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Best proximity results for Suzuki and convex type contractions

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Abstract

The aim of the paper is to introduce new Suzuki and convex type contractions and prove new best proximity results for these contractions in the setting of a metric space. As applications, we deduce similar results for such type of contractions in partially ordered metric spaces and derive new Suzuki type fixed point results. An illustrative example is provided here to highlight our findings.

Keywords: best proximity point; proximal α-admissible mapping; Suzuki type proximal contractions

1 Introduction and preliminaries

The background literature on best proximity theory and associated fixed point theory in (ordered) metric spaces, Banach spaces and fuzzy metric spaces is very abundant in the literature; see, for instance, [1–6] and references therein.

For any two nonempty sets A and B in a metric space (X, d), the point $p \in A$ is called a best proximity point of the mapping $T : A \to B$ if $d(p, Tp) = d(A, B)$, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. We shall denote the set of best proximity points of T by $\mathrm{Bpp}(T)$. For more details, we refer the reader to [7–11] and [4–6, 12–31].

We define

$$\begin{aligned} A_0 &= \bigl\{p \in A : d(p, q) = d(A, B) \text{ for some } q \in B\bigr\}, \\ B_0 &= \bigl\{q \in B : d(p, q) = d(A, B) \text{ for some } p \in A\bigr\}. \end{aligned} \tag{1.1}$$

Definition 1.1 [20] For nonempty subsets A, B of metric space (X, d) with $A_0 \neq \emptyset$, we say the pair (A, B) satisfy

(a) the P-property if

$$\begin{cases} d(x_1, y_1) = d(A, B), \\ d(x_2, y_2) = d(A, B), \end{cases} \implies d(x_1, x_2) = d(y_1, y_2)$$

for all $x_1, x_2 \in A$ and $y_1, y_2 \in B$,

(b) the weak P-property [22, 26] if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$d(x_1, y_1) = d(A, B) \quad\text{and}\quad d(x_2, y_2) = d(A, B) \quad\Rightarrow\quad d(x_1, x_2) \leq d(y_1, y_2). \tag{1.2}$$

We shall use $\Psi = \{\psi : [0,+\infty) \to [0,+\infty) : \sum_{n=1}^{\infty} \psi^n(t) < +\infty \text{ for all } t > 0\}$, where ψ is nondecreasing function.

Now we introduce new concepts of proximal mappings, for more details see [5].

Definition 1.2 If $\alpha : A \times A \to [-\infty,\infty)$, then $T : A \to B$ is called proximal α^+-admissible if

$$\begin{cases} \alpha(x_1,x_2) \geq 0, \\ d(u_1, Tx_1) = d(A,B), \\ d(u_2, Tx_2) = d(A,B), \end{cases} \implies \alpha(u_1,u_2) \geq 0$$

for all $x_1, x_2, u_1, u_2 \in A$.

Definition 1.3 The mapping $T : A \to B$ is called a Suzuki type $\alpha^+\psi$-proximal contraction, if

$$\frac{1}{2} d^*(x, Tx) \leq d(x,y) \quad \Rightarrow \quad \alpha(x,y) + d(Tx, Ty) \leq \psi\big(M(x,y)\big) \tag{1.3}$$

for all $x, y \in A$, where $d^*(x, Tx) = d(x, Tx) - d(A,B)$, $\alpha : A \times A \to [-\infty,\infty)$, $\psi \in \Psi$, and

$$M(x,y) = \max\left\{d(x,y), \frac{d(x, Tx) + d(y, Ty)}{2} - d(A,B), \frac{d(x, Ty) + d(y, Tx)}{2} - d(A,B)\right\}.$$

In this manuscript, we propose new types of Suzuki and convex proximal maps to prove best proximity results. We also derive similar results in ordered metric spaces. Several interesting consequences of our obtained results are presented here.

2 Suzuki type $\alpha^+\psi$-proximal maps

Now we prove our first main result.

Theorem 2.1 *Suppose A and B are nonempty closed subsets of a complete metric space X with $A_0 \neq \emptyset$. Let $T : A \to B$ satisfy* (1.3) *together with the following assertions*:

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the weak P-property,*
(ii) *T is proximal α^+-admissible,*
(iii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B) \quad \text{and} \quad \alpha(x_0,x_1) \geq 0,$$

(iv) *T is continuous, or*
(v) *A is α-regular, that is, if $\{x_n\}$ is a sequence in A such that $\alpha(x_n,x_{n+1}) \geq 0$ and $x_n \to x \in A$ as $n \to \infty$, then $\alpha(x_n,x) \geq 0$ for all $n \in \mathbb{N}$.*

Then $\mathrm{Bpp}(T)$ *is nonempty.*

Proof Since $T(A_0) \subseteq B_0$, we have $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A,B).$$

As T satisfies (iii) and is proximal α^+-admissible, we obtain $\alpha(x_1,x_2) \geq 0$. That is,

$$d(x_2, Tx_1) = d(A,B), \qquad \alpha(x_1,x_2) \geq 0.$$

Again, since $T(A_0) \subseteq B_0$, there exists $x_3 \in A_0$ such that

$$d(x_3, Tx_2) = d(A,B).$$

Thus we have

$$d(x_2, Tx_1) = d(A,B), \qquad d(x_3, Tx_2) = d(A,B), \qquad \alpha(x_1,x_2) \geq 0.$$

Again since T is proximal α^+-admissible, so $\alpha(x_2,x_3) \geq 0$. Hence,

$$d(x_3, Tx_2) = d(A,B), \qquad \alpha(x_2,x_3) \geq 0.$$

We continue this process, to get

$$d(x_{n+1}, Tx_n) = d(A,B), \qquad \alpha(x_{n+1},x_n) \geq 0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{2.1}$$

By using the above observations we can write

$$\begin{aligned}
\frac{1}{2}d^*(x_{n-1}, Tx_{n-1}) &= \frac{1}{2}\bigl[d(x_{n-1}, Tx_{n-1}) - d(A,B)\bigr] \\
&\leq \frac{1}{2}\bigl[d(x_{n-1},x_n) + d(x_n, Tx_{n-1}) - d(A,B)\bigr] \\
&= \frac{1}{2}d(x_{n-1},x_n) \\
&\leq d(x_n,x_{n-1}).
\end{aligned}$$

That is,

$$\frac{1}{2}d^*(x_{n-1}, Tx_{n-1}) \leq d(x_n,x_{n-1}).$$

Now from (1.3) we get

$$d(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1},x_n) + d(Tx_{n-1}, Tx_n) \leq \psi\bigl(M(x_{n-1},x_n)\bigr). \tag{2.2}$$

