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A projection method for approximating fixed points of quasinonexpansive mappings in Hadamard spaces

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Abstract
This work is devoted to analyzing the feasibility study of a Moudafi viscosity projection method with a weak contraction for a finite family of quasinonexpansive mappings in a Hadamard space. To this end, we need to construct a countable family of nonexpansive mappings satisfying AKTT condition with a weak contraction by choosing an appropriate control sequence under certain conditions.

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1 Introduction

Let C be a nonempty subset of a metric space (X,d). Suppose that, for each $x \in X$, there exists a unique point $P_C x \in C$ such that $d(x, P_C x) = d(x, C) = \inf_{y\in C} d(x,y)$. Then, the mapping P_C of X onto C is called the metric projection.

The well-known Banach contraction principle is an important tool in the theory of metric spaces; it guarantees the existence and uniqueness of fixed points of certain self-mappings of metric spaces. One generalization of the contraction principle for *weak contractions* is obtained by Alber and Guerre-Delabriere [1] in Hilbert spaces. A mapping $f : X \to X$ is called a φ-*weak contraction* if

$$d\big(f(x), f(y)\big) \le d(x,y) - \varphi\big(d(x,y)\big), \quad x, y \in X,$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function with $\varphi(t) = 0$ if and only if $t = 0$.

Let $T : C \to X$ be a mapping. If $d(Tx, Ty) \le d(x,y)$ for all $x, y \in C$, then T is nonexpansive. We denote by $\mathfrak{F}(T)$ the set o fixed points of T. The mapping T is *quasinonexpansive* if $\mathfrak{F}(T)$ is nonempty and

$$d(Tx, y) \le d(x,y), \quad x \in C, y \in \mathfrak{F}(T).$$

A point $p \in C$ is said to be a *strongly asymptotic fixed point* [2] of T if there exists a sequence $\{x_n\}$ in C that converges strongly to p and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. We denote by

$\widetilde{\mathfrak{F}}(T)$ the set of strongly asymptotic fixed points of T. It is known that the fixed point set of a quasinonexpansive mapping defined on a CAT(0) space (see Section 2 for the definition) is closed and convex.

Approximation methods for finding specific fixed points of a family of nonexpansive mappings in Hilbert, Banach, and geodesic metric spaces have been studied by many researchers; see, *e.g.*, [3–9] and the references therein. One well-known method, called the shrinking projection method, was first proposed by Takahashi *et al.* [10] and has been applied to a variety of approximation problems; see, *e.g.*, [11, 12]. In particular, Kimura and Takahashi [11] applied this method to the zero-point problem for a maximal monotone operator defined in a Banach space and obtained strong convergence theorems. To generate the iterative sequence by the shrinking projection method, they use the metric projection onto a closed convex set C_n for each $n \in \mathbb{N}$. It is noticeable that the larger the integer n, the more complicated the shape of C_n. Hence, the calculation of the projection is tedious as n gets larger. In 2011, Kimura *et al.* [2] overcome this difficulty and introduce the so-called averaged projection method of Halpern type for a family of quasinonexpansive mappings by combining the Halpern iteration. They still use the metric projection approach; nevertheless, the subsets corresponding to these projections have simpler shapes than the classical ones. Let us denote by $\mathfrak{F}(\mathfrak{T})$ the common fixed point set of all mappings in a family $\mathfrak{T}$. Their theorem is stated as follows.

Theorem 1.1 (Kimura *et al.* [2], Theorem 3.1) *Let C be a closed convex subset of a Hilbert space H, $\mathfrak{T} = \{T_j : j = 1,\dots,N\}$ a finite family of quasinonexpansive mappings of C into H with $\mathfrak{F}(\mathfrak{T}) \neq \emptyset$ and $\widetilde{\mathfrak{F}}(T_j) = \mathfrak{F}(T_j)$ for $j = 1,\dots,N$. Let $u, x_1 \in C$ and define the sequence $\{x_n\}$ by*

$$\begin{aligned}
y_n^j &= \alpha_n x_n + (1-\alpha_n)T_j x_n,\\
C_n^j &= \left\{z \in C : \left\|y_n^j - z\right\| \leq \|x_n - z\|\right\}, \quad j = 1,\dots,N,\\
v_{n,k}^j &= P_{C_k^j} x_n, \quad k = 1,\dots,n, j = 1,\dots,N,\\
w_{n,k} &= \sum_{j=1}^{N} \beta_k^j v_{n,k}^j, \quad k = 1,\dots,n,\\
x_{n+1} &= \delta_n u + (1-\delta_n)\sum_{k=1}^{n} \gamma_{n,k} w_{n,k},
\end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n^j : j = 1,\dots,N\}$, $\{\gamma_{n,k} : k \leq n\}$, and $\{\delta_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(i) $\liminf_{n\to\infty} \alpha_n < 1$,

(ii) $\beta_n^j > 0$ *for* $j = 1,\dots,N$, *and* $\sum_{j=1}^{N} \beta_n^j = 1$ *for* $n \in \mathbb{N}$,

(iii) $\sum_{k=1}^{n} \gamma_{n,k} = 1$ *for* $n \in \mathbb{N}$, $\lim_{n\to\infty} \gamma_{n,k} > 0$ *for* $k \in \mathbb{N}$, *and* $\sum_{n=1}^{\infty}\sum_{k=1}^{n} |\gamma_{n+1,k} - \gamma_{n,k}| < \infty$,

(iv) $\lim_{n\to\infty} \delta_n = 0$, $\sum_{n=1}^{\infty} \delta_n = \infty$, *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$.

