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The (α, β)-generalized convex contractive condition with approximate fixed point results and some consequence

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Abstract
The aim of this work is to introduce some new notions of generalized convex contraction mappings and establish some approximate fixed point theorems for such mappings in the setting of complete metric spaces. Examples and application to approximate fixed point results for cyclic mappings are also given in order to illustrate the effectiveness of the obtained results.

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1 Introduction

It is well known that fixed point theory is one of the important tools for solving various problems in nonlinear analysis and various fields of applied mathematical analysis. The Banach contraction mapping principle presented by Banach [1] in his thesis is one of the cornerstones in the development of fixed point theory. This principle has been used to solve several problems such as the existence and uniqueness problems for a solution of nonlinear integral equations and nonlinear differential equations. Furthermore, it can be applied to the convergence theorem for solving some problems in computational mathematics. Hence, a large number of researchers have focused on the development of this topic. For instant, one of an interesting directions is the extension of fixed point results to approximate fixed point results. Indeed, in various practical situations, some conditions in the fixed point results are too strong and thus the existence of a fixed point is not guaranteed. In this case, one can consider points close to fixed points, which we call approximate fixed points.

On the other hand, the concept of convex contractions was introduced by Istratescu [2] in 1982. He also proved that each convex contraction self mapping on a complete metric space has a unique fixed point. Recently, Miandaragh *et al.* [3, 4] introduced two more general concepts of convex contractions, which are called generalized convex contractions and generalized convex contractions of order 2 and they also discussed some approximate fixed point results for such mappings.

The purpose of this paper is to formulate the concepts of generalized convex contractions and generalized convex contractions of order 2 in general terms and prove the existence results of approximate fixed points for these mappings on a complete metric space by using the idea of cyclic (α,β)-admissible mappings due to Alizadeh *et al.* [5]. We furnish an illustrative example to demonstrate the validity of the hypotheses and the degree of utility of our results. Our result extends, unifies, and generalizes various well-known fixed point and approximate fixed point results such as the Banach contraction mapping principle [1], Kannan's fixed point results [6], fixed point and approximate fixed point results for convex contraction mappings due to Istratescu [2], and many results in the literature. As a consequence of the presented results, the approximate fixed point results for cyclic mappings are also given in order to illustrate the effectiveness of the obtained results.

2 Preliminaries

In this section, we give some definitions, examples, and remarks which are useful for the main results in this paper. Throughout this paper, $\mathbb{Z}^+$ denotes the set of positive integers and $\mathbb{R}$ denotes the set of real numbers.

Definition 2.1 ([7]) Let (X,d) be a metric space, $T:X\to X$ be a mapping and $\varepsilon>0$ be a given real number. A point $x_0\in X$ is said to be an *ε-fixed point* (*approximate fixed point*) of T if

$$d(x_0,Tx_0)<\varepsilon.$$

Remark 2.2 We observe that a fixed point is an ε-fixed point, where ε is an arbitrary positive real number. However, the converse is not true.

For a metric space (X,d) and a given $\varepsilon>0$, the set of all ε-fixed points of a mapping $T:X\to X$ is denoted by

$$F_\varepsilon(T):=\bigl\{x\in X \mid d(x,Tx)<\varepsilon\bigr\}.$$

Definition 2.3 ([8]) Let (X,d) be a metric space and $T:X\to X$ be a mapping. We say that T has the *approximate fixed point property* if for all $\varepsilon>0$, there exists an ε-fixed point of T, that is,

$$\forall\varepsilon>0,\quad F_\varepsilon(T)\neq\emptyset,$$

or, equivalently,

$$\inf_{x\in X} d(x,Tx)=0.$$

Example 2.4 Let $X=(0,1)\backslash\{\frac{\ln(2)}{2}\}$ and a metric d on X be defined by $d(x,y)=|x-y|$ for all $x,y\in X$. Define a mapping $T:X\to X$ by $Tx=\frac{x}{2}e^{2x}$ for all $x\in X$. Then T does not have a fixed point but T has the approximate fixed point property. Indeed,

$$\inf_{x\in X} d(x,Tx)=\inf_{x\in X}\left|x-\frac{x}{2}e^{2x}\right|=0.$$

Figure 1 Geometry for Example 2.4.

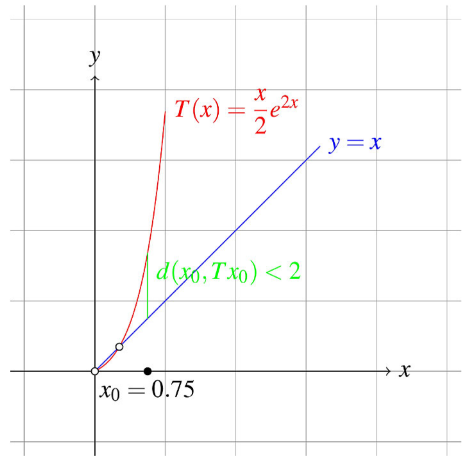
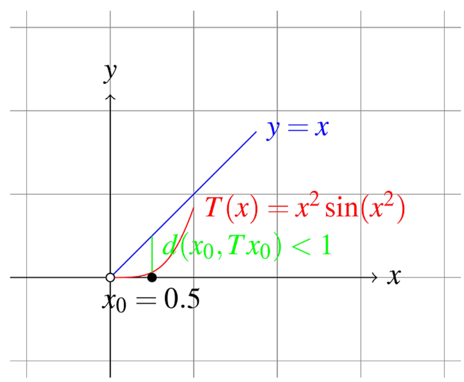


Figure 2 Geometry for Example 2.5.