By a simple calculation we obtain (see for details [2, 5]),

$$\begin{aligned}
M(x_{n-1},x_n) = {}& \max\biggl\{d(x_{n-1},x_n), \frac{d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)}{2} - d(A,B), \\
& \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2} - d(A,B)\biggr\} \\
\leq {}& \max\bigl\{d(x_{n-1},x_n), d(x_n,x_{n+1})\bigr\}.
\end{aligned} \tag{2.3}$$

By the weak P-property and (2.1) one obtains

$$d(x_n, x_{n+1}) \leq d(Tx_{n-1}, Tx_n) \quad \text{for all } n \in \mathbb{N}.$$

Equations (2.2) and (2.3) imply that

$$d(x_n, x_{n+1}) \leq \psi\big(M(x_{n-1}, x_n)\big) \leq \psi\big(\max\big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\big\}\big) \quad \text{for all } n \in \mathbb{N}. \tag{2.4}$$

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, from (2.1) one obtains

$$d(x_{n_0}, Tx_{n_0}) = d(x_{n_0+1}, Tx_{n_0}) = d(A, B),$$

that is, $x_{n_0} \in \mathrm{Bpp}(T)$. Thus, we suppose that

$$d(x_{n+1}, x_n) > 0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{2.5}$$

If, $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then (2.4) implies

$$d(x_n, x_{n+1}) \leq \psi\big(d(x_n, x_{n+1})\big) < d(x_n, x_{n+1}),$$

which is a contradiction. Thus,

$$d(x_n, x_{n+1}) \leq \psi\big(d(x_{n-1}, x_n)\big) < d(x_{n-1}, x_n) \quad \text{for all } n \in \mathbb{N}. \tag{2.6}$$

Applying the monotonicity of ψ, by induction, it follows from (2.6),

$$d(x_n, x_{n+1}) \leq \psi^n\big(d(x_1, x_0)\big) \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Suppose ϵ is any positive real number. Then there exists $N \in \mathbb{N}$ such that

$$\sum_{n \geq N} \psi^n\big(d(x_0, x_1)\big) < \epsilon \quad \text{for all } n \in \mathbb{N}.$$

If $m, n \in \mathbb{N}$ with $m > n \geq N$. We apply the triangle inequality to get

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \psi^k\big(d(x_0, x_1)\big) < \sum_{n \geq N} \psi^n\big(d(x_0, x_1)\big) < \epsilon.$$

Consequently $\lim_{m,n,\to+\infty} d(x_n, x_m) = 0$, which implies $\{x_n\}$ is Cauchy sequence. By completeness of X, $x_n \to z \in X$. If (iv) holds, then $Tx_n \to Tz$ as $n \to \infty$ and

$$d(A, B) = \lim_{n \to \infty} d(x_{n+1}, Tx_n) = d(z, Tz),$$

as required. Next, assume that (v) holds. Then $\alpha(x_n, z) \geq 0$.

If the following inequalities hold:

$$\frac{1}{2} d^*(x_n, Tx_n) > d(x_n, z) \quad \text{and} \quad \frac{1}{2} d^*(x_{n+1}, Tx_{n+1}) > d(x_{n+1}, z),$$

for some $n \in \mathbb{N}$, then by using (2.6) and definition of d^*, we obtain the following contradiction:

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, z) + d(x_{n+1}, z) \\ &< \frac{1}{2}\big[d^*(x_n, Tx_n) + d^*(x_{n+1}, Tx_{n+1})\big] \\ &\leq \frac{1}{2}\big[d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1}) - 2d(A,B)\big] \\ &= d(x_n, x_{n+1}). \end{aligned}$$

Consequently,, for any $n \in \mathbb{N}$, either

$$\frac{1}{2}d^*(x_n, Tx_n) \leq d(x_n, z) \quad \text{or} \quad \frac{1}{2}d^*(x_{n+1}, Tx_{n+1}) \leq d(x_{n+1}, z)$$

holds. Thus, we may pick a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\frac{1}{2}d^*(x_{n_k}, Tx_{n_k}) \leq d(x_{n_k}, z) \quad \text{and} \quad \alpha(x_{n_k}, x_{n_k+1}) \geq 0$$

for all $k \in \mathbb{N}$. By (1.3) we get

$$d(Tx_{n_k}, Tz) \leq \psi\big(M(x_{n_k}, z)\big). \tag{2.7}$$

Notice that

$$\begin{aligned} M(x_{n_k}, z) &= \max\left\{d(x_{n_k}, z), \frac{d(x_{n_k}, Tx_{n_k}) + d(z, Tz)}{2} - d(A,B), \right. \\ &\quad \left. \frac{d(x_{n_k}, Tz) + d(z, Tx_{n_k})}{2} - d(A,B)\right\} \\ &\leq \max\left\{d(x_{n_k}, z), \frac{d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, Tx_{n_k}) + d(z, Tz)}{2} - d(A,B), \right. \\ &\quad \left. \frac{d(x_{n_k}, z) + d(z, Tz) + d(z, x_{n_k+1}) + d(x_{n_k+1}, Tx_{n_k})}{2} - d(A,B)\right\} \\ &= \max\left\{d(x_{n_k}, z), \frac{d(x_{n_k}, x_{n_k+1}) + d(A,B) + d(z, Tz)}{2} - d(A,B), \right. \\ &\quad \left. \frac{d(x_{n_k}, z) + d(z, Tz) + d(z, x_{n_k+1}) + d(A,B)}{2} - d(A,B)\right\}, \end{aligned}$$

which implies

$$\lim_{k\to\infty} M(x_{n_k}, z) \leq \frac{d(z, Tz) - d(A,B)}{2}. \tag{2.8}$$

Further,

$$\begin{aligned} d(z, Tz) &\leq d(z, x_{n_k+1}) + d(x_{n_k+1}, Tx_{n_k}) + d(Tx_{n_k}, Tz) \\ &\leq d(z, x_{n_k+1}) + d(A,B) + d(Tx_{n_k}, Tz), \end{aligned}$$

which gives

$$d(z,Tz) - d(z,x_{n_k+1}) - d(A,B) \le d(Tx_{n_k},Tz). \tag{2.9}$$

As $k \to \infty$ in (2.9) we deduce

$$d(z,Tz) - d(A,B) \le \lim_{k\to\infty} d(Tx_{n_k},Tz). \tag{2.10}$$

Therefore from (2.7), (2.8), and (2.10)

$$\begin{aligned} d(z,Tz) - d(A,B) &\le \lim_{k\to\infty} d(Tx_{n_k},Tz) \\ &\le \psi\Bigl(\lim_{k\to\infty} M(x_{n_k},z)\Bigr) \le \psi\left(\frac{d(z,Tz) - d(A,B)}{2}\right). \end{aligned} \tag{2.11}$$