Then $\{x_n\}$ converges strongly to the point $P_{\mathfrak{F}(\mathfrak{T})}u$.

The problem of whether or not we can construct a shrinking projection method analogous to that given in Theorem 1.1 for solving a common fixed point problem for a finite family of quasinonexpansive mappings in a geodesic metric space is still open. The purpose of this paper is to analyze the feasibility study of Moudafi viscosity type of projection method with a weak contraction for a finite family of quasinonexpansive mappings in a complete CAT(0) space, also known as a Hadamard space.

This paper is organized as follows. In Section 2 we recall the definition of geodesic metric spaces and summarize some useful lemmas and the main properties of CAT(0) spaces. Besides, without vector addition as in a Banach space, we present an inequality to estimate the distance between two elements defined by finite convex combination '$\oplus$' in a CAT(0) space; see Lemma 2.2. In Section 3 we construct a sequence of nonexpansive mappings satisfying AKTT condition by choosing an appropriate control sequence under certain conditions; see Theorem 3.2. Therefore, a convergence theorem of a new Moudafi viscosity approximation follows from Theorem 3.2; see Theorem 3.3. Using Theorem 3.3, we also derive a strong convergence theorem by a Moudafi type viscosity approximation with a weak contraction for a family of quasinonexpansive mappings; see Theorem 3.4. As a particular case where a weak contraction is constant in Theorem 3.4, a strong convergence theorem by the averaged projection method of Halpern type is then obtained; see Theorem 3.5.

2 Preliminaries

Let (X,d) be a metric space. For $x,y\in X$, a *geodesic path* joining x to y (or a *geodesic* from x to y) is an isometric mapping $c:[0,\ell]\subset\mathbb{R}\to X$ such that $c(0)=x$, $c(\ell)=y$, that is, $d(c(t),c(t'))=|t-t'|$ for all $t,t'\in[0,\ell]$. Therefore, $d(x,y)=\ell$. The image of c is called a *geodesic* (*segment*) from x to y, and we shall denote a definite choice of this geodesic segment by $[x,y]$. A point $z=c(t)$ in the geodesic $[x,y]$ will be written as $z=(1-\lambda)x\oplus\lambda y$, where $\lambda=t/\ell$, and so $d(z,x)=\lambda d(x,y)$ and $d(z,y)=(1-\lambda)d(x,y)$. A subset C of X is *convex* if every pair of points $x,y\in C$ can be joined by a geodesic in X and the image of every such geodesic is contained in C.

A *geodesic triangle* $\triangle(x_1,x_2,x_3)$ in (X,d) consists of three points $x_i\in X$ $(i=1,2,3)$, its *vertices*, and a geodesic segment between each pair of vertices, its *sides*. If a point $x\in X$ lies in the union of $[x_i,x_j]$, $i,j\in\{1,2,3\}$, then we write $x\in\triangle(x_1,x_2,x_3)$. A *comparison triangle* for the geodesic triangle $\triangle(x_1,x_2,x_3)$ in X is a triangle $\triangle(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i,\bar{x}_j)=d(x_i,x_j)$ for $i,j\in\{1,2,3\}$.

A geodesic triangle $\triangle$ in X is said to satisfy the CAT(0) *inequality* if, given a comparison triangle $\overline{\triangle}$ in $\mathbb{E}^2$ for $\triangle$,

$$d(x,y)\le d_{\mathbb{E}^2}(\bar{x},\bar{y})\quad\text{for } x,y\in\triangle,$$

where $\bar{x},\bar{y}\in\overline{\triangle}$ are the corresponding comparison points of x,y. The geodesic metric space X is called a CAT(0) space if all geodesic triangles in X satisfy the CAT(0) inequality. Note that Hilbert spaces are CAT(0).

Lemma 2.1 *Let* (X,d) *be a* CAT(0) *space, and let* $\alpha,\beta\in[0,1]$. *Then*:

(i) *For* $x,y\in X$, *we have*

$$d\bigl(\alpha x\oplus(1-\alpha)y,\beta x\oplus(1-\beta)y\bigr)=|\alpha-\beta|d(x,y).$$

(ii) ([13], *Chapter* II.2. *Proposition* 2.2) *For* $x,y,p,q\in X$, *we have*

$$d\bigl(\alpha x\oplus(1-\alpha)y,\alpha p\oplus(1-\alpha)q\bigr)\le\alpha d(x,p)+(1-\alpha)d(y,q).$$

In particular, if $p = q$, this reduces to

$$d\big(\alpha x \oplus (1-\alpha)y, p\big) \le \alpha d(x,p) + (1-\alpha)d(y,p).$$

(iii) ([14], *Lemma* 2.5) *For $x, y, z \in X$, we have*

$$d\big(\alpha x \oplus (1-\alpha)y, z\big)^2 \le \alpha d(x,z)^2 + (1-\alpha)d(y,z)^2 - \alpha(1-\alpha)d(x,y)^2.$$

