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A Schauder-type theorem for discontinuous operators with applications to second-order BVPs

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Abstract

We prove a new fixed point theorem of Schauder type, which applies to discontinuous operators in noncompact domains. In order to do so, we present a modification of a recent Schauder-type theorem of Pouso. We apply our result to second-order boundary value problems with discontinuous nonlinearities. We include an example to illustrate our theory.

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1 Introduction

In the recent and interesting paper [1], Pouso proved a novel version of Schauder's theorem for discontinuous operators in compact sets. Pouso used this tool to prove new results on the existence of solutions of a widely studied second-order ordinary differential equation (ODE) subject to Dirichlet boundary conditions (BCs), namely

$$u'' = f(t, u), \quad u(0) = u(1) = 0,$$

where f is an L^1 -bounded nonlinearity. The approach in [1] relies on a careful use of ideas of set-valued analysis and viability theory.

In this manuscript, we further develop the ideas of Pouso. First, we prove that a Schauder-type theorem for discontinuous operators can be formulated for arbitrary nonempty, closed, and convex (not necessarily bounded) subsets of a Banach space. Second, we apply our new result to prove the existence of solutions of a large class of discontinuous second-order ODEs subject to separated BCs, complementing the results of [1] and improving them also in the special case of Dirichlet BCs.

2 Schauder's fixed point theorem for discontinuous operators

For completeness, we begin this section by recalling the classical Schauder fixed point theorem.

Theorem 2.1 ([2], Theorem 2.A) *Let K be a nonempty, closed, bounded, convex subset of a Banach space X and suppose that $T : K \rightarrow K$ is a compact operator (that is, T is continuous and maps bounded sets into precompact ones). Then T has a fixed point.*

A well-known consequence of Theorem 2.1 is the following.

Corollary 2.2 ([2], Corollary 2.13) *Let K be a nonempty, compact and convex subset of a Banach space X , and $T : K \rightarrow K$ a continuous operator. Then T has a fixed point.*

The main result in [1] is an improvement of Corollary 2.2, where the continuity of the operator T is replaced by a weaker assumption. We briefly describe the main idea: given a compact subset K of a Banach space X and an operator $T : K \rightarrow K$, which can be discontinuous, it is possible to construct a multivalued mapping \mathbb{T} by ‘convexifying’ T as follows:

$$\mathbb{T}u := \bigcap_{\varepsilon > 0} \overline{\text{co}}(T(B_\varepsilon(u) \cap K)) \quad \text{for every } u \in K, \tag{2.1}$$

where $B_\varepsilon(u)$ denotes the closed ball centered in u with radius ε , and $\overline{\text{co}}$ denotes the closed convex hull. The operator \mathbb{T} in (2.1) is an upper semicontinuous mapping with convex and compact values (see [3, 4]), and therefore Kakutani’s fixed point theorem guarantees that \mathbb{T} has a fixed point in K . If we impose an extra assumption that, roughly speaking, states that a fixed point of \mathbb{T} must be a fixed point of T , then we obtain the desired result.

The following characterization sheds light on the definition of the multivalued operator \mathbb{T} . It is formulated for compact subsets, but it works for arbitrary nonempty subsets of a Banach space (see also [1], Proposition 3.2).

Proposition 2.3 *Let K be a compact subset of a Banach space X , and $T : K \rightarrow K$. Then the following statements are equivalent:*

- (1) $y \in \mathbb{T}u$, where \mathbb{T} is as in (2.1);
- (2) for every $\varepsilon > 0$ and every $\rho > 0$, there exists a finite family of vectors $u_i \in B_\varepsilon(u) \cap K$ and coefficients $\lambda_i \in [0, 1]$ ($i = 1, \dots, m$) such that $\sum \lambda_i = 1$ and

$$\left\| y - \sum_{i=1}^m \lambda_i T u_i \right\| < \rho.$$

The variant of Schauder’s theorem in compact subsets given by Pouso is the following.

Theorem 2.4 ([1], Theorem 3.1) *Let K be a nonempty, compact and convex subset of a normed space X , and $T : K \rightarrow K$. Furthermore, assume that*

$$\{u\} \cap \mathbb{T}u \subset \{Tu\} \quad \text{for all } u \in K,$$

where \mathbb{T} is as in (2.1). Then T has a fixed point.