Now, if $d(z,Tz) - d(A,B) > 0$, then we get

$$d(z,Tz) - d(A,B) \le \psi\left(\frac{d(z,Tz) - d(A,B)}{2}\right) < \frac{d(z,Tz) - d(A,B)}{2}, \tag{2.12}$$

a contradiction. Hence, $d(z,Tz) = d(A,B)$ as desired. □

Example 2.1 Suppose $X = \mathbb{R}^2$ is equipped with the metric

$$d\bigl((p_1,p_2),(q_1,q_2)\bigr) = |p_1 - q_1| + |p_2 - q_2|,$$

for all $(p_1,p_2),(q_1,q_2) \in X$. Let $A_1 = \{(p,q)|p = 1, 0 \le q \le \frac{1}{2}\}$, $A_2 = \{(p,q)|p = 4, q \ge 5\}$, $A_3 = \{(p,q)p = 5, q \ge 4\}$, $A_4 = \{(p,q)|p = 3, q \ge 3\}$ and $A = A_1 \cup A_2 \cup A_3 \cup A_4$. Further define $B_1 = \{(p,q)|p = \frac{1}{2}, \frac{1}{2} \le q \le 1\}$, $B_2 = \{(p,q)|p = 0, q \le 4\}$, $B_3 = \{(P,q)|p = 4, q \le 0\}$, and $B = B_1 \cup B_2 \cup B_3$.

Note that $d(A,B) = 1$, $A_0 = \{(p,q)|p = 1, 0 \le q \le \frac{1}{2}\}$, and $B_0 = \{(p,q)|p = \frac{1}{2}, \frac{1}{2} \le q \le 1\}$. Let, for $x_1 = (1,u_1), x_2 = (1,u_2) \in A_0$ and $y_1 = (\frac{1}{2},v_1), y_2 = (\frac{1}{2},v_2) \in B_0$, us have $d(x_1,y_1) = d(A,B) = 1$ and $d(x_2,y_2) = d(A,B) = 1$. Then

$$\frac{1}{2} + |u_1 - v_1| = 1$$

and

$$\frac{1}{2} + |u_2 - v_2| = 1,$$

and so $|u_1 - v_1| = \frac{1}{2}$ and $|u_2 - v_2| = \frac{1}{2}$. Since $v_1, v_2 \ge u_1, u_2$, we have $v_1 = \frac{1}{2} + u_1$ and $v_2 = \frac{1}{2} + u_2$. This shows that $d(x_1,x_2) \le d(y_1,y_2)$. So (A,B) satisfy the weak P-property. Let $T : A \to B$ be defined by

$$T(p_1,p_2) = \begin{cases} (\frac{1}{2},\frac{1}{2}) & \text{if } p_1 = p_2, \\ (p_1,0) & \text{if } p_1 < p_2, \\ (0,p_2) & \text{if } p_1 > p_2. \end{cases}$$

Notice that $T(A_0) \subseteq B_0$.

Define the functions $\psi : [0,+\infty) \to [0,+\infty)$ and $\alpha : A \times A \to [-\infty,\infty)$ by

$$\psi(t) = \frac{8}{9}t \quad \text{and} \quad \alpha(p,q) = \begin{cases} 0, & \text{if } p,q \in \{(1,0),(4,5),(5,4)\}, \\ -\infty, & \text{otherwise.} \end{cases}$$

Assume that $\frac{1}{2}d^*(p,Tp) \leq d(p,q)$ and $\alpha(p,q) \geq 0$, for $p,q \in A$. Then

$$\begin{cases} p = (1,0), & q = (4,5) \quad \text{or} \\ p = (1,0), & q = (5,4) \quad \text{or} \\ q = (1,0), & p = (4,5) \quad \text{or} \\ q = (1,0), & p = (5,4). \end{cases}$$

Since $d(Tp,Tq) = d(Tq,Tp)$ and $M(p,q) = M(q,p)$ for all $p,q \in A$, we can suppose that

$$(p,q) = \bigl((1,0),(4,5)\bigr) \quad \text{or} \quad (p,q) = \bigl((1,0),(5,4)\bigr).$$

Now, we discuss the following cases:

(i) if $(p,q) = ((1,0),(4,5))$, then

$$d\bigl(T(1,0),T(4,5)\bigr) = 4 \leq 7 = \frac{7}{8} \cdot 8 = \psi\bigl(d\bigl((1,0),d(4,5)\bigr)\bigr) \leq \psi\bigl(M(p,q)\bigr);$$

(ii) if $(p,q) = ((1,0),(5,4))$, then

$$d\bigl(T(1,0),T(5,4)\bigr) = 4 \leq \frac{7}{8} \cdot 8 = \psi\bigl(d\bigl((1,0),(5,4)\bigr)\bigr) \leq \psi\bigl(M(p,q)\bigr).$$

Consequently, we have

$$\frac{1}{2}d^*(p,Tp) \leq d(p,q) \quad \Rightarrow \quad d(Tp,Tq) \leq \psi\bigl(M(p,q)\bigr).$$

Thus all the assumptions of Theorem 2.1 are satisfied and $\mathrm{Bpp}(T) = \{(1,0)\}$.

The next result can be deduced easily from Theorem 2.1.

Theorem 2.2 *Let X, A, A_0, and B be as in Theorem* 2.1. *Assume that $T : A \to B$ satisfies the assertions* (i)-(v) *in Theorem* 2.1 *and*

$$\alpha(p,q) + d(Tp,Tq) \leq \psi\bigl(M(p,q)\bigr)$$

holds for all $p,q \in A$. Then $\mathrm{Bpp}(T)$ *is nonempty.*

If $\alpha = 0$ on A, in Theorem 2.1, we obtain the following new result.

Corollary 2.1 *Suppose X, A, A_0, and B are as in Theorem* 2.1 *and $T : A \to B$ satisfies the following assumptions*:

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the weak P-property,*

(ii) *for all* $p, q \in A$ *with* $\frac{1}{2}d^*(p, Tp) \leq d(p,q)$ *we have*

$$d(Tp, Tq) \leq \psi\bigl(M(p,q)\bigr).$$

Then $\mathrm{Bpp}(T)$ *is nonempty.*

3 α^+ Θ-proximal maps

This section deals with best proximity theorems for Suzuki contractions involving the Θ function which was recently introduced by Jleli and Samet [27].