We will extend the equality in Lemma 2.1(i) to any finitely many elements in X. First, we recall the notion of a finite sum '$\oplus$' defined by Butsan *et al.* [4]. Fix $n \in \mathbb{N}$ with $n \ge 2$ and let $\{\alpha_1, \dots, \alpha_n\} \subset (0,1)$ with $\sum_{k=1}^n \alpha_k = 1$ and $\{x_1, \dots, x_n\} \subset X$. By induction we define

$$\bigoplus_{k=1}^n \alpha_k x_k = (1-\alpha_n)\left(\frac{\alpha_1}{1-\alpha_n}x_1 \oplus \cdots \oplus \frac{\alpha_{n-1}}{1-\alpha_n}x_{n-1}\right) \oplus \alpha_n x_n. \tag{2.1}$$

The definition of $\bigoplus$ in (2.1) is an ordered one in the sense that it depends on the order of points $x_1, \dots, x_n$. However, we occasionally use the notation $\alpha_1 x_1 \oplus \alpha_2 x_2 \oplus \cdots \oplus \alpha_n x_n$ for such a point. Lemma 2.1(ii) assures that, for $y \in X$,

$$d\left(\bigoplus_{k=1}^n \alpha_k x_k, y\right) \le \sum_{k=1}^n \alpha_k d(x_k, y). \tag{2.2}$$

Lemma 2.2 *Let (X,d) be a* CAT(0) *space, and for $n \in \mathbb{N}$ with $n \ge 2$, let $\{\alpha_k\}_{k=1}^n$ and $\{\beta_k\}_{k=1}^n \subset (0,1)$ be two sequences such that $\sum_{k=1}^n \alpha_k = \sum_{k=1}^n \beta_k = 1$. Then, for $x_1, \dots, x_n \in X$, we have*

$$\begin{aligned}
&d\left(\bigoplus_{k=1}^n \alpha_k x_k, \bigoplus_{k=1}^n \beta_k x_k\right) \\
&\quad\le \left|\frac{\alpha_2}{\alpha_1+\alpha_2} - \frac{\beta_2}{\beta_1+\beta_2}\right|(\alpha_1+\alpha_2)d(x_1,x_2) \\
&\qquad+ \left|\frac{\alpha_3}{\sum_{k=1}^3 \alpha_k} - \frac{\beta_3}{\sum_{k=1}^3 \beta_k}\right| \sum_{k=1}^3 \alpha_k \cdot \sum_{k=1}^2 \frac{\beta_k}{\beta_1+\beta_2} d(x_k, x_3) \\
&\qquad+ \cdots + \left|\frac{\alpha_j}{\sum_{k=1}^j \alpha_k} - \frac{\beta_j}{\sum_{k=1}^j \beta_j}\right| \sum_{k=1}^j \alpha_j \cdot \sum_{k=1}^{j-1} \frac{\beta_k}{\beta_1+\cdots+\beta_{j-1}} d(x_k, x_j) \\
&\qquad+ \cdots + |\alpha_n - \beta_n| \sum_{k=1}^{n-1} \frac{\beta_k}{1-\beta_n} d(x_k, x_n).
\end{aligned}$$

Proof We will prove the result by induction.

Step 1. According to Lemma 2.1(ii), (2.1), and (2.2), we derive

$$\begin{aligned}
&d\left(\bigoplus_{k=1}^n \alpha_k x_k, \bigoplus_{k=1}^n \beta_k x_k\right) \\
&\quad\le d\left((1-\alpha_n)\left(\bigoplus_{k=1}^{n-1} \frac{\alpha_k}{1-\alpha_n}x_k\right) \oplus \alpha_n x_n, (1-\alpha_n)\left(\bigoplus_{k=1}^{n-1} \frac{\beta_k}{1-\beta_n}x_k\right) \oplus \alpha_n x_n\right)
\end{aligned}$$

$$
\begin{aligned}
&+ d\left((1-\alpha_n)\left(\bigoplus_{k=1}^{n-1}\frac{\beta_k}{1-\beta_n}x_k\right) \oplus \alpha_n x_n, (1-\beta_n)\left(\bigoplus_{k=1}^{n-1}\frac{\beta_k}{1-\beta_n}x_k\right) \oplus \beta_n x_n\right) \\
&\leq (1-\alpha_n) d\left(\bigoplus_{k=1}^{n-1}\frac{\alpha_k}{1-\alpha_n}x_k, \bigoplus_{k=1}^{n-1}\frac{\beta_k}{1-\beta_n}x_k\right) + |\alpha_n - \beta_n| d\left(\bigoplus_{k=1}^{n-1}\frac{\beta_k}{1-\beta_n}x_k, x_n\right) \\
&\leq (1-\alpha_n) d\left(\bigoplus_{k=1}^{n-1}\frac{\alpha_k}{1-\alpha_n}x_k, \bigoplus_{k=1}^{n-1}\frac{\beta_k}{1-\beta_n}x_k\right) \\
&\quad + |\alpha_n - \beta_n| \sum_{k=1}^{n-1}\frac{\beta_k}{1-\beta_n} d(x_k, x_n).
\end{aligned}
$$

