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Some extensions of the Meir-Keeler theorem

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Abstract
Meir and Keeler formulated their fixed point theorem for contractive mappings with purely metric condition. This idea was extended by numerous mathematicians. In this paper, we present a simple method of proving such theorems and give new results.

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1 Introduction

In theorems cited further, we apply notations that are better suited to the results of the next sections of our paper.

Meir and Keeler proved the following theorem.

Theorem 1.1 ([1], Theorem) *Let (X,ρ) be a complete metric space, and f a mapping of X into itself. If*

$$\text{given } \alpha > 0, \text{ there exists } \epsilon > 0 \text{ such that}$$
$$\alpha \le \rho(y,x) < \alpha + \epsilon \quad \text{implies} \quad \rho(fy,fx) < \alpha,$$

then f has a unique fixed point x. Moreover, $\lim_{n\to\infty} f^n x_0 = x$ *for any* $x_0 \in X$.

Let us recall that if (X,ρ) is a metric space, then a selfmapping f on X is contractive if the following condition is satisfied:

$$\rho(fy,fx) < \rho(y,x), \quad x \ne y, x, y \in X. \tag{1}$$

The result of Meir and Keeler was extended by Matkowski and Ćirić.

Theorem 1.2 ([2], Theorem 1.5.1, [3]) *Let f be a contractive selfmapping on a complete metric space (X,ρ) that satisfies the following condition*:

$$\text{for any } \alpha > 0, \text{ there exists } \epsilon > 0 \text{ such that}$$
$$\alpha < \rho(y,x) < \alpha + \epsilon \quad \text{implies} \quad \rho(fy,fx) \le \alpha, \quad x, y \in X.$$

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Then f has a unique fixed point x, and $f^n x_0 \to x$ for each $x_0 \in X$.

Both theorems are more general than the next theorem of Boyd and Wong [4].

Theorem 1.3 ([4], Theorem 1) *Let (X,ρ) be a complete metric space, and let $f\colon X \to X$ satisfy*

$$\rho(fy,fx) \le \varphi\bigl(\rho(y,x)\bigr), \quad x,y \in X,$$

where $\varphi\colon [0,\infty) \to [0,\infty)$ is upper semicontinuous from the right and such that $\varphi(t) < t$ for all $t \in (0,\infty)$. Then, f has a unique fixed point x, and $f^n x_0 \to x$ for each $x_0 \in X$.

In an extension of this theorem ([5], Theorem 3.1), we assumed that $\varphi(t) < t$ for all $t \in (0,\infty)$ and

$$\begin{aligned} &\text{for each } \alpha > 0, \text{ there exists } \epsilon > 0 \\ &\text{such that } \varphi(\cdot) \le \alpha \text{ on } (\alpha, \alpha+\epsilon). \end{aligned} \tag{2}$$

In addition, the metric was replaced by a dislocated metric (for definitions, see Section 2). Clearly, our theorem is more general than the Boyd-Wong one even for metric spaces.

Jachymski [6] obtained the following more general result for metric spaces.

Theorem 1.4 ([6], Corollary) *Let f be a selfmapping of a complete metric space (X,ρ) such that $\rho(fy,fx) < d(y,x)$ for $x \neq y$ and $\rho(fy,fx) \le \varphi(\rho(y,x))$ for all $x,y \in X$, where $\varphi\colon [0,\infty) \to [0,\infty)$ satisfies condition (2). Then f has a unique fixed point x, and $f^n x_0 \to x$ for any $x_0 \in X$.*

It appears that the simple reasoning presented in [5] applies to conditions of the Meir-Keeler type. Consequently, we easily obtain extensions of the well-known theorems to the case of dislocated metric spaces or partial metric spaces. In addition, new results for cyclic mappings are proved. Also, the next theorem of Proinov is strongly extended in Section 3 (see Theorems 3.6, 3.7).