For instance, for $\varepsilon = 2$ there is $x_0 = 0.75$ such that

$$d(x_0, Tx_0) = \left|0.75 - \frac{0.75}{2}e^{2(0.75)}\right| < 2 = \varepsilon$$

(see Figure 1).

Example 2.5 Let $X = (0,1)$ and a metric d on X be defined by $d(x,y) = |x-y|$ for all $x, y \in X$. Define a mapping $T : X \to X$ by $Tx = x^2\sin(x^2)$ for all $x \in X$. Then T does not have a fixed point but T has the approximate fixed point property. Indeed,

$$\inf_{x\in X} d(x, Tx) = \inf_{x\in X}\left|x - x^2\sin(x^2)\right| = 0.$$

For instance, for $\varepsilon = 1$ there is $x_0 = 0.5$ such that

$$d(x_0, Tx_0) = \left|0.5 - (0.5)^2\sin\bigl((0.5)^2\bigr)\right| < 1 = \varepsilon$$

(see Figure 2).

In 1996, Browder and Petryshyn [9] defined the following notions.

Definition 2.6 ([9]) A self mapping T on a metric space (X,d) is said to be *asymptotically regular* at a point $x \in X$ if

$$d\big(T^n x, T^{n+1}x\big) \to 0 \quad \text{as } n \to \infty,$$

where $T^n x$ denotes the nth iterate of T at x.

It is not hard to prove the following results.

Lemma 2.7 *Let (X,d) be a metric space and $T : X \to X$ be an asymptotically regular at some point $z \in X$. Then T has the approximate fixed point property.*

In 2014, Alizadeh *et al.* [5] introduced the notions of cyclic (α,β)-admissible mappings as follows.

Definition 2.8 Let X be a nonempty set and $\alpha,\beta : X \to [0,\infty)$ be two given functions. A mapping $T : X \to X$ is said to be a *cyclic* (α,β)*-admissible mapping* if the following conditions hold:

(i) $\alpha(x) \geq 1$ for some $x \in X$ implies $\beta(Tx) \geq 1$;
(ii) $\beta(y) \geq 1$ for some $y \in X$ implies $\alpha(Ty) \geq 1$.

Example 2.9 Let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = \begin{cases} -x^3, & x \in [0,\infty), \\ -\frac{x}{2}, & \text{otherwise.} \end{cases}$$

Assume that $\alpha,\beta : \mathbb{R} \to [0,\infty)$ are defined by

$$\alpha(x) = \begin{cases} e^x, & x \in [0,\infty), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\beta(y) = \begin{cases} e^{-y}, & y \in (-\infty,0], \\ 0, & \text{otherwise.} \end{cases}$$

If $\alpha(x) = e^x \geq 1$, then $x \geq 0$, which implies $Tx \leq 0$. Therefore, $\beta(Tx) = e^{-Tx} \geq 1$. Also, if $\beta(y) = e^{-y} \geq 1$, then $y \leq 0$, which implies $Ty \geq 0$. So, $\alpha(Ty) = e^{Ty} \geq 1$. Then T is a cyclic (α,β)-admissible mapping.

3 Main results

In this section, we introduce the concepts of (α,β)-generalized convex contraction and (α,β)-generalized convex contraction of order 2 and prove the approximate fixed point theorems for such mappings.

Definition 3.1 Let (X,d) be a metric space. The mapping $T : X \to X$ is called an (α,β)-*generalized convex contraction* if there exist mappings $\alpha,\beta : X \to [0,\infty)$ and $a,b \in [0,\infty)$ with $a+b<1$, satisfying the following condition:

$$\text{for all } x,y \in X, \quad \alpha(x)\beta(y) \geq 1 \quad \Longrightarrow \quad d\bigl(T^2x,T^2y\bigr) \leq ad(Tx,Ty) + bd(x,y). \tag{3.1}$$

Now, we establish a new approximate fixed point theorem for (α,β)-generalized convex contraction mappings in complete metric spaces.