Step 2. Apply the inequality in Step 1 for the case $n-1$ to obtain

$$
\begin{aligned}
&d\left(\bigoplus_{k=1}^{n-1}\frac{\alpha_k}{1-\alpha_n}x_k, \bigoplus_{k=1}^{n-1}\frac{\beta_k}{1-\beta_n}x_k\right) \\
&\quad \leq \frac{1-\alpha_{n-1}-\alpha_n}{1-\alpha_n} d\left(\bigoplus_{k=1}^{n-2}\frac{\alpha_k}{1-\alpha_{n-1}-\alpha_n}x_k, \bigoplus_{k=1}^{n-2}\frac{\beta_k}{1-\beta_{n-1}-\beta_n}x_k\right) \\
&\qquad + \left|\frac{\alpha_{n-1}}{1-\alpha_n} - \frac{\beta_{n-1}}{1-\beta_n}\right| \sum_{k=1}^{n-2}\frac{\beta_k}{1-\beta_{n-1}-\beta_n} d(x_k, x_{n-1}).
\end{aligned}
$$

Step 3. Recall that $\sum_{k=1}^{n}\alpha_k = \sum_{k=1}^{n}\beta_k = 1$. Hence, the two inequalities in Step 1 and Step 2 imply that

$$
\begin{aligned}
&d\left(\bigoplus_{k=1}^{n}\alpha_k x_k, \bigoplus_{k=1}^{n}\beta_k x_k\right) \\
&\quad \leq (1-\alpha_{n-1}-\alpha_n) d\left(\bigoplus_{k=1}^{n-2}\frac{\alpha_k}{1-\alpha_{n-1}-\alpha_n}x_k, \bigoplus_{k=1}^{n-2}\frac{\beta_k}{1-\beta_{n-1}-\beta_n}x_k\right) \\
&\qquad + \left|\frac{\alpha_{n-1}}{\sum_{k=1}^{n-1}\alpha_k} - \frac{\beta_{n-1}}{\sum_{k=1}^{n-1}\beta_k}\right| \sum_{k=1}^{n-1}\alpha_k \cdot \sum_{k=1}^{n-2}\frac{\beta_k}{1-\beta_{n-1}-\beta_n} d(x_k, x_{n-1}) \\
&\qquad + |\alpha_n - \beta_n| \sum_{k=1}^{n-1}\frac{\beta_k}{1-\beta_n} d(x_k, x_n).
\end{aligned}
$$

Continuing the process in Step 1 to estimate the first term of this inequality on the right-hand side, after $n-2$ steps, we have

$$
\begin{aligned}
&d\left(\bigoplus_{k=1}^{n}\alpha_k x_k, \bigoplus_{k=1}^{n}\beta_k x_k\right) \\
&\quad \leq \left|\frac{\alpha_2}{\alpha_1+\alpha_2} - \frac{\beta_2}{\beta_1+\beta_2}\right| (\alpha_1+\alpha_2) d(x_1, x_2) \\
&\qquad + \left|\frac{\alpha_3}{\sum_{k=1}^{3}\alpha_k} - \frac{\beta_3}{\sum_{k=1}^{3}\beta_k}\right| \sum_{k=1}^{3}\alpha_k \cdot \sum_{k=1}^{2}\frac{\beta_k}{\beta_1+\beta_2} d(x_k, x_3)
\end{aligned}
$$

$$+\cdots+\left|\frac{\alpha_j}{\sum_{k=1}^{j}\alpha_k}-\frac{\beta_j}{\sum_{k=1}^{j}\beta_j}\right|\sum_{k=1}^{j}\alpha_j\cdot\sum_{k=1}^{j-1}\frac{\beta_k}{\beta_1+\cdots+\beta_{j-1}}d(x_k,x_j)$$
$$+\cdots+|\alpha_n-\beta_n|\sum_{k=1}^{n-1}\frac{\beta_k}{1-\beta_n}d(x_k,x_n).$$
□

Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty}\alpha_n=1$. For notational convenience, let

$$\bar{\alpha}_k=\frac{\alpha_k}{\sum_{j=1}^{k}\alpha_j},\qquad \alpha'_k=\sum_{j=k+1}^{\infty}\alpha_j\quad\text{for } k\in\mathbb{N},$$

The following result is an immediate consequence of Lemma 2.2.

Lemma 2.3 *Let (X,d) be a* CAT(0) *space, and for $n\in\mathbb{N}$ $(n\geq 2)$, let $\{\alpha_k\}_{k=1}^{n},\{\beta_k\}_{k=1}^{n}\subset(0,1)$ be such that $\sum_{k=1}^{n}\alpha_k=\sum_{k=1}^{n}\beta_k=1$. Then for $x_1,\dots,x_n\in X$, we have*

$$d\left(\bigoplus_{k=1}^{n}\alpha_kx_k,\bigoplus_{k=1}^{n}\beta_kx_k\right)\leq M\sum_{k=1}^{n}|\bar{\alpha}_k-\bar{\beta}_k|,$$

where $M=\max\{d(x_i,x_j):i,j=1,\dots,n\}$.

It is remarkable that Dhompongsa *et al.* [5] define an infinite sum '$\oplus$' as follows. Let $\{\alpha_n\}\subset(0,1)$ with $\sum_{n=1}^{\infty}\alpha_n=1$, and let $\{x_n\}$ be a bounded sequence in a complete metric space X. Choose arbitrary $u\in X$. Suppose that $\lim_{n\to\infty}\sum_{k=n}^{\infty}\alpha'_k=0$. Define the sequence $\{y_n\}$ in X by

$$y_n=\alpha_1x_1\oplus\alpha_2x_2\oplus\cdots\oplus\alpha_nx_n\oplus\alpha'_nu.$$

Then, according to (2.1),

$$y_n=\left(\sum_{k=1}^{n}\alpha_k\right)z_n\oplus\alpha'_nu,\tag{2.3}$$

where

$$z_n=\frac{\alpha_1}{\sum_{k=1}^{n}\alpha_k}x_1\oplus\cdots\oplus\frac{\alpha_n}{\sum_{k=1}^{n}\alpha_k}x_n.$$

Recall that $\{y_n\}$ is a Cauchy sequence [5] and therefore converges to some point $x\in X$. We can write

$$x=\bigoplus_{n=1}^{\infty}\alpha_nx_n.$$

By (2.3), $d(y_n,z_n)=\alpha'_nd(z_n,u)$. Hence, $\{z_n\}$ also converges to x, and the limit x is independent of the choice of u.