Let Δ_Θ denote the set of all functions $\Theta : (0,\infty) \to [1,\infty)$ with the following conditions:

(Θ_1) Θ is increasing;
(Θ_2) for all sequences $\{\alpha_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} \Theta(\alpha_n) = 1$;
(Θ_3) there exist $0 < r < 1$ and $\ell \in (0,\infty]$ such that $\lim_{n\to 0^+} \frac{\Theta(t)-1}{t^r} = \ell$.

Definition 3.1 A mapping $T : A \to B$ is called a Suzuki type $\alpha^+\Theta$-proximal contraction, if for all $x, y \in A$ with $\frac{1}{2}d^*(x, Tx) \leq d(x,y)$ and $d(Tx, Ty) > 0$,

$$\Rightarrow \quad \alpha(x,y) + \Theta\bigl(d(Tx, Ty)\bigr) \leq \bigl[\Theta\bigl(M(x,y)\bigr)\bigr]^k, \tag{3.1}$$

where $\alpha : A \times A \to [-\infty,\infty)$, $0 \leq k < 1$, and $\Theta \in \Delta_\Theta$.

Theorem 3.1 *Assume that* X, A, A_0, *and* B *are as in Theorem* 2.1 *and* $T : A \to B$ *satisfy* (3.1) *and the assertions* (i)-(v) *in Theorem* 2.1. *Then* $\mathrm{Bpp}(T)$ *is nonempty.*

Proof As in the proof of Theorem 2.1, we can construct a sequence $\{x_n\}$ satisfying

$$d(x_{n+1}, Tx_n) = d(A,B), \qquad \alpha(x_n, x_{n+1}) \geq 0, \quad n \in \mathbb{N} \cup \{0\} \tag{3.2}$$

and

$$\frac{1}{2}d^*(x_{n-1}, Tx_{n-1}) \leq d(x_n, x_{n-1}) \quad \text{and} \quad d(x_n, x_{n-1}) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Now (3.1) implies

$$\Theta\bigl(d(Tx_{n-1}, Tx_n)\bigr) \leq \alpha(x_{n-1}, x_n) + \Theta\bigl(d(Tx_{n-1}, Tx_n)\bigr) \leq \bigl[\Theta\bigl(M(x_{n-1}, x_n)\bigr)\bigr]^k. \tag{3.3}$$

In Theorem 2.1 we obtain

$$M(x_{n-1}, x_n) \leq \max\bigl\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\bigr\} \tag{3.4}$$

and

$$d(x_n, x_{n+1}) \leq d(Tx_{n-1}, Tx_n) \quad \text{for all } n \in \mathbb{N}.$$

Therefore from (3.3) and (3.4) we get

$$\begin{aligned}\Theta\big(d(x_n,x_{n+1})\big) &\le \Theta\big(d(Tx_{n-1},Tx_n)\big)\\ &\le \big[\Theta\big(M(x_{n-1},x_n)\big)\big]^k\\ &\le \big[\Theta\big(\max\big\{d(x_{n-1},x_n),d(x_n,x_{n+1})\big\}\big)\big]^k \quad \text{for all } n\in\mathbb{N}.\end{aligned} \tag{3.5}$$

Now if $\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$, then from (3.5) we get

$$\Theta\big(d(x_n,x_{n+1})\big) \le \big[\Theta\big(d(x_n,x_{n+1})\big)\big]^k < \Theta\big(d(x_n,x_{n+1})\big),$$

which is a contradiction. Hence,

$$\Theta\big(d(x_n,x_{n+1})\big) \le \big[\Theta\big(d(x_{n-1},x_n)\big)\big]^k \quad \text{for all } n\in\mathbb{N}. \tag{3.6}$$

Therefore,

$$\begin{aligned}1 &\le \Theta\big(d(x_n,x_{n+1})\big) \le \Theta\big(d(x_{n-1},x_n)\big)^k\\ &\le \Theta\big(d(x_{n-2},x_{n-1})\big)^{k^2} \le \cdots \le \Theta\big(d(x_0,x_1)\big)^{k^n}.\end{aligned} \tag{3.7}$$

Taking the limit as $n\to\infty$ in (3.7) we have

$$\lim_{n\to\infty}\Theta\big(d(x_n,x_{n+1})\big) = 1,$$

and since $\Theta\in\Delta_\Theta$ we obtain

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{3.8}$$

Again since $\Theta\in\Delta_\Theta$, there exist $0<r<1$ and $0<\ell\le\infty$ with

$$\lim_{n\to\infty}\frac{\Theta(d(x_n,x_{n+1}))-1}{[d(x_n,x_{n+1})]^r} = \ell. \tag{3.9}$$

Assume that $\ell<\infty$. Let $C=\frac{\ell}{2}$. Thus there exists $n_0\in\mathbb{N}$ such that

$$\left|\frac{\Theta(d(x_n,x_{n+1}))-1}{[d(x_n,x_{n+1})]^r}-\ell\right| \le C \quad \text{for all } n\ge n_0,$$

hence

$$\frac{\Theta(d(x_n,x_{n+1}))-1}{[d(x_n,x_{n+1})]^r} \ge \ell - C = C \quad \text{for all } n\ge n_0,$$

and so

$$n\big[d(x_n,x_{n+1})\big]^r \le nD\big[\Theta\big(d(x_n,x_{n+1})\big)-1\big] \quad \text{for all } n\ge n_0,$$

where $D = \frac{1}{C}$. If $\ell = \infty$, then there exists $n_0 \in \mathbb{N}$,

$$\frac{\Theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} \geq C \quad \text{for all } n \geq n_0,$$

which implies

$$n\bigl[d(x_n, x_{n+1})\bigr]^r \leq nD\bigl[\Theta\bigl(d(x_n, x_{n+1})\bigr) - 1\bigr] \quad \text{for all } n \geq n_0,$$

where $D = \frac{1}{C}$. Hence, in all cases there exist $D > 0$ and $n_0 \in \mathbb{N}$ such that

$$n\bigl[d(x_n, x_{n+1})\bigr]^r \leq nD\bigl[\Theta\bigl(d(x_n, x_{n+1})\bigr) - 1\bigr] \quad \text{for all } n \geq n_0.$$

