n,p) \quad \textit{for all } n \in \mathbb{N}.$$

In particular, $\{x_n\}$ is $(\pi/2, \mathrm{F}(T))$-Fejér monotone.

Proof Let $p \in \mathrm{F}(T)$ be such that $d(x_1,p) < \pi/2$. Since $z_1 \in Tx_1$ and T is quasi-nonexpansive, we get $d(z_1,p) \le H(Tx_1,\{p\}) \le d(x_1,p) < \pi/2$. By Lemma 3.4, we see that y_1 is well defined and $d(y_1,p) \le d(x_1,p)$. Thus $d(z_1',p) \le H(Ty_1,\{p\}) \le d(y_1,p) < \pi/2$. Again, by Lemma 3.4, we see that x_2 is well defined and $d(x_2,p) \le d(x_1,p)$. Hence, the result follows from mathematical induction. □

Lemma 3.6 *Let X, T, and $\{x_n\}$ be the same as Lemma* 3.5.

(i) *If $\sum_{n=1}^{\infty} \alpha_n\beta_n(1-\beta_n) = \infty$, then $\liminf_{n\to\infty} d(x_n,v_n) = 0$.*
(ii) *If $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then $\liminf_{n\to\infty} d(x_n,u_n) = 0$.*

Proof We first show the assertion (i). Suppose that $\liminf_{n\to\infty} d(x_n,v_n) > 0$. So, we may assume that

$$0 < \delta \le d(x_n,v_n) < \pi \quad \text{for all } n \in \mathbb{N}.$$

By Lemma 2.1, we get

$$\begin{aligned} \cos d(x_{n+1},p) &= \cos d\bigl(\alpha_n u_n \oplus (1-\alpha_n)x_n, p\bigr) \\ &\ge \alpha_n \cos d(u_n,p) + (1-\alpha_n)\cos d(x_n,p). \end{aligned}$$

Since $u_n \in Ty_n$ and T is quasi-nonexpansive, $d(u_n,p) \le H(Ty_n,\{p\}) \le d(y_n,p)$. This implies that

$$\cos d(x_{n+1},p) \ge \alpha_n \cos d(y_n,p) + (1-\alpha_n)\cos d(x_n,p).$$

It follows from Lemma 2.1 that

$$\begin{aligned}
&\cos d(y_n,p)\sin d(x_n,v_n)\\
&\quad\geq \cos d(v_n,p)\sin\bigl(\beta_n d(x_n,v_n)\bigr)+\cos d(x_n,p)\sin\bigl((1-\beta_n)d(x_n,v_n)\bigr).
\end{aligned}$$

In particular,

$$\begin{aligned}
&\cos d(x_{n+1},p)\sin d(x_n,v_n)\\
&\quad\geq \alpha_n\cos d(v_n,p)\sin\bigl(\beta_n d(x_n,v_n)\bigr)+\alpha_n\cos d(x_n,p)\sin\bigl((1-\beta_n)d(x_n,v_n)\bigr)\\
&\qquad+(1-\alpha_n)\cos d(x_n,p)\sin d(x_n,v_n).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\cos d(x_{n+1},p)-\cos d(x_n,p)\\
&\quad\geq \alpha_n\cos d(x_n,p)\biggl(\frac{\sin(\beta_n d(x_n,v_n))+\sin((1-\beta_n)d(x_n,v_n))}{\sin d(x_n,v_n)}-1\biggr).
\end{aligned}$$

Since $0<\delta\leq d(x_n,v_n)<\pi$, we have

$$\begin{aligned}
&\frac{\sin(\beta_n d(x_n,v_n))+\sin((1-\beta_n)d(x_n,v_n))}{\sin d(x_n,v_n)}-1\\
&\quad=\frac{2\sin(\beta_n d(x_n,v_n)/2)\sin((1-\beta_n)d(x_n,v_n)/2)}{\cos(d(x_n,v_n)/2)}\\
&\quad\geq\frac{\sin(\beta_n\delta/2)\sin((1-\beta_n)\delta/2)}{\cos(\delta/2)}\\
&\quad\geq\frac{\beta_n(1-\beta_n)\sin^2(\delta/2)}{\cos(\delta/2)}.
\end{aligned}$$