To verify our main results in Section 3, the following property is required and crucial.

Lemma 2.4 (Dhompongsa *et al.* [5], Lemma 3.8) *Let C be a closed convex subset of a complete* $\mathrm{CAT}(0)$ *space X, $\{T_n\}$ a sequence of nonexpansive mappings on C with $\bigcap_{n=1}^{\infty} \mathfrak{F}(T_n) \neq \emptyset$, and $\{\alpha_n\}$ a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} \alpha_n = 1$ and $\lim_{n\to\infty} \sum_{k=n}^{\infty} \alpha'_k = 0$. Define the mapping $S : C \to C$ by $Sx = \bigoplus_{n=1}^{\infty} \alpha_n T_n x$, $x \in C$. Then S is nonexpansive, and $\mathfrak{F}(S) = \bigcap_{n=1}^{\infty} \mathfrak{F}(T_n)$.*

3 Projection method

Let C be a closed convex subset of a complete metric space X. A family $\{T_n\}$ of nonexpansive self-mappings of C is said to satisfy *AKTT condition* [3] if for every bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup\bigl\{d(T_{n+1}x, T_n x) : x \in B\bigr\} < \infty.$$

In this case, the sequence $\{T_n x\}$ is Cauchy for each $x \in C$ and so converges in X. We recall the following convergence theorem with a weak contraction for a sequence of nonexpansive mappings with AKTT condition.

Theorem 3.1 (Huang [15], Theorem 4.11) *Let X be a complete* $\mathrm{CAT}(0)$ *space, C a closed convex subset of X, $\{T_n\}$ a family of nonexpansive mappings on C satisfying AKTT condition such that $\bigcap_{n=1}^{\infty} \mathfrak{F}(T_n) \neq \emptyset$, f a φ-weak contraction on C, where φ is strictly increasing, and $\{\alpha_n\}$ is a sequence in $(0,1]$ satisfying*

- (C1) $\lim_{n\to\infty} \alpha_n = 0$;
- (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (C3) *either* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, *or* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$.

Define the mapping $S : C \to C$ by $Sx = \lim_{n\to\infty} T_n x$ for $x \in C$. Suppose that $\mathfrak{F}(S) = \bigcap_{n=1}^{\infty} \mathfrak{F}(T_n)$. Then the sequence $\{x_n\}$ defined by $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n) T_n x_n$$

converges strongly to a point $\hat{x} \in C$ such that $\hat{x} = P_{\mathfrak{F}(S)} f(\hat{x})$.

We now construct a sequence of nonexpansive mappings satisfying AKTT condition by choosing an appropriate control sequence under certain conditions.

Theorem 3.2 *Let C be a closed convex subset of a complete* $\mathrm{CAT}(0)$ *space X, $\mathfrak{T} = \{T_n\}$ a family of nonexpansive mappings on C with $\mathfrak{F}(\mathfrak{T}) \neq \emptyset$, and $\{\gamma_{n,k} : k \leq n\} \subset (0,1)$ a sequence satisfying*

- (D1) $\sum_{k=1}^{n} \gamma_{n,k} = 1$, $\forall n \in \mathbb{N}$;
- (D2) $\lambda_k = \lim_{n\to\infty} \gamma_{n,k} > 0$, $\forall k \in \mathbb{N}$, *and* $\lim_{n\to\infty} \sum_{k=n}^{\infty} \lambda'_k = 0$;
- (D3) $\sum_{n=1}^{\infty} \sum_{k=1}^{n+1} |\bar{\gamma}_{n+1,k} - \bar{\gamma}_{n,k}| < \infty$, *where* $\gamma_{n,n+1} = 0$ *and*

$$\bar{\gamma}_{n,k} = \frac{\gamma_{n,k}}{\gamma_{n,1} + \cdots + \gamma_{n,k}}, \quad k = 1, \ldots, n+1.$$

For each $n \in \mathbb{N}$, define the mapping $S_n : C \to C$ by

$$S_n x = \bigoplus_{k=1}^{n} \gamma_{n,k} T_k x.$$

Then $\{S_n\}$ *is a family of nonexpansive mappings satisfying AKTT condition and*

$$\bigcap_{n=1}^{\infty} \mathfrak{F}(S_n) = \bigcap_{n=1}^{\infty} \mathfrak{F}(T_n).$$

Moreover, the mapping $S : C \to C$ *defined by* $Sx = \lim_{n\to\infty} S_n x$ *is also nonexpansive, and* $\mathfrak{F}(S) = \bigcap_{n=1}^{\infty} \mathfrak{F}(S_n)$.