Theorem 3.2 *Let (X,d) be a metric space and $T : X \to X$ be an (α,β)-generalized convex contraction mapping. Assume that T is a cyclic (α,β)-admissible mapping and there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. Then T has the approximate fixed point property. Moreover, if T is continuous and (X,d) is a complete metric space, then T has a fixed point.*

Proof First of all, let $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = T^{n+1}x_0$ for all $n \in \mathbb{Z}^+ \cup \{0\}$. If $x_{n'} = x_{n'+1}$ for some $n' \in \mathbb{Z}^+ \cup \{0\}$, then $x_{n'}$ is a fixed point of T. So, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{Z}^+ \cup \{0\}$. By the definition of a cyclic (α,β)-admissible mapping, we have

$$\begin{aligned}
&\alpha(x_0) \geq 1 \quad \Longrightarrow \quad \beta(x_1) \geq 1 \quad \Longrightarrow \quad \alpha(x_2) \geq 1 \quad \Longrightarrow \quad \cdots, \\
&\beta(x_0) \geq 1 \quad \Longrightarrow \quad \alpha(x_1) \geq 1 \quad \Longrightarrow \quad \beta(x_2) \geq 1 \quad \Longrightarrow \quad \cdots.
\end{aligned}$$

Therefore, $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{Z}^+ \cup \{0\}$. Let us denote

$$\nu := d\bigl(Tx_0,T^2x_0\bigr) + d(x_0,Tx_0)$$

and

$$\gamma := a+b.$$

Since $\alpha(x_0)\beta(x_1) \geq 1$, we get

$$\begin{aligned}
d\bigl(T^2x_0,T^3x_0\bigr) &\leq ad\bigl(Tx_0,T^2x_0\bigr) + bd(x_0,Tx_0) \\
&= \gamma\nu.
\end{aligned}$$

Also, since $\alpha(x_1)\beta(x_2) \geq 1$, we get

$$\begin{aligned}
d\bigl(T^3x_0,T^4x_0\bigr) &\leq ad\bigl(T^2x_0,T^3x_0\bigr) + bd\bigl(Tx_0,T^2x_0\bigr) \\
&\leq ad\bigl(Tx_0,T^2x_0\bigr) + bd(x_0,Tx_0) + bd\bigl(Tx_0,T^2x_0\bigr) \\
&\leq ad(x_0,Tx_0) + ad\bigl(Tx_0,T^2x_0\bigr) + bd(x_0,Tx_0) + bd\bigl(Tx_0,T^2x_0\bigr) \\
&= \gamma\nu.
\end{aligned}$$

By continuing this process, we get

$$d\bigl(T^mx_0,T^{m+1}x_0\bigr) \leq \begin{cases} \gamma^{(m-1)/2}\nu & \text{if } m = 1,3,5,\ldots, \\ \gamma^{m/2}\nu & \text{if } m = 2,4,6,\ldots. \end{cases}$$

Thus it follows that $d(T^m x_0, T^{m+1}x_0) \to 0$ as $m \to \infty$. So, T is an asymptotically regular at a point $x_0 \in X$. By using Lemma 2.7, we see that T has the approximate fixed point property.

Next, we will show that T has a fixed point provided that (X, d) is a complete metric space and T is continuous. Now, we will prove that $\{x_n\}$ is a Cauchy sequence in X. Without loss of generality, we may assume that $m, n \in \mathbb{Z}^+$ such that $n > m > 1$. We distinguish two cases as follows.

Case 1: Let m be an even number. That is, $m = 2l$, where $l \in \mathbb{Z}^+$. Therefore,

$$\begin{aligned} d\bigl(T^m x_0, T^n x_0\bigr) &\le d\bigl(T^m x_0, T^{m+1}x_0\bigr) + d\bigl(T^{m+1}x_0, T^{m+2}x_0\bigr) + \cdots + d\bigl(T^{n-1}x_0, T^n x_0\bigr) \\ &\le \gamma^l \nu + \gamma^l \nu + \gamma^{l+1}\nu + \gamma^{l+1}\nu + \gamma^{l+2}\nu + \cdots \\ &\le 2\gamma^l \nu + 2\gamma^{l+1}\nu + 2\gamma^{l+2}\nu + \cdots \\ &= \frac{2\gamma^l \nu}{1-\gamma}. \end{aligned}$$

Case 2: Let m be an odd number. That is, $m = 2l + 1$, where $l \in \mathbb{Z}^+$. Therefore,

$$\begin{aligned} d\bigl(T^m x_0, T^n x_0\bigr) &\le d\bigl(T^m x_0, T^{m+1}x_0\bigr) + d\bigl(T^{m+1}x_0, T^{m+2}x_0\bigr) + \cdots + d\bigl(T^{n-1}x_0, T^n x_0\bigr) \\ &\le \gamma^l \nu + \gamma^{l+1} \nu + \gamma^{l+1}\nu + \gamma^{l+2}\nu + \gamma^{l+2}\nu + \cdots \\ &\le 2\gamma^l \nu + 2\gamma^{l+1}\nu + 2\gamma^{l+2}\nu + \cdots \\ &= \frac{2\gamma^l \nu}{1-\gamma}. \end{aligned}$$

It follows that

$$d\bigl(T^m x_0, T^n x_0\bigr) \le \begin{cases} \frac{2\gamma^{m/2}\nu}{1-\gamma} & \text{if } m = 2, 4, 6, \ldots, \\ \frac{2\gamma^{(m-1)/2}\nu}{1-\gamma} & \text{if } m = 3, 5, 7, \ldots. \end{cases}$$

Therefore, $d(T^m x_0, T^n x_0) \to 0$ as $m, n \to \infty$, that is, $\{x_n\}$ is a Cauchy sequence in X. By using the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since T is continuous, we obtain

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx^*$$

and thus T has a fixed point. This completes the proof. □

Next we give an example to illustrate the usability of Theorem 3.2.