Theorem 2.4 is very interesting and powerful; however, when we want to look for solutions for a certain boundary value problem (BVP), the fact of working in a compact domain could be quite restrictive. In order to overcome this difficulty, we first recall that Theorem 2.1 admits the following extension to unbounded domains.

Theorem 2.5 ([5], Theorem 4.4.10) *Let M be nonempty, closed and convex subset of a Banach space X , and $T : M \rightarrow M$ a continuous operator. If $T(M)$ is precompact, then T has a fixed point.*

Secondly, we recall the following result due to Bohnenblust and Karlin.

Theorem 2.6 ([2], Corollary 9.8) *Let M be a nonempty, closed and convex subset of a Banach space X and suppose that*

- (i) $T : M \rightarrow 2^M$ is upper semicontinuous;
- (ii) $T(M)$ is relatively compact in X ;
- (iii) $T(u)$ is nonempty, closed, and convex for all $u \in M$.

Then T has a fixed point.

Now we introduce the main result in this section, which is an extension of Theorem 2.5 to the case of discontinuous operators.

Theorem 2.7 *Let M be a nonempty, closed, and convex subset of a Banach space X , and $T : M \rightarrow M$ a mapping satisfying*

- (i) $T(M)$ is relatively compact in X ;
- (ii) $\{u\} \cap \mathbb{T}u \subset \{Tu\}$ for all $u \in M$, where \mathbb{T} is as in (2.1).

Then T has a fixed point in M .

Proof The multivalued operator \mathbb{T} is upper semicontinuous with $\mathbb{T}u$ nonempty, convex and compact for each $u \in M$. Now we show that condition (i) implies that $\mathbb{T}(M)$ is relatively compact on X . Indeed, for each $u \in M$ and all $\varepsilon > 0$, we have that

$$\overline{\text{co}}T(B_\varepsilon(u) \cap M) \subset \overline{\text{co}}T(M),$$

and therefore $\mathbb{T}(M)$ is a closed subset of the compact set $\overline{\text{co}}T(M)$ (note that the closed convex hull of a compact set in a Banach space is also compact; see, for example, [6], Theorem 5.35).

Since $\mathbb{T}(M)$ is relatively compact, we obtain by application of Theorem 2.6 that \mathbb{T} has a fixed point. Finally, condition (ii) implies that the obtained fixed point of \mathbb{T} is a fixed point of T . □

Remark 2.8 Notice that if T is continuous then $\mathbb{T}u = \{Tu\}$ for all u , and so previous results regarding operator \mathbb{T} actually generalize known results about single-valued operators.

3 Second-order BVPs with separated BCs

In this section, we apply the previous abstract result on fixed points for discontinuous operators in order to look for $W^{2,1}$ -solutions for the following singular second-order ODE with separated BCs:

$$\begin{cases} u''(t) + g(t)f(t, u(t)) = 0 & \text{for almost every (a.e.) } t \in I = [0, 1], \\ \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{cases} \tag{3.1}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ and $\Gamma = \gamma\beta + \alpha\gamma + \alpha\delta > 0$.

This kind of second-order BVPs have received a lot of attention in the literature. For example, in the monograph [7] the method of lower and upper solutions is used to look for C^2 -solutions in the case of continuous nonlinearities and $W^{2,1}$ -solutions in the case of Carathéodory ones. This method is also applied in [8] to a continuous φ -Laplacian problem with separated BCs. On the other hand, a monotone method is applied in [9] in order to look for extremal solutions for a functional problem with derivative dependence in the nonlinearity. As a main novelty of the present work, we allow the nonlinearity f to have a countable number of discontinuities with respect to its spatial variable, and we require no monotonicity conditions. Moreover, the function g can be singular.

To apply our new fixed point theorem to the BVP (3.1), we recall that $u \in W^{2,1}(I)$ is a solution of (3.1) if (and only if) u is a solution of the following Hammerstein integral equation:

$$u(t) = \int_0^1 k(t,s)g(s)f(s, u(s)) ds, \tag{3.2}$$

whenever the integral in (3.2) has sense and where k is the corresponding Green function, which is given by (see, for example, [10])

$$k(t,s) = \frac{1}{\Gamma} \begin{cases} (\gamma + \delta - \gamma t)(\beta + \alpha s) & \text{if } 0 \leq s \leq t \leq 1, \\ (\beta + \alpha t)(\gamma + \delta - \gamma s) & \text{if } 0 \leq t < s \leq 1. \end{cases} \tag{3.3}$$

It is known [10] that k is nonnegative. Furthermore, note that k is continuous (and therefore bounded) in the square $[0, 1] \times [0, 1]$ and that its partial derivatives $\frac{\partial k}{\partial t}$ and $\frac{\partial k}{\partial s}$ can be discontinuous on the diagonal $t = s$. However, these partial derivatives are essentially bounded on the square.