Proof Fix any $n \in \mathbb{N}$. We may assume that $\gamma_{n,k} = 0$ for all $k > n$. Then Lemma 2.4 states that S_n is nonexpansive and $\mathfrak{F}(S_n) = \bigcap_{k=1}^{n} \mathfrak{F}(T_k)$. Thus,

$$\bigcap_{n=1}^{\infty} \mathfrak{F}(S_n) = \bigcap_{k=1}^{\infty} \mathfrak{F}(T_k) \neq \emptyset.$$

For every bounded subset B of C, the set $\{T_k x : x \in B, k \in \mathbb{N}\}$ is bounded since $\bigcap_{k=1}^{\infty} \mathfrak{F}(T_k) \neq \emptyset$. Let

$$M = \operatorname{diam}\{T_k x : x \in B, k \in \mathbb{N}\},$$

so that by Lemma 2.3, for $x \in B$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
d(S_{n+1}x, S_n x) &\le d\left(\bigoplus_{k=1}^{n} \gamma_{n+1,k} T_k x, \bigoplus_{k=1}^{n} \gamma_{n,k} T_k x\right) + \gamma_{n+1,n+1} d\left(T_{n+1}x, \bigoplus_{k=1}^{n} \gamma_{n,k} T_k x\right) \\
&\le M \sum_{k=1}^{n} |\bar{\gamma}_{n+1,k} - \bar{\gamma}_{n,k}| + M\gamma_{n+1,n+1} \sum_{k=1}^{n} \gamma_{n,k} \\
&= M \sum_{k=1}^{n} |\bar{\gamma}_{n+1,k} - \bar{\gamma}_{n,k}| + M\bar{\gamma}_{n+1,n+1} \\
&= M \sum_{k=1}^{n+1} |\bar{\gamma}_{n+1,k} - \bar{\gamma}_{n,k}|.
\end{aligned}$$

It follows that

$$\sum_{n=1}^{\infty} \sup\bigl\{d(S_{n+1}x, S_n x) : x \in B\bigr\} \le M \sum_{n=1}^{\infty} \sum_{k=1}^{n+1} |\bar{\gamma}_{n+1,k} - \bar{\gamma}_{n,k}| < \infty.$$

Therefore, $\{S_n\}$ is a family of nonexpansive mappings on C satisfying AKTT condition such that $\bigcap_{n=1}^{\infty} \mathfrak{F}(S_n) \neq \emptyset$. It follows that $\{S_n x\}$ converges for all $x \in C$, and thus S is well defined.

If $m, n \in \mathbb{N}$ and $m > n$, then we get

$$\begin{aligned}
\sum_{k=1}^{n} |\bar{\gamma}_{m,k} - \bar{\gamma}_{n,k}| &\le \sum_{k=1}^{n} \bigl(|\bar{\gamma}_{n+1,k} - \bar{\gamma}_{n,k}| + |\bar{\gamma}_{n+2,k} - \bar{\gamma}_{n+1,k}| + \cdots + |\bar{\gamma}_{m,k} - \bar{\gamma}_{m-1,k}|\bigr) \\
&= \sum_{j=n}^{m-1} \sum_{k=1}^{n} |\bar{\gamma}_{j+1,k} - \bar{\gamma}_{j,k}| \\
&\le \sum_{j=n}^{m-1} \sum_{k=1}^{j+1} |\bar{\gamma}_{j+1,k} - \bar{\gamma}_{j,k}|.
\end{aligned}$$

Recall that $\bar{\lambda}_k = \lim_{n\to\infty} \bar{\gamma}_{n,k}$ for $k \in \mathbb{N}$. We take the limit as $m \to \infty$ to obtain

$$\sum_{k=1}^{n} |\bar{\lambda}_k - \bar{\gamma}_{n,k}| \leq \sum_{j=n}^{\infty} \sum_{k=1}^{j+1} |\bar{\gamma}_{j+1,k} - \bar{\gamma}_{j,k}|$$

and then take the limit as $n \to \infty$ to obtain

$$\lim_{n\to\infty} \sum_{k=1}^{n} |\bar{\lambda}_k - \bar{\gamma}_{n,k}| = 0. \tag{3.1}$$

On the other hand, the absolute convergence of the series

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n+1} (\bar{\gamma}_{n+1,k} - \bar{\gamma}_{n,k})$$

implies the convergence of its partial sums

$$\sum_{n=1}^{m} \sum_{k=1}^{n+1} (\bar{\gamma}_{n+1,k} - \bar{\gamma}_{n,k}) = \left(\sum_{k=1}^{m+1} \bar{\gamma}_{m+1,k} \right) - \bar{\gamma}_{1,1} = \left(\sum_{k=1}^{m+1} \bar{\gamma}_{m+1,k} \right) - 1.$$

Hence, by (3.1), $\sum_{k=1}^{\infty} \bar{\lambda}_k$ converges (in fact, to $\sum_{k=1}^{\infty} \bar{\gamma}_{n,k}$), and so does $\sum_{k=1}^{\infty} \lambda_k$ because $\lambda_k \leq \bar{\lambda}_k$. Let $\lambda = \sum_{k=1}^{\infty} \lambda_k$. Define the mapping $W : C \to C$ by

$$Wx = \bigoplus_{n=1}^{\infty} \frac{\lambda_n}{\lambda} T_n x.$$

Then by (D2) Lemma 2.4 guarantees that W is nonexpansive and $\mathfrak{F}(W) = \bigcap_{n=1}^{\infty} \mathfrak{F}(T_n)$. If

$$W_n x = \bigoplus_{k=1}^{n} \frac{\lambda_k}{\sum_{j=1}^{n} \lambda_j} T_k x, \quad x \in C,$$

then $\{W_n x\}$ converges to Wx. Recall that

$$\overline{\left(\frac{\lambda_k}{\sum_{j=1}^{n} \lambda_j} \right)} = \bar{\lambda}_k \quad \text{for } k = 1, \ldots, n.$$