Example 3.3 Let $X = [1, \infty)$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Define $T : X \to X$ and $\alpha, \beta : X \to [0, \infty)$ by

$$Tx = \begin{cases} \frac{x+6}{4}, & \text{if } x \in [1, 2], \\ \sqrt{2x}, & \text{if } x \in (2, 8), \\ x^2 - 8x + 4, & \text{otherwise}, \end{cases} \qquad \alpha(x) = \begin{cases} \frac{x+2}{2}, & \text{if } x \in [1, 2], \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\beta(x) = \begin{cases} \frac{x+4}{3}, & \text{if } x \in [1,2], \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that (X,d) is a complete metric space and T is continuous.

Now, we show that Theorem 3.2 can guarantee the existence of fixed point of T. First of all, we will show that T is an (α,β)-generalized convex contraction mapping with $a = \frac{2}{3}$ and $b = \frac{1}{9}$.

For $x, y \in X$ with $\alpha(x)\beta(y) \geq 1$, we have $x, y \in [1,2]$ and thus

$$\begin{aligned} d(T^2x, T^2y) &= \left| T\left(\frac{x+6}{4}\right) - T\left(\frac{y+6}{4}\right) \right| \\ &= \left| \frac{x+30}{16} - \frac{y+30}{16} \right| \\ &= \frac{1}{16}|x-y| \\ &\leq \frac{5}{18}|x-y| \\ &= \frac{2}{3}\left| \frac{x+3}{4} - \frac{y+3}{4} \right| + \frac{1}{9}|x-y| \\ &= ad(Tx,Ty) + bd(x,y). \end{aligned}$$

This shows that T is an (α,β)-generalized convex contraction mapping with $a = \frac{2}{3}$ and $b = \frac{1}{9}$. Clearly, T is a cyclic (α,β)-admissible mapping. It is easy to see that there is $x_0 = 1 \in X$ such that

$$\alpha(x_0) = \alpha(1) = 1.5 \geq 1 \quad \text{and} \quad \beta(x_0) = \beta(1) = 5/3 \geq 1. \tag{3.2}$$

By using Theorem 3.2, we see that T has a fixed point in X.

Corollary 3.4 *Let (X,d) be a metric space, $\alpha,\beta : X \to [0,\infty)$ be two mappings and $T : X \to X$ be a mapping such that*

$$\alpha(x)\beta(y)d(T^2x, T^2y) \leq ad(Tx,Ty) + bd(x,y) \tag{3.3}$$

for all $x, y \in X$, where $a, b \in [0,1)$ with $a + b < 1$. Assume that T is a cyclic (α,β)-admissible mapping and there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. Then T has the approximate fixed point property. Moreover, if T is continuous and (X,d) is a complete metric space, then T has a fixed point.

Proof We will show that T is an (α,β)-generalized convex contraction mapping. Suppose that $x, y \in X$ with $\alpha(x)\beta(y) \geq 1$ and then

$$\begin{aligned} d(T^2x, T^2y) &\leq \alpha(x)\beta(y)d(T^2x, T^2y) \\ &\leq ad(Tx,Ty) + bd(x,y). \end{aligned}$$

This implies that T is an (α,β)-generalized convex contraction mapping. By Theorem 3.2, we get the desired result. □

Corollary 3.5 *Let (X,d) be a metric space, $\alpha,\beta : X \to [0,\infty)$ be two mappings and $T : X \to X$ be a mapping such that*

$$\left[d\left(T^2x,T^2y\right)+\tau\right]^{\alpha(x)\beta(y)} \le ad(Tx,Ty)+bd(x,y)+\tau \tag{3.4}$$

for all $x,y \in X$, where $a,b \in [0,1)$ with $a+b<1$ and $\tau \ge 1$. Assume that T is an (α,β)-admissible mapping and there exists $x_0 \in X$ such that $\alpha(x_0) \ge 1$ and $\beta(Tx_0) \ge 1$. Then T has the approximate fixed point property. Moreover, if T is continuous and (X,d) is a complete metric space, then T has a fixed point.