In the sequel, we consider the Banach space $X = C^1(I)$ of continuously differentiable functions defined on I with the norm

$$\|u\| = \sup_{t \in I} |u(t)| + \sup_{t \in I} |u'(t)|.$$

Lemma 3.1 *Assume that:*

- (H₁) $g \in L^1(I)$;
- (H₂) *there exist $R > 0$ and $H_R \in L^\infty(I)$ such that for a.e. $t \in I$ and all $u \in [-R, R]$ we have $|f(t, u)| \leq H_R(t)$;*
- (H₃) *the following estimate holds:*

$$\|H_R\|_{L^\infty} (M_1 + M_2) \leq R,$$

where

$$M_1 = \sup_{t \in I} \int_0^1 k(t,s)|g(s)| ds, \quad M_2 = \sup_{t \in I} \int_0^1 \left| \frac{\partial k}{\partial t}(t,s)g(s) \right| ds; \tag{3.4}$$

- (H₄) *for each $u \in \overline{B}_R = \{u \in X : \|u\| \leq R\}$ the composition $t \in I \mapsto f(t, u(t))$ is a measurable function.*

Then the operator $T : \bar{B}_R \rightarrow X$ given by

$$Tu(t) = \int_0^1 k(t,s)g(s)f(s,u(s)) ds$$

is well defined and maps \bar{B}_R into itself.

Remark 3.2 Since k is the Green’s function related to a homogeneous second-order BVP, $Tu \in W^{2,1}(I)$ for all u , so, in particular, $(Tu)'$ is absolutely continuous (then $Tu \in X$), and $(Tu)''$ exists almost everywhere on I . This will be used later in our argumentations.

Proof of Lemma 3.1 Let $R > 0$ given by condition (H_2) . First, note that the kernel k has the form (3.3). Therefore, for each $t \in [0, 1]$, $k(t, \cdot)$ is a continuous function, and for $s \neq t$, the function $s \in [0, 1] \rightarrow \frac{\partial k}{\partial t}(t, s)$ is well defined and integrable. Then, conditions (H_1) , (H_2) and (H_4) imply that for $u \in \bar{B}_R$ the function Tu is well defined.

On the other hand, for $u \in \bar{B}_R$ we have

$$\begin{aligned} \|Tu\| &\leq \sup_{t \in I} \int_0^1 k(t,s)|g(s)||f(s,u(s))| ds + \sup_{t \in I} \int_0^1 \left| \frac{\partial k}{\partial t}(t,s) \right| |g(s)||f(s,u(s))| ds \\ &\leq \|H_R\|_\infty (M_1 + M_2), \end{aligned}$$

and then condition (H_3) implies that $\|Tu\| \leq R$. □

Lemma 3.3 Under the assumptions of Lemma 3.1, $T(\bar{B}_R)$ is relatively compact in X .

Proof We have shown in Lemma 3.1 that $T(\bar{B}_R) \subset \bar{B}_R$. Therefore, the set $T(\bar{B}_R)$ is totally bounded in X . Now, to see that $T(\bar{B}_R)$ is equicontinuous, we only have to notice that, for a.e. $t \in I$ and every $u \in \bar{B}_R$, we have

$$|(Tu)''(t)| \leq |g(t)|H_R(t),$$

which implies that

$$|(Tu)'(t) - (Tu)'(s)| \leq \int_t^s |(Tu)''(r)| dr \leq \int_t^s |g(r)|H_R(r) dr.$$

Then $T(\bar{B}_R)$ is relatively compact in X . □

In a similar way as in Definition 4.1 of [1], we introduce the admissible discontinuities for our nonlinearities.