Fix any $x \in C$. Then by Lemma 2.3 and (3.1) we get

$$d(S_n x, W_n x) \leq K \sum_{k=1}^{n} |\bar{\gamma}_{n,k} - \bar{\lambda}_k| \to 0 \quad \text{as } n \to \infty,$$

where $K = \max\{d(T_i x, T_j x) : i, j = 1, \ldots, n\}$. This shows that $Wx = Sx$ for all $x \in C$, as required. □

The following result follows immediately from Theorems 3.1 and 3.2.

Theorem 3.3 *Let C be a closed convex subset of a complete* CAT(0) *space X, $\mathfrak{T} = \{T_n\}$ a family of nonexpansive mappings on C such that $\mathfrak{F}(\mathfrak{T}) \neq \emptyset$, and f a φ-weak contraction on C, where φ is strictly increasing. Let $\{\alpha_n\} \subset (0,1]$ and $\{\gamma_{n,k} : k \leq n\} \subset (0,1)$ be two sequences such that $\{\alpha_n\}$ satisfies* (C1)-(C3) *and $\{\gamma_{n,k} : k \leq n\}$ satisfies* (D1)-(D3). *Let $x_1 \in C$ and define the sequence $\{x_n\}$ by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) \bigoplus_{k=1}^{n} \gamma_{n,k} T_k x_n.$$

Then $\{x_n\}$ converges strongly to a point $\hat{x} \in C$ such that $\hat{x} = P_{\mathfrak{F}(\mathfrak{T})} f(\hat{x})$.

Proof For each $n \in \mathbb{N}$, let $S_n : C \to C$ be the mapping defined by

$$S_n x = \bigoplus_{k=1}^{n} \gamma_{n,k} T_k x.$$

Then by Theorem 3.2, $\{S_n\}$ is a family of nonexpansive mappings satisfying the AKTT condition and $\bigcap_{n=1}^{\infty} \mathfrak{F}(S_n) = \bigcap_{n=1}^{\infty} \mathfrak{F}(T_n)$. We can write

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) S_n x_n.$$

Define the mapping $S : C \to C$ by $Sx = \lim_{n\to\infty} S_n x$ for $x \in C$, so that S is nonexpansive and $\mathfrak{F}(S) = \bigcap_{n=1}^{\infty} \mathfrak{F}(S_n)$. Consequently, Theorem 3.1 assures the strong convergence of $\{x_n\}$ with limit $\hat{x}$, say, such that $\hat{x} = P_{\mathfrak{F}(S)} f(\hat{x})$. □

Using Theorem 3.3, we establish a strong convergence theorem by a Moudafi type of shrinking projection method for a family of quasinonexpansive mappings as follows.

Theorem 3.4 *Let C be a closed convex subset of a complete* CAT(0) *space X such that $\{z \in C : d(u,z) \leq d(v,z)\}$ is a convex subset of C for every $u, v \in C$. Let $\mathfrak{T} = \{T_j : j = 1, \ldots, N\}$ be a finite family of quasinonexpansive mappings of C into X with $\mathfrak{F}(\mathfrak{T}) \neq \emptyset$ and $\widetilde{\mathfrak{F}}(T_j) = \mathfrak{F}(T_j)$ for $j = 1, \ldots, N$, and f a φ-weak contraction on C, where φ is strictly increasing. Let $\{\alpha_n\}$, $\{\delta_n\}$ be sequences in $(0,1]$, and $\{\beta_n^j : j = 1, \ldots, N\}$ and $\{\gamma_{n,k} : k \leq n\}$ be sequences in $(0,1)$. Let $x_1 \in C$ and define the sequence $\{x_n\}$ by*

$$\begin{aligned}
&y_n^j = \delta_n x_n \oplus (1-\delta_n) T_j x_n, \\
&C_n^j = \bigl\{z \in C : d\bigl(y_n^j, z\bigr) \leq d(x_n, z)\bigr\}, \quad j = 1, \ldots, N, \\
&v_{n,k}^j = P_{C_k^j} x_n, \quad k = 1, \ldots, n, j = 1, \ldots, N, \\
&w_{n,k} = \bigoplus_{j=1}^{N} \beta_k^j v_{n,k}^j, \quad k = 1, \ldots, n, \\
&x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) \bigoplus_{k=1}^{n} \gamma_{n,k} w_{n,k},
\end{aligned}$$

where $\{\alpha_n\}$ satisfies (C1)-(C3), *$\{\gamma_{n,k} : k \leq n\}$ satisfies* (D1)-(D3), *and $\{\delta_n\}$, $\{\beta_n^j\}$ satisfy the following conditions*:

(i) $\liminf_{n\to\infty} \delta_n < 1$;
(ii) $\sum_{j=1}^{N} \beta_n^j = 1$ *for* $n \in \mathbb{N}$.