Proof We will show that T is an (α,β)-generalized convex contraction mapping. Suppose that $x,y \in X$ with $\alpha(x)\beta(y) \ge 1$ and hence

$$\begin{aligned} d\left(T^2x,T^2y\right)+\tau &\le \left[d\left(T^2x,T^2y\right)+\tau\right]^{\alpha(x)\beta(y)} \\ &\le ad(Tx,Ty)+bd(x,y)+\tau. \end{aligned}$$

This implies that

$$d\left(T^2x,T^2y\right) \le ad(Tx,Ty)+bd(x,y),$$

that is, T is an (α,β)-generalized convex contraction mapping. By Theorem 3.2, we get the desired result. □

Corollary 3.6 *Let (X,d) be a metric space, $\alpha,\beta : X \to [0,\infty)$ be two mappings and $T : X \to X$ be a mapping such that*

$$\left[\tau-1+\alpha(x)\beta(y)\right]^{d(T^2x,T^2y)} \le \tau^{ad(Tx,Ty)+bd(x,y)} \tag{3.5}$$

for all $x,y \in X$, where $a,b \in [0,1)$ with $a+b<1$ and $\tau > 1$. Assume that T is a cyclic (α,β)-admissible mapping and there exists $x_0 \in X$ such that $\alpha(x_0)\beta(Tx_0) \ge 1$. Then T has the approximate fixed point property. Moreover, if T is continuous and (X,d) is a complete metric space, then T has a fixed point.

Proof We will show that T is an (α,β)-generalized convex contraction mapping. Suppose that $x,y \in X$ with $\alpha(x)\beta(y) \ge 1$ and hence

$$\begin{aligned} \tau^{d(T^2x,T^2y)} &\le \left[\tau-1+\alpha(x)\beta(y)\right]^{d(T^2x,T^2y)} \\ &\le \tau^{ad(Tx,Ty)+bd(x,y)}. \end{aligned}$$

This implies that

$$d\left(T^2x,T^2y\right) \le ad(Tx,Ty)+bd(x,y),$$

that is, T is an (α,β)-generalized convex contraction mapping. By Theorem 3.2, we get the desired result. □

We observe that the Banach contractive condition due to Banach [1] implies the contractive condition (3.1) whenever $\alpha,\beta : X \to [0,\infty)$ defined by $\alpha(x) = \beta(x) = 1$ for all $x \in X$. From the previous observation, we get the following result.

Corollary 3.7 *Let (X,d) be a metric space and $T : X \to X$ be a Banach contraction mapping, i.e., there exists $k \in [0,1)$ such that*

$$d(Tx,Ty) \leq kd(x,y) \tag{3.6}$$

for all $x,y \in X$. Then T has the approximate fixed point property. Moreover, if (X,d) is a complete metric space, then T has a fixed point.

Next, we introduce the concept of an (α,β)-generalized convex contraction of order 2 and also establish a new approximate fixed point theorem for such mappings in complete metric spaces.

Definition 3.8 Let (X,d) be a metric space. The mapping $T : X \to X$ is called an (α,β)-*generalized convex contraction of order 2* if there exist mappings $\alpha,\beta : X \to [0,\infty)$ and $a_1,a_2,b_1,b_2 \in [0,1)$ with $a_1 + a_2 + b_1 + b_2 < 1$, satisfying the following condition:

$$\begin{aligned} &\text{for all } x,y \in X, \quad \alpha(x)\beta(y) \geq 1 \\ &\quad \Longrightarrow \quad d\bigl(T^2x,T^2y\bigr) \leq a_1d(x,Tx) + a_2d\bigl(Tx,T^2x\bigr) + b_1d(y,Ty) + b_2d\bigl(Ty,T^2y\bigr). \end{aligned} \tag{3.7}$$

Theorem 3.9 *Let (X,d) be a metric space and $T : X \to X$ be an (α,β)-generalized convex contraction mapping of order 2. Assume that T is a cyclic (α,β)-admissible mapping and there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. Then T has the approximate fixed point property. Moreover, if T is continuous and (X,d) is a complete metric space, then T has a fixed point.*

Proof First of all, let $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = T^{n+1}x_0$ for all $n \in \mathbb{Z}^+ \cup \{0\}$. If $x_{n'} = x_{n'+1}$ for some $n' \in \mathbb{Z}^+ \cup \{0\}$, then $x_{n'}$ is a fixed point of T. So, we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{Z}^+ \cup \{0\}$. It follows from T being a cyclic (α,β)-admissible mapping that

$$\begin{aligned} &\alpha(x_0) \geq 1 \quad \Longrightarrow \quad \beta(x_1) \geq 1 \quad \Longrightarrow \quad \alpha(x_2) \geq 1 \quad \Longrightarrow \quad \cdots, \\ &\beta(x_0) \geq 1 \quad \Longrightarrow \quad \alpha(x_1) \geq 1 \quad \Longrightarrow \quad \beta(x_2) \geq 1 \quad \Longrightarrow \quad \cdots. \end{aligned}$$