Definition 3.4 We say that $\gamma : [a, b] \subset I \rightarrow \mathbb{R}$, $\gamma \in W^{2,1}([a, b])$, is an admissible discontinuity curve for the differential equation $u''(t) + g(t)f(t, u(t)) = 0$ if one of the following conditions holds:

- (i) $-\gamma''(t) = g(t)f(t, \gamma(t))$ for a.e. $t \in [a, b]$;
- (ii) there exist $\psi \in L^1([a, b])$, $\psi > 0$ almost everywhere, and $\varepsilon > 0$ such that

$$\begin{aligned} &\text{either } -\gamma''(t) + \psi(t) < g(t)f(t, \gamma) \\ &\text{for a.e. } t \in [a, b] \text{ and all } \gamma \in [\gamma(t) - \varepsilon, \gamma(t) + \varepsilon], \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\text{or } -\gamma''(t) - \psi(t) > g(t)f(t, \gamma) \\ &\text{for a.e. } t \in [a, b] \text{ and all } \gamma \in [\gamma(t) - \varepsilon, \gamma(t) + \varepsilon]. \end{aligned} \tag{3.6}$$

If (i) holds, then we say that γ is *viable* for the differential equation; if (ii) holds, we say that γ is *inviable*.

The previous definition says, roughly speaking, that a time-dependent discontinuity curve γ is admissible if one of the following holds: either γ solves the differential equation on its domain, or, if it does not, the solutions are pushed ‘far away’ from γ .

To prove our main result on the existence of solutions for problem (3.1) by using admissible discontinuity curves, we need some auxiliary theoretical results on integrable functions. The reader can see their proofs in [1].

Lemma 3.5 ([1], Lemma 4.1) *Let $a, b \in \mathbb{R}, a < b$, and let $g, h \in L^1(a, b), g \geq 0$ a.e., and $h > 0$ a.e. in (a, b) .*

For every measurable set $J \subset (a, b)$ with $m(J) > 0$, there is a measurable set $J_0 \subset J$ with $m(J \setminus J_0) = 0$ such that, for every $\tau_0 \in J_0$, we have

$$\lim_{t \rightarrow \tau_0^+} \frac{\int_{[\tau_0, t] \setminus J} g(s) ds}{\int_{\tau_0}^t h(s) ds} = 0 = \lim_{t \rightarrow \tau_0^-} \frac{\int_{[t, \tau_0] \setminus J} g(s) ds}{\int_t^{\tau_0} h(s) ds}. \tag{3.7}$$

Corollary 3.6 ([1], Corollary 4.2) *Let $a, b \in \mathbb{R}, a < b$, and let $h \in L^1(a, b)$ be such that $h > 0$ a.e. in (a, b) .*

For every measurable set $J \subset (a, b)$ with $m(J) > 0$, there is a measurable set $J_0 \subset J$ with $m(J \setminus J_0) = 0$ such that, for all $\tau_0 \in J_0$, we have

$$\lim_{t \rightarrow \tau_0^+} \frac{\int_{[\tau_0, t] \cap J} h(s) ds}{\int_{\tau_0}^t h(s) ds} = 1 = \lim_{t \rightarrow \tau_0^-} \frac{\int_{[t, \tau_0] \cap J} h(s) ds}{\int_t^{\tau_0} h(s) ds}. \tag{3.8}$$

Corollary 3.7 ([1], Corollary 4.3) *Let $a, b \in \mathbb{R}, a < b$, and let $f, f_n : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous functions on $[a, b]$ ($n \in \mathbb{N}$) such that $f_n \rightarrow f$ uniformly on $[a, b]$ and, for a measurable set $A \subset [a, b]$ with $m(A) > 0$, we have*

$$\lim_{n \rightarrow \infty} f'_n(t) = g(t) \quad \text{for a.a. } t \in A.$$

If there exists $M \in L^1(a, b)$ such that $|f'(t)| \leq M(t)$ a.e. in $[a, b]$ and also $|f'_n(t)| \leq M(t)$ a.e. in $[a, b]$ ($n \in \mathbb{N}$), then $f'(t) = g(t)$ for a.a. $t \in A$.

Now we can show the main result in this section.