Now (3.7) implies

$$n\bigl[d(x_n, x_{n+1})\bigr]^r \leq nD\bigl[\Theta\bigl(d(x_0, x_1)\bigr)^{k^n} - 1\bigr] \quad \text{for all } n \geq n_0,$$

and on letting $n \to \infty$ we obtain

$$\lim_{n\to\infty} n\bigl[d(x_n, x_{n+1})\bigr]^r = 0. \tag{3.10}$$

It follows from (3.10) that there is $n_1 \in \mathbb{N}$ with

$$n\bigl[d(x_n, x_{n+1})\bigr]^r \leq 1$$

for all $n > n_1$. This implies

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/r}}$$

for all $n > n_1$. If $m > n > n_1$, then

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/r}}.$$

Since $0 < r < 1$, $\sum_{i=n}^{\infty} \frac{1}{i^{1/r}}$ is convergent. Thus, $d(x_n, x_m) \to 0$ as $m, n \to \infty$, which shows that $\{x_n\}$ is a Cauchy sequence. Thus there is $z \in X$ such that $x_n \to z$ as $n \to \infty$. Assume that (iv) holds. Thus $Tx_n \to Tz$ as $n \to \infty$, which implies

$$d(A, B) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(z, Tz),$$

as required. Next, assume that (v) holds. As in the proof of Theorem 2.1 we can deduce there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying

$$\frac{1}{2} d^*(x_{n_k}, Tx_{n_k}) \leq d(x_{n_k}, z) \quad \text{and} \quad \alpha(x_{n_k}, x_{n_k+1}) \geq 0$$

for all $k \in \mathbb{N}$. By (3.1) we get

$$\Theta\bigl(d(Tx_{n_k}, Tz)\bigr) \leq \bigl[\Theta\bigl(M(x_{n_k}, z)\bigr)\bigr]^k < \Theta\bigl(M(x_{n_k}, z)\bigr),$$

which implies

$$d(Tx_{n_k}, Tz) \leq M(x_{n_k}, z).$$

As in Theorem 2.1 we obtain

$$\lim_{k\to\infty} M(x_{n_k}, z) \leq \frac{d(z, Tz) - d(A, B)}{2}$$

and

$$d(z, Tz) - d(A, B) \leq \lim_{k\to\infty} d(Tx_{n_k}, Tz);$$

therefore,

$$d(z, Tz) - d(A, B) \leq \frac{d(z, Tz) - d(A, B)}{2},$$

which is a contradiction when $d(z, Tz) > d(A, B)$. So, $d(z, Tz) = d(A, B)$, that is, $\mathrm{Bpp}(T)$ is nonempty. □

Corollary 3.1 *Suppose X, A, A_0, and B are as in Theorem* 2.1 *and $T : A \to B$ satisfies the assertions* (i)-(v) *in Theorem* 3.1. *If*

$$\alpha(p, q) + \Theta\big(d(Tp, Tq)\big) \leq \big[\Theta\big(M(p, q)\big)\big]^k$$

holds for all $p, q \in A$ where $\alpha : A \times A \to [-\infty, \infty)$ and $\Theta \in \Delta_\Theta$, then $\mathrm{Bpp}(T)$ *is nonempty.*

Corollary 3.2 *Suppose X, A, A_0, and B are as in Theorem* 2.1 *and $T : A \to B$ satisfies the following assertions*:

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the weak P-property*;

(ii) *for all $p, q \in A$ with $\frac{1}{2}d^*(p, Tp) \leq d(p, q)$ we have*

$$\Theta\big(d(Tp, Tq)\big) \leq \big[\Theta\big(M(p, q)\big)\big]^k,$$

where $\Theta \in \Delta_\Theta$.

Then $\mathrm{Bpp}(T)$ *is nonempty.*

Remark 3.1

(a) The results proved in the above sections generalize the corresponding results of Zhang *et al.* [26], Suzuki [22], Hussain *et al.* [2, 3] and many others.

(b) Several more best proximity point theorems can be obtained using more choices for the function Θ, and some other concrete choices of α and $\psi \in \Psi$ in the results of the above sections.

4 Best proximity results for convex type contractions

We discuss two new and general types of proximal convex contractions and establish corresponding best proximity results (see also [8]).

Definition 4.1 Suppose $T : A \to B$ be a mapping where A and B are two nonempty subsets of a metric space X. Then T is an

(1) α^+-convex proximal contractive map of the first type if, for $x, y, u, u^*, v \in A$,

$$\left.\begin{array}{l} \alpha(x,y) \geq 0, \\ d(u,Tx) = d(A,B), \\ d(u^*,Tu) = d(A,B), \\ d(v,Ty) = d(A,B), \\ d(v^*,Tv) = d(A,B) \end{array}\right\} \quad \Longrightarrow \quad d\bigl(u^*,v^*\bigr) \leq r_1 d(u,v) + r_2 d(x,y) \tag{4.1}$$

holds where $r_1, r_2 \geq 0$, $r_1 + r_2 < 1$;

(2) α^+-convex proximal contractive map of second type if for $x, y, u, u^*, v \in A$,

$$\left.\begin{array}{l} \alpha(x,y) \geq 0, \\ d(u,Tx) = d(A,B), \\ d(u^*,Tu) = d(A,B), \\ d(v,Ty) = d(A,B), \\ d(v^*,Tv) = d(A,B) \end{array}\right\}$$

$$\Longrightarrow \quad d\bigl(u^*,v^*\bigr) \leq r_1 d(x,u) + r_2 d\bigl(u,u^*\bigr) + r_3 d(y,v) + r_4 d\bigl(v,v^*\bigr) \tag{4.2}$$

holds where $r_1, r_2, r_3, r_4 \geq 0$, $r_1 + r_2 + r_3 + r_4 < 1$.

Theorem 4.1 *Suppose X, A, A_0, and B are as in Theorem* 2.1 *and $T : A \to B$ satisfy* (4.1) *with $T(A_0) \subseteq B_0$ and the conditions* (ii)-(iv) *in Theorem* 2.1. *Then* $\mathrm{Bpp}(T)$ *is nonempty. Moreover,* $\mathrm{Bpp}(T)$ *is a singleton if $\alpha(x,y) \geq 0$ for all $x, y \in \mathrm{Bpp}(T)$.*

Proof Following the technique of the proof in Theorem 2.1, one can find a sequence $\{x_n\}$ such that

$$d(x_{n+1}, Tx_n) = d(A,B), \qquad \alpha(x_n, x_{n+1}) \geq 0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{4.3}$$

For

$$x = x_{n-2}, \qquad u = x_{n-1}, \qquad u^* = x_n, \qquad y = x_{n-1}, \qquad v = x_n \quad \text{and} \quad v^* = x_{n+1},$$

equation (4.1) implies

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \alpha(x_{n-2}, x_{n-1}) + d(x_n, x_{n+1}) \\ &\leq r_1 d(x_{n-1}, x_n) + r_2 d(x_{n-2}, x_{n-1}). \end{aligned}$$