Therefore, $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{Z}^+ \cup \{0\}$. Let us denote

$$\begin{aligned} &w := d\bigl(Tx_0,T^2x_0\bigr) + d(x_0,Tx_0), \\ &\delta := 1 - b_2, \end{aligned}$$

and

$$\epsilon := a_1 + a_2 + b_1.$$

Since $\alpha(x_0)\beta(x_1) \geq 1$, we get

$$\begin{aligned} d\big(T^2x_0, T^3x_0\big) &\leq a_1 d(x_0, Tx_0) + a_2 d\big(Tx_0, T^2x_0\big) + b_1 d\big(Tx_0, T^2x_0\big) + b_2 d\big(T^3x_0, T^2x_0\big) \\ &\leq a_1 w + (a_2 + b_1)w + b_2 d\big(T^3x_0, T^2x_0\big). \end{aligned}$$

This implies that $d(T^2x_0, T^3x_0) \leq \frac{\epsilon}{\delta}w$. Also, since $\alpha(x_1)\beta(x_2) \geq 1$, we get

$$\begin{aligned} d\big(T^3x_0, T^4x_0\big) &\leq a_1 d\big(Tx_0, T^2x_0\big) + a_2 d\big(T^2x_0, T^3x_0\big) \\ &\quad + b_1 d\big(T^2x_0, T^3x_0\big) + b_2 d\big(T^3x_0, T^4x_0\big) \\ &\leq a_1 w + (a_2 + b_1)\frac{\epsilon}{\delta}w + b_2 d\big(T^3x_0, T^4x_0\big) \\ &\leq a_1 w + (a_2 + b_1)w + b_2 d\big(T^3x_0, T^2x_0\big). \end{aligned}$$

It means that $d(T^3x_0, T^4x_0) \leq \frac{\epsilon}{\delta}w$. By continuing this process, we get

$$d\big(T^m x_0, T^{m+1}x_0\big) \leq \begin{cases} (\frac{\epsilon}{\delta})^{(m-1)/2}\nu & \text{if } m = 1, 3, 5, \ldots, \\ (\frac{\epsilon}{\delta})^{m/2}\nu & \text{if } m = 2, 4, 6, \ldots. \end{cases}$$

This follows that $d(T^m x_0, T^{m+1}x_0) \to 0$ as $m \to \infty$ and thus T is an asymptotically regular at a point $x_0 \in X$. By using Lemma 2.7, we see that T has the approximate fixed point property.

Next, we show that T has a fixed point provided that (X, d) is a complete metric space and T is continuous. First, we will claim that $\{x_n\}$ is a Cauchy sequence in X. Let $m, n \in \mathbb{Z}^+$ such that $n > m > 1$. Now, we distinguish the following cases.

Case 1: m is even number such that $m = 2l$, where $l \in \mathbb{Z}^+$. Therefore,

$$\begin{aligned} d\big(T^m x_0, T^n x_0\big) &\leq d\big(T^m x_0, T^{m+1}x_0\big) + d\big(T^{m+1}x_0, T^{m+2}x_0\big) + \cdots + d\big(T^{n-1}x_0, T^n x_0\big) \\ &\leq \left(\frac{\epsilon}{\delta}\right)^l w + \left(\frac{\epsilon}{\delta}\right)^l w + \left(\frac{\epsilon}{\delta}\right)^{l+1} w + \left(\frac{\epsilon}{\delta}\right)^{l+1} w + \left(\frac{\epsilon}{\delta}\right)^{l+2} w + \cdots \\ &\leq 2\left(\frac{\epsilon}{\delta}\right)^l w + 2\left(\frac{\epsilon}{\delta}\right)^{l+1} w + 2\left(\frac{\epsilon}{\delta}\right)^{l+2} w + \cdots \\ &= \frac{2(\frac{\epsilon}{\delta})^l w}{1 - \frac{\epsilon}{\delta}}. \end{aligned}$$

Case 2: m is odd number such that $m = 2l + 1$, where $l \in \mathbb{Z}^+$. Therefore,

$$\begin{aligned} d\big(T^m x_0, T^n x_0\big) &\leq d\big(T^m x_0, T^{m+1}x_0\big) + d\big(T^{m+1}x_0, T^{m+2}x_0\big) + \cdots + d\big(T^{n-1}x_0, T^n x_0\big) \\ &\leq \left(\frac{\epsilon}{\delta}\right)^l w + \left(\frac{\epsilon}{\delta}\right)^{l+1} w + \left(\frac{\epsilon}{\delta}\right)^{l+1} w + \left(\frac{\epsilon}{\delta}\right)^{l+2} w + \left(\frac{\epsilon}{\delta}\right)^{l+2} w + \cdots \end{aligned}$$

$$\begin{aligned}&\leq 2\left(\frac{\epsilon}{\delta}\right)^{l} w+2\left(\frac{\epsilon}{\delta}\right)^{l+1} w+2\left(\frac{\epsilon}{\delta}\right)^{l+2} w+\cdots \\ &=\frac{2(\frac{\epsilon}{\delta})^{l} w}{1-\frac{\epsilon}{\delta}}.\end{aligned}$$