Theorem 3.8 *Let f and g satisfy (H₁)-(H₄) and the following:*

(H₅) *there exist admissible discontinuity curves $\gamma_n : I_n = [a_n, b_n] \rightarrow \mathbb{R}, n \in \mathbb{N}$, such that for a.e. $t \in I$ the function $f(t, \cdot)$ is continuous in $[-R, R] \setminus \bigcup_{n:t \in I_n} \{\gamma_n(t)\}$.*

Then problem (3.1) has at least one solution in \overline{B}_R .

Proof We consider the multivalued operator \mathbb{T} associated to T as in (2.1). Therefore, \mathbb{T} is upper semicontinuous with nonempty, convex and compact values and, as T , maps \bar{B}_R into itself. Moreover, $\mathbb{T}(\bar{B}_R)$ is relatively compact in X by Lemma 3.3. Therefore, if we show that $\{u\} \cap \mathbb{T}u \subset \{Tu\}$, then we obtain by Theorem 2.7 that T has a fixed point in \bar{B}_R , which corresponds to a solution of the BVP (3.1). This part of the proof now follows the lines of [1], Theorem 4.4, but we include it for completeness and for highlighting the main differences between the two results. Thus, we fix $u \in \bar{B}_R$ and consider three cases.

Case 1: $m(\{t \in I_n : u(t) = \gamma_n(t)\}) = 0$ for all $n \in \mathbb{N}$.

Then we have that $f(t, \cdot)$ is continuous for a.e. $t \in I$, and therefore if $u_k \rightarrow u$ in \bar{B}_R then we obtain $f(t, u_k(t)) \rightarrow f(t, u(t))$ for a.e. $t \in I$. This, together with (H₂) and (H₃), implies that Tu_k converges uniformly to Tu in X . Then, T is continuous at u , and therefore we obtain $\mathbb{T}u = \{Tu\}$.

Case 2: there exists $n \in \mathbb{N}$ such that γ_n is inviable and $m(\{t \in I_n : u(t) = \gamma_n(t)\}) > 0$. We will show that, in this case, $u \notin \mathbb{T}u$.

To do this, assume that γ_n satisfies (3.6) (the other case is similar). Let $\psi \in L^1(I)$ and $\varepsilon > 0$ given by (3.6) and set

$$J = \{t \in I_n : u(t) = \gamma_n(t)\}, \quad M(t) = |g(t)|H_R(t).$$

Notice that (H₁) and (H₂) imply that $M \in L^1(I)$, and so we deduce from Lemma 3.5 that there is a measurable set $J_0 \subset J$ with $m(J_0) = m(J) > 0$ such that, for all $\tau_0 \in J_0$, we have

$$\lim_{t \rightarrow \tau_0^+} \frac{2 \int_{[\tau_0, t] \setminus J} M(s) ds}{(1/4) \int_{\tau_0}^t \psi(s) ds} = 0 = \lim_{t \rightarrow \tau_0} \frac{2 \int_{[t, \tau_0] \setminus J} M(s) ds}{(1/4) \int_t^{\tau_0} \psi(s) ds}. \tag{3.9}$$

By Corollary 3.6 there exists $J_1 \subset J_0$ with $m(J_0 \setminus J_1) = 0$ such that, for all $\tau_0 \in J_1$, we have

$$\lim_{t \rightarrow \tau_0^+} \frac{\int_{[\tau_0, t] \cap J_0} \psi(s) ds}{\int_{\tau_0}^t \psi(s) ds} = 1 = \lim_{t \rightarrow \tau_0} \frac{\int_{[t, \tau_0] \cap J_0} \psi(s) ds}{\int_t^{\tau_0} \psi(s) ds}. \tag{3.10}$$

Let us now fix a point $\tau_0 \in J_1$. From (3.9) and (3.10) we deduce that there exist $t_- < \tau_0$ and $t_+ > \tau_0$, t_{\pm} sufficiently close to τ_0 , such that the following inequalities are satisfied:

$$2 \int_{[\tau_0, t_+] \setminus J} M(s) ds < \frac{1}{4} \int_{\tau_0}^{t_+} \psi(s) ds, \tag{3.11}$$

$$\int_{[\tau_0, t_+] \cap J} \psi(s) ds \geq \int_{[\tau_0, t_+] \cap J_0} \psi(s) ds > \frac{1}{2} \int_{\tau_0}^{t_+} \psi(s) ds, \tag{3.12}$$

$$2 \int_{[t_-, \tau_0] \setminus J} M(s) ds < \frac{1}{4} \int_{t_-}^{\tau_0} \psi(s) ds, \tag{3.13}$$

$$\int_{[t_-, \tau_0] \cap J} \psi(s) ds > \frac{1}{2} \int_{t_-}^{\tau_0} \psi(s) ds. \tag{3.14}$$