Since $d(x_n,p)\leq d(x_1,p)$, we have

$$\begin{aligned}
\cos d(x_{n+1},p)-\cos d(x_n,p)&\geq\alpha_n\cos d(x_n,p)\biggl(\frac{\beta_n(1-\beta_n)\sin^2(\delta/2)}{\cos(\delta/2)}\biggr)\\
&\geq\frac{2\alpha_n\beta_n(1-\beta_n)\cos d(x_1,p)\sin^2(\delta/2)}{\cos(\delta/2)}.
\end{aligned}$$

By Lemma 3.5, we get $\sum_{n=1}^{\infty}(\cos d(x_{n+1},p)-\cos d(x_n,p))<\infty$. Notice that $\sin(\delta/2)$, $\cos(\delta/2)$, $\cos d(x_1,p)$ are positive. Consequently, $\sum_{n=1}^{\infty}\alpha_n\beta_n(1-\beta_n)<\infty$, which is a contradiction.

We next show the assertion (ii). Suppose that $\liminf_{n\to\infty}d(x_n,u_n)>0$. Thus, we may assume that

$$0<\delta\leq d(x_n,u_n)<\pi\quad\text{for all } n\in\mathbb{N}.$$

It follows from Lemma 2.1 and $d(u_n,p)\leq d(y_n,p)\leq d(x_n,p)$ that

$$\begin{aligned}
&\cos d(x_{n+1},p)\sin d(x_n,u_n)\\
&\quad=\cos d\bigl(\alpha_n u_n\oplus(1-\alpha_n)x_n,p\bigr)\sin d(x_n,u_n)
\end{aligned}$$

$$\geq \cos d(u_n,p)\sin\bigl(\alpha_n d(x_n,u_n)\bigr) + \cos d(x_n,p)\sin\bigl((1-\alpha_n)d(x_n,u_n)\bigr)$$
$$\geq \cos d(x_n,p)\bigl(\sin\bigl(\alpha_n d(x_n,u_n)\bigr) + \sin\bigl((1-\alpha_n)d(x_n,u_n)\bigr)\bigr).$$

Hence

$$\begin{aligned}&\cos d(x_{n+1},p) - \cos d(x_n,p)\\ &\quad\geq \cos d(x_n,p)\left(\frac{\sin(\alpha_n d(x_n,u_n)) + \sin((1-\alpha_n)d(x_n,u_n))}{\sin d(x_n,u_n)} - 1\right).\end{aligned}$$

Since $0 < \delta \leq d(x_n,u_n) < \pi$, we get

$$\frac{\sin(\alpha_n d(x_n,u_n)) + \sin((1-\alpha_n)d(x_n,u_n))}{\sin d(x_n,u_n)} - 1 \geq \frac{2\alpha_n(1-\alpha_n)\sin^2(\delta/2)}{\cos(\delta/2)}.$$

This implies that

$$\cos d(x_{n+1},p) - \cos d(x_n,p) \geq \frac{2\alpha_n(1-\alpha_n)\cos d(x_1,p)\sin^2(\delta/2)}{\cos(\delta/2)}.$$

Consequently, $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n) < \infty$, which is a contradiction. □

Now, we obtain the following convergence theorem, which improves the results of Panyanak [1].

Theorem 3.7 *Let X be a complete* CAT(1) *space, $T : X \to \mathrm{CB}(X)$ be a quasi-nonexpansive mapping with the endpoint condition, and let $\{x_n\}$ be the Ishikawa algorithm. Assume that*

(A) *T satisfies condition* (I) *with respect to each bounded subset of X, and* $\sum_{n=1}^{\infty}\alpha_n\beta_n(1-\beta_n) = \infty$;

(B) $d(x_1, \mathrm{F}(T)) < \pi/2$.

Then $\{x_n\}$ converges to an element of $\mathrm{F}(T)$.

Proof By Lemma 3.5, we see that $\{x_n\}$ is bounded. Then T satisfies condition (I) with respect to $\{x_n\}$. Using Lemma 3.6 gives $\liminf_{n\to\infty} d(x_n, v_n) = 0$. In particular,

$$\liminf_{n\to\infty} d(x_n, Tx_n) = 0.$$

Notice that $\mathrm{F}(T)$ is closed. It follows from Lemma 3.2 that $\liminf_{n\to\infty} d(x_n, \mathrm{F}(T)) = 0$. Then the result follows from Lemma 3.1. □

Remark 3.8 Our Theorem 3.7 improves both Theorems 3.2 and 3.3 of [1] (see Theorem 1.2) in the following aspects.