Therefore, we can conclude that

$$d\left(T^{m} x_{0}, T^{n} x_{0}\right) \leq \begin{cases}\frac{2(\frac{\epsilon}{\delta})^{m/2} w}{1-\frac{\epsilon}{\delta}} & \text{if } m=2,4,6,\ldots, \\ \frac{2(\frac{\epsilon}{\delta})^{(m-1)/2} w}{1-\frac{\epsilon}{\delta}} & \text{if } m=3,5,7,\ldots.\end{cases}$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. As follows from T being continuous, we get $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$. By the uniqueness of the limit $\{x_n\}$, we obtain $Tx^* = x^*$ and thus T has a fixed point. This completes the proof. □

Corollary 3.10 *Let (X,d) be a metric space, $\alpha,\beta : X \to [0,\infty)$ be two mappings and $T : X \to X$ be a mapping such that*

$$\alpha(x)\beta(y)d\left(T^2x,T^2y\right) \leq a_1 d(x,Tx) + a_2 d\left(Tx,T^2x\right) + b_1 d(y,Ty) + b_2 d\left(Ty,T^2y\right) \tag{3.8}$$

for all $x, y \in X$, where $a_1, a_2, b_1, b_2 \in [0,1)$ with $a_1 + a_2 + b_1 + b_2 < 1$. Assume that T is a cyclic (α,β)-admissible mapping and there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. Then T has the approximate fixed point property. Moreover, if T is continuous and (X,d) is a complete metric space, then T has a fixed point.

Corollary 3.11 *Let (X,d) be a metric space, $\alpha,\beta : X \to [0,\infty)$ be two mappings and $T : X \to X$ be a mapping such that*

$$\begin{aligned}&\left[d\left(T^2x,T^2y\right)+\tau\right]^{\alpha(x)\beta(y)} \\ &\quad\leq a_1 d(x,Tx) + a_2 d\left(Tx,T^2x\right) + b_1 d(y,Ty) + b_2 d\left(Ty,T^2y\right) + \tau\end{aligned} \tag{3.9}$$

for all $x, y \in X$, where $a_1, a_2, b_1, b_2 \in [0,1)$ with $a_1 + a_2 + b_1 + b_2 < 1$ and $\tau \geq 1$. Assume that T is a cyclic (α,β)-admissible mapping and there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. Then T has the approximate fixed point property. Moreover, if T is continuous and (X,d) is a complete metric space, then T has a fixed point.

Corollary 3.12 *Let (X,d) be a metric space, $\alpha,\beta : X \to [0,\infty)$ be two mappings and $T : X \to X$ be a mapping such that*

$$\left[\tau-1+\alpha(x)\beta(y)\right]^{d(T^2x,T^2y)} \leq \tau^{a_1 d(x,Tx)+a_2 d(Tx,T^2x)+b_1 d(y,Ty)+b_2 d(Ty,T^2y)} \tag{3.10}$$

for all $x, y \in X$, where $a_1, a_2, b_1, b_2 \in [0,1)$ with $a_1 + a_2 + b_1 + b_2 < 1$ and $\tau > 1$. Assume that T is a cyclic (α,β)-admissible mapping and there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and

$\beta(x_0) \geq 1$. *Then T has the approximate fixed point property. Moreover, if T is continuous and (X,d) is a complete metric space, then T has a fixed point.*

The following result is a special case of Theorem 3.9 because the Kannan contractive condition due to Kannan [6] implies the contractive condition (3.7) if $\alpha, \beta : X \to [0,\infty)$ defined by $\alpha(x) = \beta(x) = 1$ for all $x \in X$.

Corollary 3.13 *Let (X,d) be a metric space and $T : X \to X$ be a Kannan contraction mapping, i.e., there exists $k \in [0,1/2)$ such that*

$$d(Tx,Ty) \leq k\big[d(x,Tx) + d(y,Ty)\big] \tag{3.11}$$

for all $x, y \in X$. Then T has the approximate fixed point property. Moreover, if T is continuous and (X,d) is a complete metric space, then T has a fixed point.

Corollaries 3.7 and 3.13, are interesting for defining the concepts of other classes of nonlinear mappings which are generalizations of several well-known mappings due to Chatterjea [10], Ćirić [11], Geraghty [12], Meir and Keeler [13], Mizoguchi and Takahashi [14], Suzuki [15], *etc.*

4 Some cyclic contractions via cyclic (α,β)-admissible mappings

In this section, we introduce the concept of cyclic generalized convex contraction mappings and prove some approximate fixed point results for such mappings in complete metric spaces.

Definition 4.1 Let A and B be two nonempty closed subsets of a metric space (X,d). The mapping $T : A \cup B \to A \cup B$ is called a *cyclic generalized convex contraction* if the following conditions hold:

(i) $T(A) \subseteq B$ and $T(B) \subseteq A$,

(ii) there exist $a, b \in [0,\infty)$ with $a + b < 1$, satisfying the following condition:

$$d\big(T^2x, T^2y\big) \leq ad(Tx,Ty) + bd(x,y) \tag{4.1}$$

for all $x \in A$, $y \in B$.