Finally, we define the positive number

$$\rho = \min \left\{ \frac{1}{4} \int_{t_-}^{\tau_0} \psi(s) ds, \frac{1}{4} \int_{\tau_0}^{t_+} \psi(s) ds \right\}, \tag{3.15}$$

and we are now in a position to prove that $u \notin \mathbb{T}u$. By Proposition 2.3 it suffices to prove the following claim.

Claim *Let $\varepsilon > 0$ be given by our assumptions over γ_n , and let ρ be as in (3.15). For every finite family $u_i \in B_\varepsilon(u) \cap \bar{B}_R$ and $\lambda_i \in [0, 1]$ ($i = 1, 2, \dots, m$) with $\sum \lambda_i = 1$, we have*

$$\left\| u - \sum \lambda_i Tu_i \right\|_{C^1} \geq \rho.$$

Let u_i and λ_i be as in the claim and, for simplicity, denote $v = \sum \lambda_i Tu_i$. Then for a.a. $t \in J = \{t \in I_n : u(t) = \gamma_n(t)\}$ we have

$$v''(t) = \sum_{i=1}^m \lambda_i (Tu_i)''(t) = \sum_{i=1}^m \lambda_i g(t) f(t, u_i(t)). \tag{3.16}$$

On the other hand, for every $i \in \{1, 2, \dots, m\}$ and every $t \in J$, we have

$$|u_i(t) - \gamma_n(t)| = |u_i(t) - u(t)| < \varepsilon,$$

and then the assumptions on γ_n ensure that, for a.a. $t \in J$, we have

$$v''(t) = \sum_{i=1}^m \lambda_i g(t) f(t, u_i(t)) < \sum_{i=1}^m \lambda_i (\gamma_n''(t) - \psi(t)) = u''(t) - \psi(t). \tag{3.17}$$

Now we compute

$$\begin{aligned} v'(\tau_0) - v'(t_-) &= \int_{t_-}^{\tau_0} v''(s) ds = \int_{[t_-, \tau_0] \cap J} v''(s) ds + \int_{[t_-, \tau_0] \setminus J} v''(s) ds \\ &< \int_{[t_-, \tau_0] \cap J} u''(s) ds - \int_{[t_-, \tau_0] \cap J} \psi(s) ds \\ &\quad + \int_{[t_-, \tau_0] \setminus J} M(s) ds \quad (\text{by (3.17), (3.16) and (H}_2\text{)}) \\ &= u'(\tau_0) - u'(t_-) - \int_{[t_-, \tau_0] \setminus J} u''(s) ds - \int_{[t_-, \tau_0] \cap J} \psi(s) ds + \int_{[t_-, \tau_0] \setminus J} M(s) ds \\ &\leq u'(\tau_0) - u'(t_-) - \int_{[t_-, \tau_0] \cap J} \psi(s) ds + 2 \int_{[t_-, \tau_0] \setminus J} M(s) ds \\ &< u'(\tau_0) - u'(t_-) - \frac{1}{4} \int_{t_-}^{\tau_0} \psi(s) ds \quad (\text{by (3.13) and (3.14)}). \end{aligned}$$

Hence, $\|u - v\|_{C^1} \geq v'(t_-) - u'(t_-) \geq \rho$, provided that $v'(\tau_0) \geq u'(\tau_0)$.

Similar computations with t_+ instead of t_- show that if $v'(\tau_0) \leq u'(\tau_0)$, then we also have $\|u - v\|_{C^1} \geq \rho$. The claim is proven.

Case 3: $m(\{t \in I_n : u(t) = \gamma_n(t)\}) > 0$ only for some of those $n \in \mathbb{N}$ such that γ_n is viable. We will show that, in this case, $u \in \mathbb{T}u$ implies $u = Tu$.