By taking $\vartheta = d(x_2, x_1) + d(x_1, x_0)$ and $r = r_1 + r_2$ we have

$$d(x_m, x_{m+1}) \leq r^J \vartheta,$$

where $m = 2l$ or $m = 2l + 1$. Let $m = 2l$. Then for $n = 2p$ with $p > 2$ and $l \geq 1$ and $m < n$ we deduce

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+3}) + \cdots + d(x_{n-1}, x_n) \\ &= d(x_2 l, x_{2l+1}) + d(x_{2l+1}, x_{2l+2}) + d(x_{2l+2}, x_{2l+3}) + \cdots + d(x_{2p-1}, x_{2p}) \\ &\leq r^l \vartheta + r^l \vartheta + r^{l+1} \vartheta + \cdots + r^{p-1} \vartheta \\ &= 2r^l \vartheta + 2r^{l+1} \vartheta + \cdots + 2r^{p-1} \vartheta \leq \frac{2r^l}{1-r} \vartheta. \end{aligned}$$

Similarly, for $m = 2l$ and $n = 2p + 1$ with $p \geq 1$ and $l \geq 1$ and $m < n$ we get

$$d(x_m, x_n) \leq \frac{2r^l}{1-r} \vartheta.$$

Now, assume that $m = 2l + 1$. Then for $n = 2p$ with $p \geq 2$ and $l \geq 1$ and $m < n$ we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+3}) + \cdots + d(x_{n-1}, x_n) \\ &= d(x_2 l, x_{2l+1}) + d(x_{2l+1}, x_{2l+2}) + d(x_{2l+2}, x_{2l+3}) + \cdots + d(x_{2p-1}, x_{2p}) \\ &\leq r^l \vartheta + r^{l+l} \vartheta + r^{l+1} \vartheta + \cdots + r^p \vartheta \\ &\leq 2r^l \vartheta + 2r^{l+1} \vartheta + \cdots + 2r^p \vartheta \leq \frac{2r^l}{1-r} \vartheta. \end{aligned}$$

Similarly, for $m = 2l + 1$ and $n = 2p + 1$ with $p \geq 1$ and $l \geq 1$ and $m < n$ we deduce

$$d(x_m, x_n) \leq \frac{2r^l}{1-r} \vartheta.$$

Hence, for all $m, n \in \mathbb{N}$ with $m < n$ we have

$$d(x_m, x_n) \leq \frac{2r^l}{1-r} \vartheta,$$

which letting $l \to \infty$ implies $d(x_m, x_n) \to 0$. That is, $\{x_n\}$ is a Cauchy sequence and hence there is $z \in X$ such that $x_n \to z$ as $n \to \infty$. Continuity of T implies $Tx_n \to Tz$ as $n \to \infty$. Hence,

$$d(A, B) = \lim_{n \to \infty} d(x_{n+1}, Tx_n) = d(z, Tz).$$

Let $w, z \in \mathrm{Bpp}(T)$ with $w \neq z$. Then $\alpha(w, z) \geq 0$. Now with

$$w = x = u = u^*, \qquad z = y = v = v^*$$

(4.1) implies

$$d(w, z) \leq \alpha(w, z) + d(w, z) \leq r_1 d(w, z) + r_2 d(w, z),$$

which is a contradiction and hence $d(w, z) = 0$. *i.e.*, $w = z$. Thus $\mathrm{Bpp}(T)$ is a singleton. □

By taking, $\alpha(x, y) = 0$, in the above theorem we deduce the following result.

Corollary 4.1 *Suppose X, A, A_0, and B are as in Theorem* 2.1 *and $T : A \to B$ is a continuous convex proximal contractive mapping of the first type satisfying $T(A_0) \subseteq B_0$. Then* $\mathrm{Bpp}(T)$ *is nonempty.*

Theorem 4.2 *Suppose X, A, A_0, and B are as in Theorem* 2.1 *and $T : A \to B$ is an α^+-convex proximal contractive map of second type with $T(A_0) \subseteq B_0$ and satisfying conditions* (ii)-(iv) *in Theorem* 2.1. *Then* $\mathrm{Bpp}(T)$ *is nonempty. Moreover,* $\mathrm{Bpp}(T)$ *is a singleton if $\alpha(x,y) \geq 0$ for all $x, y \in \mathrm{Bpp}(T)$.*

Proof As in Theorem 2.1, one may find a sequence $\{x_n\}$ such that

$$d(x_{n+1}, Tx_n) = d(A,B), \qquad \alpha(x_n, x_{n+1}) \geq 0 \quad \text{for all } n \in \mathbb{N} \cup 0. \tag{4.4}$$

For

$$x = x_{n-2}, \qquad u = x_{n-1}, \qquad u^* = x_n, \qquad y = x_{n-1}, \qquad v = x_n \quad \text{and} \quad v^* = x_{n+1}$$

with $r = r_1 + r_2 + r_3$, $\eta = 1 - r_4$, and $\vartheta = d(x_2, x_1) + d(x_1, x_0)$, (4.2) implies

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \alpha(x_{n-2}, x_{n-1}) + d(x_n, x_{n+1}) \\ &\leq r_1 d(x_{n-2}, x_{n-1}) + r_2 d(x_{n-1}, x_n) + r_3 d(x_{n-1}, x_n) + r_4 d(x_n, x_{n+1}). \end{aligned} \tag{4.5}$$

Now if $n = 2$, then

$$\begin{aligned} d(x_2, x_3) &\leq r_1 d(x_0, x_1) + r_2 d(x_1, x_2) + r_3 d(x_1, x_2) + r_4 d(x_2, x_3) \\ &\leq r\vartheta + r_4 d(x_2, x_3), \end{aligned}$$

which implies $(1 - r_4) d(x_2, x_3) \leq r\vartheta$. That is, $d(x_2, x_3) \leq \frac{r}{\eta}\vartheta$. Again by taking $n = 3$ in (4.5) we get

$$\begin{aligned} d(x_3, x_4) &\leq r_1 d(x_1, x_2) + r_2 d(x_2, x_3) + r_3 d(x_2, x_3) + r_4 d(x_3, x_4) \\ &\leq r\vartheta + r_4 d(x_3, x_4), \end{aligned}$$

which implies $d(x_3, x_4) \leq \frac{r}{\eta}\vartheta$. Similarly, $d(x_4, x_5) \leq (\frac{r}{\eta})^2\vartheta$ and $d(x_5, x_6) \leq (\frac{r}{\eta})^2\vartheta$. By continuing this process, we get $d(x_m, x_{m+1}) \leq (\frac{r}{\eta})^l\vartheta$ when $m = 2l$ or $m = 2l + 1$. Let $m = 2l$. Then for $n = 2p$ with $p > 2$ and $l \geq 1$ and $m < n$ we deduce