(1) We do not assume convexity of the metric.
(2) The restrictions on the mapping T and the parameters $\{\alpha_n\}$, $\{\beta_n\}$ are relaxed. In fact, it is easy to see that either the condition (A′) or (A*) implies the condition (A).
(3) The condition $\mathrm{diam}(X) < \pi/2$ is weakened. It is clear that the condition $\mathrm{diam}(X) < \pi/2$ implies the condition (B).

Remark 3.9 Our condition (B) is *sharp* in the sense that the condition $d(x_1, \mathrm{F}(T)) < \pi/2$ cannot be improved to $d(x_1, \mathrm{F}(T)) \leq \pi/2$.

There exist a quasi-nonexpansive single-valued mapping T and an iterative sequence $\{x_n\}$ generated (1.1) such that

- $d(x_1, \mathrm{F}(T)) = \pi/2$;
- T satisfies condition (I) with respect to $\{x_n\}$;
- $\{x_n\}$ does not converge to a fixed point of T.

Example 3.10 Let T be a mapping from the unit sphere $\mathbb{S}^2$ into itself defined by

$$T(u, v, w) = \left(\frac{u - v}{\sqrt{2}}, \frac{u + v}{\sqrt{2}}, w \right) \quad \text{for } (u, v, w) \in \mathbb{S}^2.$$

Obviously, T is quasi-nonexpansive with $\mathrm{F}(T) = \{(0, 0, -1), (0, 0, 1)\}$. Define a sequence $\{x_n\}$ by $x_1 = (1, 0, 0)$ and

$$x_{n+1} = \alpha_n T\big(\beta_n Tx_n \oplus (1 - \beta_n)x_n\big) \oplus (1 - \alpha_n)x_n \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$. Then $d(x_1, \mathrm{F}(T)) = \pi/2$ and

$$x_{n+1} = \left(\cos\left(\sum_{i=1}^{n} \frac{(\alpha_i + \beta_i)\pi}{4} \right), \sin\left(\sum_{i=1}^{n} \frac{(\alpha_i + \beta_i)\pi}{4} \right), 0 \right) x_n \quad \text{for all } n \in \mathbb{N}.$$

Notice that $\{x_n\}$ is bounded. Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$. We may assume that

$$\lim_{k\to\infty} x_{n_k} = (a, b, 0) \quad \text{for some } a, b \in [0, 1].$$

Hence, $\{x_n\}$ does not converge to a fixed point of T.

Based on Lemma 3.6(ii), we present a convergence theorem which can be reduced to the Mann algorithm.

Theorem 3.11 *Let X, T, and $\{x_n\}$ be the same as Theorem* 3.7. *Assume that*

- *T satisfies condition* (I) *with respect to each bounded subset of X, and* $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ *and* $\limsup_{n\to\infty} \beta_n < 1$;
- $d(x_1, \mathrm{F}(T)) < \pi/2$.

Then $\{x_n\}$ converges to an element of $\mathrm{F}(T)$.

Proof By Lemma 3.6(ii), we have $\liminf_{n\to\infty} d(x_n, u_n) = 0$. Let $\{n_k\}$ be an increasing sequence of positive integers such that $\lim_{k\to\infty} d(x_{n_k}, u_{n_k}) = 0$.

Case 1: $\liminf_{k\to\infty} \beta_{n_k} = 0$. Without loss of generality, we may assume that $\lim_{k\to\infty} \beta_{n_k} = 0$. It follows that $\lim_{k\to\infty} d(y_{n_k}, x_{n_k}) = \lim_{k\to\infty} \beta_{n_k} d(v_{n_k}, x_{n_k}) = 0$. Then $\lim_{k\to\infty} d(y_{n_k}, u_{n_k}) = 0$, and so $\lim_{k\to\infty} d(y_{n_k}, Ty_{n_k}) = 0$. Notice that $\mathrm{F}(T)$ is closed, $\{y_n\}$ is bounded, and T satisfies condition (I) with respect to $\{y_n\}$. Using Lemma 3.2 gives $\lim_{k\to\infty} d(x_{n_k}, \mathrm{F}(T)) = 0$. Consequently, the conclusion follows from Lemma 3.1.