To see that, we consider the subsequence of all viable admissible discontinuity curves in the conditions of Case 3, which we can denote without loss of generality by $\{\gamma_n\}_{n \in \mathbb{N}}$. We

have $m(J_n) > 0$ for all $n \in \mathbb{N}$, where

$$J_n = \{t \in I_n : u(t) = \gamma_n(t)\}.$$

For each $n \in \mathbb{N}$ and for a.a. $t \in J_n$, we have

$$u''(t) = \gamma_n''(t) = g(t)f(t, \gamma_n(t)) = g(t)f(t, u(t)),$$

and therefore $u''(t) = f(t, u(t))$ a.e. in $J = \bigcup_{n \in \mathbb{N}} J_n$.

Now we assume that $u \in \mathbb{T}u$, and we prove that it also implies that $u''(t) = g(t)f(t, u(t))$ a.e. in $I \setminus J$, thus showing that $u = Tu$.

Since $u \in \mathbb{T}u$, for each $k \in \mathbb{N}$, we can use Proposition 2.3 with $\varepsilon = \rho = 1/k$ to guarantee that we can find functions $u_{k,i} \in B_{1/k}(u) \cap \bar{B}_R$ and coefficients $\lambda_{k,i} \in [0, 1]$ ($i = 1, 2, \dots, m(k)$) such that $\sum \lambda_{k,i} = 1$ and

$$\left\| u - \sum_{i=1}^{m(k)} \lambda_{k,i} Tu_{k,i} \right\|_{C^1} < \frac{1}{k}.$$

Let us denote $v_k = \sum_{i=1}^{m(k)} \lambda_{k,i} Tu_{k,i}$. Notice that $v'_k \rightarrow u'$ uniformly in I and

$$\|u_{k,i} - u\|_{C^1} \leq \frac{1}{k}$$

for all $k \in \mathbb{N}$ and all $i \in \{1, 2, \dots, m(k)\}$.

For a.a. $t \in I \setminus J$, we have that $g(t)f(t, \cdot)$ is continuous at $u(t)$, so for any $\varepsilon > 0$, there is some $k_0 = k_0(t) \in \mathbb{N}$ such that, for all $k \in \mathbb{N}$, $k \geq k_0$, we have

$$|g(t)f(t, u_{k,i}(t)) - g(t)f(t, u(t))| < \varepsilon \quad \text{for all } i \in \{1, 2, \dots, m(k)\},$$

and therefore

$$|v'_k(t) - g(t)f(t, u(t))| \leq \sum_{i=1}^{m(k)} \lambda_{k,i} |g(t)f(t, u_{k,i}(t)) - g(t)f(t, u(t))| < \varepsilon.$$

Hence, $v'_k(t) \rightarrow g(t)f(t, u(t))$ for a.a. $t \in I \setminus J$, and then Corollary 3.7 guarantees now that $u''(t) = g(t)f(t, u(t))$ for a.a. $t \in I \setminus J$.

Then, we have proven that $\{u\} \cap \mathbb{T}u \subset \{Tu\}$ for all $u \in \bar{B}_R$. By application of Theorem 2.7 we obtain that T has at least one fixed point in \bar{B}_R , which corresponds to a solution of the BVP (3.1) in \bar{B}_R . □

Remark 3.9 Note that if $g(t)f(t, 0) = 0$ for almost all $t \in [0, 1]$, then 0 is a solution of the BVP (3.1). Therefore, when $g(t)f(t, 0) \neq 0$ in a set of positive measure, then Theorem 3.8 provides the existence of a nontrivial solution. In this case, since the kernel k is nonnegative and if, moreover, $g(t)f(t, u) \geq 0$ almost everywhere, then we obtain the existence of a *nonnegative* solution with a nontrivial norm.

Remark 3.10 The improvement with respect to Theorem 4.4 of [1] relies not only on the fact that we can deal with a more general set of BCs but also on the fact that we do not require *global* L^1 estimates on f , allowing a more general class of nonlinearities. On the other hand, notice that our result can be extended to other type of BCs whenever condition (H_3) makes sense for the corresponding Green’s function.

Finally, we illustrate our results by an example.