Example 4.2 Let $X = \mathbb{R}$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x,y) = |x - y|$ for all $x, y \in X$. Define $A = [-1,0]$, $B = [0,1]$, and $T : A \cup B \to A \cup B$ by $Tx = -\frac{x}{2}$. We will show that the mapping T is a cyclic generalized convex contraction with $a = \frac{2}{7}$ and $b = \frac{1}{3}$. Assume that $x \in A$ and $y \in B$. Then we have

$$\begin{aligned} d\big(T^2x, T^2y\big) &= \left|T\left(\frac{-x}{2}\right) - T\left(\frac{-y}{2}\right)\right| \\ &= \left|\frac{x}{4} - \frac{y}{4}\right| \\ &= \frac{1}{4}|x-y| \\ &\leq \frac{10}{21}|x-y| \end{aligned}$$

$$\begin{aligned} &= \frac{2}{7}\left|\frac{x}{2} - \frac{y}{2}\right| + \frac{1}{3}|x-y| \\ &= ad(Tx, Ty) + bd(x,y). \end{aligned}$$

Therefore, T is a cyclic generalized convex contraction mapping with $a = \frac{2}{7}$ and $b = \frac{1}{3}$.

Now, we establish approximate fixed point theorems for cyclic generalized convex contraction mappings as follows.

Theorem 4.3 *Let A and B be two nonempty closed subsets of a metric space (X,d) such that $A \cap B \neq \emptyset$ and $T : A \cup B \to A \cup B$ be a cyclic generalized convex contraction mapping. Then T has the approximate fixed point property. Moreover, if T is continuous and (X,d) is a complete metric space, then T has a fixed point in $A \cap B$.*

Proof Define two mappings $\alpha, \beta : X \to [0,\infty)$ by

$$\alpha(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\beta(x) = \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{otherwise.} \end{cases}$$

Next, we will show that T is an (α,β)-generalized convex contraction mapping. Assume that $x, y \in A \cup B$ such that $\alpha(x)\beta(y) \geq 1$ and then $x \in A$ and $y \in B$. Therefore, we have

$$d\bigl(T^2x, T^2y\bigr) \leq ad(Tx, Ty) + bd(x,y)$$

and so the contractive condition (3.1) holds.

Now we will claim that T is a cyclic (α,β)-admissible mapping. Assume that $\alpha(x) \geq 1$ for some $x \in X$ and then $x \in A$. By condition (i) in Definition 4.1, we get $Tx \in B$ and so $\beta(Tx) \geq 1$. On the other hand, we may assume that $\beta(y) \geq 1$ for some $y \in X$ and so $y \in B$. Again, by using the condition (i) in Definition 4.1, we get $Tx \in A$ and then $\alpha(Tx) \geq 1$. Therefore T is a cyclic (α,β)-admissible mapping. Since $A \cap B$ is nonempty, there exists $x_0 \in A \cap B$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. From Theorem 3.2, we can conclude that T has the approximate fixed point property.

Finally, we will show that T has a fixed point provided that T is continuous and (X,d) is a complete metric space. Since A and B are two closed subsets of complete metric space (X,d), we see that $A \cup B$ is also a complete metric space. From Theorem 3.2, we see that T has a fixed point in $A \cup B$, say z. If $z \in A$, then we have $z = Tz \in B$. Also, if $z \in B$, we have $z = Tz \in A$. Therefore, $z \in A \cap B$. This completes the proof. □

Next, we introduce the concept of cyclic generalized convex contraction mappings of order 2 and establish the approximate fixed point theorem for such mappings in complete metric spaces.

Definition 4.4 Let A and B be two closed subsets of a metric space (X, d). The mapping $T : A \cup B \to A \cup B$ is called a *cyclic generalized convex contraction of order 2* if the following conditions hold:

(i) $T(A) \subseteq B$ and $T(B) \subseteq A$,

(ii) there exist $a_1, a_2, b_1, b_2 \in [0, 1)$ with $a_1 + a_2 + b_1 + b_2 < 1$, satisfying the following condition:

$$d\bigl(T^2x, T^2y\bigr) \leq a_1 d(x, Tx) + a_2 d\bigl(Tx, T^2x\bigr) + b_1 d(y, Ty) + b_2 d\bigl(Ty, T^2y\bigr) \tag{4.2}$$

for all $x \in A$, $y \in B$.

By a similar technique to the proof of Theorem 4.3, we get the following result.

Theorem 4.5 *Let A and B be two nonempty closed subsets of a metric space (X, d) such that $A \cap B \neq \emptyset$ and $T : A \cup B \to A \cup B$ be a cyclic generalized convex contraction mapping of order 2. Then T has the approximate fixed point property. Moreover, if T is continuous and (X, d) is a complete metric space, then T has a fixed point in $A \cap B$.*

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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