Case 2: $\liminf_{k\to\infty} \beta_{n_k} > 0$. In this case, $\liminf_{k\to\infty} \beta_{n_k}(1-\beta_{n_k}) > 0$. Let $p \in \mathrm{F}(T)$ be such that $d(x_1, p) < \pi/2$. By Lemma 3.5, we have

$$d(x_{n+1}, p) \le d(x_n, p) \quad \text{and} \quad d(y_n, p) \le d(x_n, p) \quad \text{for all } n \in \mathbb{N}.$$

Then $r := \lim_{n\to\infty} d(x_n, p)$ exists and

$$d(u_n, p) \le d(y_n, p) \le d(x_n, p) \quad \text{for all } n \in \mathbb{N}.$$

Since $\lim_{k\to\infty} d(x_{n_k}, u_{n_k}) = 0$, we get $\lim_{k\to\infty} d(u_{n_k}, p) = r$, and hence

$$\lim_{k\to\infty} d(y_{n_k}, p) = r. \tag{3.1}$$

We have

$$\limsup_{k\to\infty} d(v_{n_k}, p) \le \lim_{n\to\infty} d(x_{n_k}, p) = r. \tag{3.2}$$

Using (3.1), (3.2), and Lemma 3.3 gives

$$\lim_{k\to\infty} d(v_{n_k}, x_{n_k}) = 0.$$

Therefore, the proof of the remaining part follows from that of Theorem 3.7. □

The next result follows as an immediate consequence of the above theorem.

Corollary 3.12 *Let X and T be the same as Theorem* 3.7, *and let $\{x_n\}$ be the Mann algorithm. Assume that*

- *T satisfies condition* (I) *with respect to each bounded subset of X, and* $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$;
- $d(x_1, \mathrm{F}(T)) < \pi/2$.

Then $\{x_n\}$ converges to an element of $\mathrm{F}(T)$.

The following result is a strong convergence theorem in a $\mathrm{CAT}(\kappa)$ space where κ is any real number. It should be noted that if $\kappa \le 0$, then $D_\kappa = \infty$ and the condition $d(x_1, \mathrm{F}(T)) < D_\kappa/2$ is automatically satisfied.

Theorem 3.13 *Suppose that X is a complete* $\mathrm{CAT}(\kappa)$ *space where κ is a real number. Let $T : X \to \mathrm{CB}(X)$ be a quasi-nonexpansive mapping with the endpoint condition, and let $\{x_n\}$ be generated by* (1.1) *where $d(x_1, \mathrm{F}(T)) < D_\kappa/2$. Assume that the following conditions hold*:

- *T satisfies condition* (I) *with respect to each bounded subset of X*;
- *either* (i) $\sum_{n=1}^{\infty} \alpha_n\beta_n(1-\beta_n) = \infty$ *or* (ii) $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ *and* $\limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges to an element of $\mathrm{F}(T)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Acknowledgements
The first author is supported by the Thailand Research Fund under grant TRG5880026. He is also partially supported by the research grant from the Faculty of Science and the Research and Technology Transfer Affairs Division of Khon Kaen University. The research of the second author is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

Received: 2 September 2015 Accepted: 11 December 2015 Published online: 30 December 2015

References

1. Panyanak, B: On the Ishikawa iteration processes for multivalued mappings in some CAT(κ) spaces. Fixed Point Theory Appl. **2014**, 1 (2014)
2. Ishikawa, S: Fixed points by a new iteration method. Proc. Amer. Math. Soc. **44**, 147-150 (1974)
3. Mann, WR: Mean value methods in iteration. Proc. Amer. Math. Soc. **4**, 506-510 (1953)
4. Shahzad, N, Zegeye, H: On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces. Nonlinear Anal. **71**, 838-844 (2009)
5. Bridson, MR, Haefliger, A: Metric Spaces of Non-positive Curvature. Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 319. Springer, Berlin (1999)
6. Kimura, Y, Satô, K: Convergence of subsets of a complete geodesic space with curvature bounded above. Nonlinear Anal. **75**, 5079-5085 (2012)
7. Senter, HF, Dotson, WG: Approximating fixed points of nonexpansive mappings. Proc. Amer. Math. Soc. **44**, 375-380 (1974)