Example 3.11 For $n \in \mathbb{N}$, we denote by $\phi(n)$ the function such that $\phi(1) = 2$ and, for $n \geq 2$, $\phi(n)$ counts the number of divisors of n . Thus defined, $\phi(n) \geq 2$ for all $n \in \mathbb{N}$, ϕ is not bounded, and, since there are infinitely many prime numbers, $\liminf_{n \rightarrow \infty} \phi(n) = 2$. Now we define the function

$$(t, u) \in (0, 1] \times \mathbb{R} \mapsto \tilde{f}(t, u) = \phi^\lambda(n(t, u)), \quad \lambda \in (0, 1), \tag{3.18}$$

where

$$n(t, u) := \begin{cases} 1 & \text{if } u \in (-\infty, -t), \\ n & \text{if } -\frac{t}{n} \leq u < -\frac{t}{n+1} \text{ and } -t \leq u < 0, \\ n & \text{if } (n-1)\sqrt{t} \leq u < n\sqrt{t} \text{ and } u \geq 0. \end{cases}$$

We are concerned with the ODE

$$u''(t) = \frac{\phi^\lambda(n(t, u))}{\sqrt{t}} \quad \text{for a.e. } t \in I = [0, 1], \tag{3.19}$$

coupled with separated BCs.

We claim that this problem has at least one solution. In order to show this, note that we can rewrite the ODE (3.19) in the form $u''(t) + g(t)f(t, u(t)) = 0$, where $g(t) = \frac{1}{\sqrt{t}}$ and $f = -\tilde{f}$ with \tilde{f} as in (3.18). We now show that the functions g and f satisfy conditions (H_1) - (H_5) .

First, it is clear that $g \in L^1(I)$, and so (H_1) holds. On the other hand, since for all $n \in \mathbb{N}$ it is $\phi(n) \leq \max\{2, n\}$, we obtain that we have $u \in [-n, n] \Rightarrow |f(t, u)| \leq \max\{2, n\}^\lambda$ for each $n \in \mathbb{N}$. Then, if we take $R \in \mathbb{N}, R \geq 2$, large enough such that $M_1 + M_2 \leq R^{1-\lambda}$ (with M_1, M_2 as in (3.4)), then we can guarantee that (H_2) and (H_3) hold.

To check (H_4) , note that for every continuous function u , we can write the composition $t \in I \mapsto f(t, u(t))$ as

$$t \mapsto f(t, u(t)) = - \sum_{n=1}^{\infty} \phi^\lambda(n)(\chi_{I_n}(t) + \chi_{J_n}(t)) + \phi(1)\chi_K(t), \tag{3.20}$$

where χ denotes the characteristic function, and I_n, J_n, K are the following measurable sets:

$$\begin{cases} I_n = u^{-1}([(n-1)\sqrt{t}, n\sqrt{t}] \cap [0, +\infty)), & n \in \mathbb{N}, \\ J_n = u^{-1}([\frac{-1}{n}t, \frac{-1}{n+1}t] \cap [-t, 0)), & n \in \mathbb{N}, \\ K = u^{-1}((-\infty, -t)). \end{cases}$$

Then (3.20) is a measurable function, and therefore condition (H_4) is satisfied.

Finally, we check condition (H₅). For a.a. $t \in I$, the function $f(t, \cdot)$ has a countable number of discontinuities of the form $\gamma_k(t) = k\sqrt{t}$, and $\hat{\gamma}_k(t) = \frac{-1}{k+1}t$, $k \in N \subset \mathbb{N}$, but all these discontinuity curves are inviable for the differential equation. Indeed, notice that, for $k \in N$ and $t \in I$, we have $-\gamma_k''(t) = \frac{k}{4t^{3/2}} > 0$, $-\hat{\gamma}_k''(t) = 0$ and

$$g(t)f(t, y) \leq -\frac{2^\lambda}{\sqrt{t}} \leq -2^\lambda \leq -1 \quad \text{for all } y \in \mathbb{R},$$

taking into account that $\phi(n) \geq 2$ for all $n \in \mathbb{N}$. Then, condition (H₅) holds (it suffices to take, for example, the same function $\psi \equiv \frac{1}{2}$ for all discontinuity curves).

We can conclude that the differential equation (3.19), coupled with separated BCs, has at least one solution in \bar{B}_R provided that $M_1 + M_2 \leq R^{1-\lambda}$. Note that the solution is nontrivial since the zero function does not satisfy the ODE.

In the special case of $\alpha = \beta = \gamma = \delta = 1$ and $\lambda = 1/3$, we obtain (rounded to the third decimal place) $M_1 + M_2 = 2,336$ and $R = 4$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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