Then $\{x_n\}$ *converges strongly to a point* $\hat{x} \in C$ *such that* $\hat{x} = P_{\mathfrak{F}(\mathfrak{T})} f(\hat{x})$.

Proof First, we can see that every C_n^j is closed and convex by the assumption on the space. To prove that the metric projection $P_{C_k^j}$ is well defined, let $z \in \mathfrak{F}(\mathfrak{T})$. Since T_j is quasinonexpansive, we have

$$d(y_n^j, z) \le \delta_n d(x_n, z) + (1 - \delta_n) d(T_j x_n, z) \le d(x_n, z),$$

and so $z \in C_n^j$. This implies that

$$\emptyset \neq \mathfrak{F}(\mathfrak{T}) \subset C_n^j, \quad j = 1, \dots, N, n \in \mathbb{N}.$$

Thus, the metric projection onto C_n^j is well defined. For $n \in \mathbb{N}$, define $Q_n : C \to C$ by

$$Q_n x = \bigoplus_{j=1}^{N} \beta_n^j P_{C_n^j} x, \quad x \in C.$$

It follows from Lemma 2.4 and condition (ii) that Q_n is nonexpansive and $\mathfrak{F}(Q_n) = \bigcap_{j=1}^{N} C_n^j$. According to our construction, we can write

$$w_{n,k} = Q_k x_n, \quad k = 1, \dots, n,$$
$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n) \bigoplus_{j=1}^{n} \gamma_{n,k} Q_k x_n, \quad n \in \mathbb{N}.$$

Hence, Theorem 3.3 and conditions (C1)-(C3) and (D1)-(D3) assure the strong convergence of $\{x_n\}$ to a point $\hat{x} \in C$ such that $\hat{x} = P_F f(\hat{x})$, where

$$F = \bigcap_{n=1}^{\infty} \mathfrak{F}(Q_n) = \bigcap_{n=1}^{\infty} \bigcap_{j=1}^{N} C_n^j = \bigcap_{j=1}^{N} \bigcap_{n=1}^{\infty} C_n^j.$$

Notice that $\mathfrak{F}(\mathfrak{T}) \subset F$. Condition (i) asserts that there exists a convergent subsequence $\{\delta_{n_i}\}$ of $\{\delta_n\}$ such that $\lim_{i\to\infty} \delta_{n_i} < 1$. Since $\hat{x} \in C_n^j$ for all $j = 1, \dots, N$ and $n \in \mathbb{N}$, we obtain

$$\begin{aligned} d(x_{n_i}, \hat{x}) &\ge d(y_{n_i}, \hat{x}) \\ &= d(\delta_{n_i} x_{n_i} \oplus (1 - \delta_{n_i}) T_j x_{n_i}, \hat{x}) \\ &\ge d(x_{n_i}, \delta_{n_i} x_{n_i} \oplus (1 - \delta_{n_i}) T_j x_{n_i}) - d(x_{n_i}, \hat{x}) \\ &= (1 - \delta_{n_i}) d(x_{n_i}, T_j x_{n_i}) - d(x_{n_i}, \hat{x}), \end{aligned}$$

which yields

$$\frac{2}{1 - \delta_{n_i}} d(x_{n_i}, \hat{x}) \ge d(x_{n_i}, T_j x_{n_i}).$$

We then take the limit as $i \to \infty$ and get

$$\lim_{i\to\infty} d(x_{n_i}, T_j x_{n_i}) = 0, \quad j = 1, \ldots, N.$$

This shows that $\hat{x} \in \widetilde{\mathfrak{F}}(T_j) = \mathfrak{F}(T_j)$ for $j = 1, \ldots, N$, that is, $\hat{x} \in \mathfrak{F}(\mathfrak{T})$. Since $\mathfrak{F}(\mathfrak{T}) \subset F$, we then have $\hat{x} = P_F f(\hat{x}) = P_{\mathfrak{F}(\mathfrak{T})} f(\hat{x})$, which completes the proof. □

Consequently, when f is constant in Theorem 3.4, we obtain the following strong convergence theorem by a new Halpern type of shrinking projection method.

Theorem 3.5 *Let X, C, $\mathfrak{T} = \{T_j : j = 1, \ldots, N\}$, and the sequences $\{\alpha_n\}$, $\{\delta_n\}$, $\{\beta_n^j : j = 1, \ldots, N\}$, $\{\gamma_{n,k} : k \le n\}$ be as in Theorem 3.4. Let $u, x_1 \in C$ and define the sequence $\{x_n\}$ by*

$$\begin{aligned}
y_n^j &= \delta_n x_n \oplus (1 - \delta_n) T_j x_n, \\
C_n^j &= \bigl\{ z \in C : d\bigl(y_n^j, z\bigr) \le d(x_n, z) \bigr\}, \quad j = 1, \ldots, N, \\
v_{n,k}^j &= P_{C_k^j} x_n, \quad k = 1, \ldots, n, j = 1, \ldots, N, \\
w_{n,k} &= \bigoplus_{j=1}^{N} \beta_k^j v_{n,k}^j, \quad k = 1, \ldots, n, \\
x_{n+1} &= \alpha_n u \oplus (1 - \alpha_n) \bigoplus_{k=1}^{n} \gamma_{n,k} w_{n,k}.
\end{aligned}$$

Then $\{x_n\}$ converges strongly to the point $P_{\mathfrak{F}(\mathfrak{T})} u$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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