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+3}) + \cdots + d(x_{n-1}, x_n) \\ &= d(x_{2l}, x_{2l+1}) + d(x_{2l+1}, x_{2l+2}) + d(x_{2l+2}, x_{2l+3}) + \cdots + d(x_{2p-1}, x_{2p}) \\ &\leq \left(\frac{r}{\eta}\right)^l \vartheta + \left(\frac{r}{\eta}\right)^l \vartheta + \left(\frac{r}{\eta}\right)^{l+1} \vartheta + \cdots + \left(\frac{r}{\eta}\right)^{p-1} \vartheta \\ &= 2\left(\frac{r}{\eta}\right)^l \vartheta + 2\left(\frac{r}{\eta}\right)^{l+1} \vartheta + \cdots + 2\left(\frac{r}{\eta}\right)^{p-1} \vartheta \leq \frac{2(\frac{r}{\eta})^l}{1 - (\frac{r}{\eta})}\vartheta. \end{aligned}$$

Similarly, for $m = 2l$ and $n = 2p + 1$ with $p \geq 1$ and $l \geq 1$ and $m < n$ we get

$$d(x_m, x_n) \leq \frac{2(\frac{r}{\eta})^l}{1 - (\frac{r}{\eta})}\vartheta.$$

Now, assume that $m = 2l + 1$. Then for $n = 2p$ with $p \geq 2$ and $l \geq 1$ and $m < n$ we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+3}) + \cdots + d(x_{n-1}, x_n) \\ &= d(x_2 l, x_{2l+1}) + d(x_{2l+1}, x_{2l+2}) + d(x_{2l+2}, x_{2l+3}) + \cdots + d(x_{2p-1}, x_{2p}) \\ &\leq \left(\frac{r}{\eta}\right)^l \vartheta + \left(\frac{r}{\eta}\right)^{l+l} \vartheta + \left(\frac{r}{\eta}\right)^{l+1} \vartheta + \cdots + \left(\frac{r}{\eta}\right)^p \vartheta \\ &\leq 2\left(\frac{r}{\eta}\right)^l \vartheta + 2\left(\frac{r}{\eta}\right)^{l+1} \vartheta + \cdots + 2\left(\frac{r}{\eta}\right)^p \vartheta \leq \frac{2(\frac{r}{\eta})^l}{1 - (\frac{r}{\eta})}\vartheta. \end{aligned}$$

Similarly, for $m = 2l + 1$ and $n = 2p + 1$ with $p \geq 1$ and $l \geq 1$ and $m < n$ we deduce

$$d(x_m, x_n) \leq \frac{2(\frac{r}{\eta})^l}{1 - (\frac{r}{\eta})}\vartheta.$$

Hence, for all $m, n \in \mathbb{N}$ with $m < n$ we have

$$d(x_m, x_n) \leq \frac{2(\frac{r}{\eta})^l}{1 - (\frac{r}{\eta})}\vartheta.$$

Letting $l \to \infty$, we get $d(x_m, x_n) \to 0$. That is, $\{x_n\}$ is a Cauchy sequence and so there is $z \in X$ such that $x_n \to z$ as $n \to \infty$. BY continuity of T, $Tx_n \to Tz$ as $n \to \infty$. Hence,

$$d(A, B) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(z, Tz).$$

The proof that $\mathrm{Bpp}(T)$ is a singleton is similar to the above theorem and so is omitted. □

By taking, $\alpha(x, y) = 0$, in the above theorem, we deduce the following result.

Corollary 4.2 *Suppose X, A, A_0, and B are as in Theorem* 2.1 *and $T : A \to B$ is a continuous convex proximal contractive mapping of the second type satisfying $T(A_0) \subseteq B_0$. Then* $\mathrm{Bpp}(T)$ *is a singleton.*

5 Results in partially ordered sets

In this section, we deduce best proximity theorems for Suzuki and convex proximal maps in partially ordered sets.

Definition 5.1 [18] Let $(X,d,\preceq)$ be a partially ordered metric space. A map $T:A\to B$ is called proximally order-preserving if, for all $x_1,x_2,u_1,u_2\in A$,

$$\begin{cases} x_1\preceq x_2, \\ d(u_1,Tx_1)=d(A,B), \\ d(u_2,Tx_2)=d(A,B). \end{cases} \implies u_1\preceq u_2.$$

Definition 5.2 A mapping $T:A\to B$ is said to be Suzuki type ordered ψ-proximal contraction, if for $x,y\in A$

$$\frac{1}{2}d^*(x,Tx)\le d(x,y) \quad \text{and} \quad x\preceq y \quad \Rightarrow \quad d(Tx,Ty)\le \psi\bigl(M(x,y)\bigr).$$

Similarly, we can define order versions of other maps discussed in above sections.

Theorem 5.1 *Let A and B be nonempty closed subsets of a complete partially ordered metric space $(X,d,\preceq)$ such that A_0 is nonempty and $T:A\to B$ be a Suzuki type ordered ψ-proximal map satisfying the following assertions*:

(i) *$T(A_0)\subseteq B_0$ and (A,B) satisfies the weak P-property,*
(ii) *T is proximally ordered-preserving,*
(iii) *there are x_0 and x_1 in A_0 such that*

$$d(x_1,Tx_0)=d(A,B) \quad \textit{and} \quad x_0\preceq x_1,$$

(iv) *T is continuous, or*
(v) *if $\{x_n\}$ is a increasing sequence in A with $x_n\to x\in A$ as $n\to\infty$, then $x_n\preceq x$ for all $n\in\mathbb{N}$.*

Then $\operatorname{Bpp}(T)$ *is nonempty.*

Proof Define $\alpha: A\times A\to[-\infty,+\infty)$ by

$$\alpha(x,y)=\begin{cases} 0, & \text{if } x\preceq y, \\ -\infty, & \text{otherwise.} \end{cases}$$

T is proximal α^+-admissible mapping as follows.

$$\begin{cases} \alpha(x,y)\ge 0, \\ d(u,Tx)=d(A,B), \\ d(v,Ty)=d(A,B) \end{cases}$$

implies

$$\begin{cases} x\preceq y, \\ d(u,Tx)=d(A,B), \\ d(v,Ty)=d(A,B). \end{cases}$$

Since T is proximally ordered-preserving, $u \preceq v$, that is, $\alpha(u,v) \geq 0$. Further, by (ii) we have

$$d(x_1, Tx_0) = d(A,B) \quad \text{and} \quad \alpha(x_0,x_1) \geq 0.$$

Note that, if $x \preceq y$, then $\alpha(x,y) = 0$ and otherwise, $\alpha(x,y) = -\infty$. Since T is a Suzuki type ordered ψ-proximal map, we have the following inequality:

$$\frac{1}{2}d^*(x, Tx) \leq d(x,y), \qquad \alpha(x,y) \geq 0 \quad \Rightarrow \quad \alpha(x,y) + d(Tx, Ty) \leq \psi\bigl(M(x,y)\bigr).$$

Further, let $\{x_n\}$ be a sequence, such that $\alpha(x_n,x_{n+1}) \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$ with $x_n \to x$ as $n \to \infty$, then $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ with $x_n \to x$ as $n \to \infty$. That is, $\{x_n\}$ is an increasing sequence, with $x_n \to x$ as $n \to \infty$. So from (v) we have $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$. That is, $\alpha(x_n,x_n) \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$. Thus all the assumptions of Theorem 2.1 hold and $\mathrm{Bpp}(T)$ is nonempty. □

Similarly we can prove the following theorems.

Theorem 5.2 *Suppose X, A, A_0, and B are as in Theorem* 5.1 *and $T : A \to B$ is a Suzuki type ordered Θ-proximal contraction where we have the assumptions* (i)-(v) *of Theorem* 5.1. *Then* $\mathrm{Bpp}(T)$ *is nonempty.*

Theorem 5.3 *Suppose X, A, A_0, and B are as in Theorem* 5.1 *and $T : A \to B$ is an ordered convex proximal contractive mapping of the first type* (*or the second type*) *satisfying $T(A_0) \subseteq B_0$ and the conditions* (ii)-(iv) *of Theorem* 5.1. *Then* $\mathrm{Bpp}(T)$ *is nonempty. Moreover,* $\mathrm{Bpp}(T)$ *is singleton if $\alpha(x,y) \geq 0$ for all $x, y \in$* $\mathrm{Bpp}(T)$.

6 Application to fixed point theory

Here we deduce certain new and general fixed point results for Suzuki and convex contractions. Our results contain properly the main theorem due to Suzuki [24] and many of its extensions [23] (see also [28]).

If $A = B = X$, then definition (1.2) reduces to the following.

Definition 6.1 A map $T : X \to X$, is called α^+-admissible if

$$\alpha(x,y) \geq 0 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 0$$

for all $x, y \in X$.

Definition 6.2 A mapping $T : X \to X$ is called a Suzuki type $\alpha^+\psi$-contraction, if

$$\frac{1}{2}d(x, Tx) \leq d(x,y) \quad \Rightarrow \quad \alpha(x,y) + d(Tx, Ty) \leq \psi\bigl(M(x,y)\bigr)$$

for all $x, y \in X$.

Definition 6.3 A mapping $T : X \to X$ is called a Suzuki type $\alpha^+\Theta$-contraction, if

$$\frac{1}{2}d(x,Tx) \le d(x,y) \quad \Rightarrow \quad \alpha(x,y) + \Theta\bigl(d(Tx,Ty)\bigr) \le \bigl[\Theta\bigl(M(x,y)\bigr)\bigr]^k$$

for all $x, y \in X$, $\alpha : X \times X \to [-\infty, \infty)$ and $\Theta \in \Delta_\Theta$.

Now from Theorems 2.1, 2.2 and 3.1, we derive the following new fixed point theorems.

Theorem 6.1 *Assume that X is a complete metric space and $T : X \to X$ is a Suzuki type $\alpha^+\psi$-contraction with the following assertions*:
(i) *T is α^+-admissible,*
(ii) *there is x_0 with $\alpha(x_0, Tx_0) \ge 0$,*
(iii) *T is continuous or,*
(iv) *X is α-regular.*
Then $F(T)$ is nonempty.

Theorem 6.2 *Assume that X is a complete metric space and $T : X \to X$ is a Suzuki type $\alpha^+\Theta$-contraction satisfying the conditions* (i)-(iv) *in Theorem* 6.1. *Then $F(T)$ is nonempty.*

Theorem 6.3 *Suppose X is a complete metric space and $T : X \to X$ is an α^+-convex contractive mapping of the first* (*or the second*) *type with the following assertions*:
(i) *T is α^+-admissible,*
(ii) *there exists x_0 such $\alpha(x_0, Tx_0) \ge 0$,*
(iii) *T is continuous.*
Then $F(T)$ is nonempty.

By taking $\alpha(x,y) = 0$ for all $x, y \in X$ in the above theorem, we obtain the main results of Istrăţescu [29] as corollaries.

Definition 6.4 A mapping $T : X \to X$ is called a Suzuki type ordered ψ-contraction, if

$$\frac{1}{2}d(x,Tx) \le d(x,y) \quad \text{and} \quad x \preceq y \quad \Rightarrow \quad d(Tx,Ty) \le \psi\bigl(M(x,y)\bigr)$$

for $x, y \in X$, $\psi \in \Psi$.

Definition 6.5 A mapping $T : X \to X$ is called a Suzuki type ordered Θ-contraction, if

$$\frac{1}{2}d(x,Tx) \le d(x,y) \quad \text{and} \quad x \preceq y \quad \Rightarrow \quad \Theta\bigl(d(Tx,Ty)\bigr) \le \bigl[\Theta\bigl(M(x,y)\bigr)\bigr]^k$$

for $x, y \in X$ and $\Theta \in \Delta_\Theta$.

Theorem 6.4 *Suppose $(X, d, \preceq)$ is a complete partially ordered metric space and $T : X \to X$ is a Suzuki type ordered ψ-contraction with the following assertions*:
(i) *T is an increasing mapping,*
(ii) *there is $x_0 \in X$ such that $x_0 \preceq Tx_0$,*
(iii) *T is continuous or,*
(iv) *X is regular.*
Then $F(T)$ is nonempty.

Theorem 6.5 *Suppose $(X, d, \preceq)$ is a complete partially ordered metric space and $T : X \to X$ is a Suzuki type ordered Θ-contraction satisfying the conditions* (i)-(iv) *in Theorem* 6.4. *Then $F(T)$ is nonempty.*

Theorem 6.6 *Assume that $(X, d, \preceq)$ is a complete partially ordered metric space and $T : X \to X$ is an ordered convex contractive mapping of the first* (*or the second*) *type with the following assertions*:

(i) *T is increasing,*

(ii) *there is x_0 such $x_0 \preceq Tx_o$,*

(iii) *T is continuous.*

Then $F(T)$ is singleton.

Remark 6.1 Several more fixed point theorems can be obtained using more choices for function Θ, and/or some other concrete choices of α and $\psi \in \Psi$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approve the final manuscript.

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