Theorem 1.5 ([7], Theorem 4.2) *Let (X,ρ) be a complete metric space, and let f be a continuous selfmapping such that $\lim_{n\to\infty} \rho(f^{n+1}x_0, f^n x_0) = 0$, $x_0 \in X$, and for*

$$D(x,y) = \rho(y,x) + \gamma\bigl[\rho(fy,y) + \rho(fx,x)\bigr] \quad (a\ \gamma \ge 0), x,y \in X,$$

the following conditions are satisfied:

$$\begin{aligned} &\rho(fy,fx) < D(y,x), \quad x,y \in X, \\ &\textit{for any } \alpha > 0, \textit{ there exists } \epsilon > 0 \textit{ such that} \\ &\alpha < D(y,x) < \alpha + \epsilon \quad \textit{implies} \quad \rho(fy,fx) \le \alpha, \quad x,y \in X. \end{aligned}$$

Then f has a unique fixed point x, and $f^n x_0 \to x$ for each $x_0 \in X$.

2 Lemmas

Lemma 2.1 *Let* $(a_n)_{n\in\mathbb{N}}$ *be a nonnegative sequence such that*

$$a_{n+1} > 0 \quad \textit{yields} \quad a_{n+1} < a_n, \quad n \in \mathbb{N}. \tag{3}$$

Then $\lim_{n\to\infty} a_n = 0$ *iff the following condition is satisfied*:

$$\begin{aligned} &\textit{for each } \alpha > 0, \textit{ there exists } \epsilon > 0 \textit{ such that} \\ &\alpha < a_n < \alpha + \epsilon \quad \textit{implies} \quad a_{n+1} \leq \alpha, \quad n \in \mathbb{N}. \end{aligned} \tag{4}$$

Proof Assume that (4) holds and suppose $\lim_{n\to\infty} a_n = 0$ is false. If $a_n = 0$, then (3) yields $a_{n+k} = 0$, $k \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = 0$. Therefore, $a_n > 0$, $n \in \mathbb{N}$, and $(a_n)_{n\in\mathbb{N}}$ decreases to an $\alpha > 0$. We have $\alpha < a_{n+1} < a_n$, $n \in \mathbb{N}$, and $a_n < \alpha + \epsilon$ for large n. Now, from (4) it follows that $\alpha < a_{n+1} \leq \alpha$, a contradiction. In turn, assume that $\lim_{n\to\infty} a_n = 0$. If $\alpha > 0$ is such that $a_n \leq \alpha$, $n \in \mathbb{N}$, then (4) is satisfied. If there exists $a_n > \alpha$, then $a_{n+1} < a_n$ (see (3)), and for some n and ϵ, a_n is the unique element of $(a_n)_{n\in\mathbb{N}}$ in $(\alpha, \alpha + \epsilon)$, that is, (4) is satisfied. □

We use the term of *dislocated metric* (*i.e.*, *d-metric*) following Hitzler and Seda [8]; d-metric differs from metric since $p(x, y) = 0$ yields $x = y$ (no equivalence). The topology of a d-metric space is generated by balls. If p is a d-metric, then the pair (X, p) was first defined by Matthews as a metric domain (see [9], Definition, p.13).

In the present section, we put $x_n = f^n x_0$, $n \in N$.

Lemma 2.2 *Let* (X, p) *be a d-metric space, and let* f *be a selfmapping satisfying*

$$p(x_{n+2}, x_{n+1}) > 0 \quad \textit{implies} \quad p(x_{n+2}, x_{n+1}) < p(x_{n+1}, x_n), \quad n \in \mathbb{N}. \tag{5}$$

Then $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$ *iff the following condition holds*:

$$\begin{aligned} &\textit{for each } \alpha > 0, \textit{ there exists } \epsilon > 0 \textit{ such that} \\ &\alpha < p(x_{n+1}, x_n) < \alpha + \epsilon \quad \textit{implies} \quad p(x_{n+2}, x_{n+1}) \leq \alpha, \quad n \in \mathbb{N}. \end{aligned} \tag{6}$$

Proof We apply Lemma 2.1 to $a_n = p(x_{n+1}, x_n)$, $n \in \mathbb{N}$. □

The subsequent definition is equivalent to the classical one (see (1)) if p is a metric.

Definition 2.3 Let (X, p) be a d-metric space. Then a selfmapping f on X is *contractive* if the following condition is satisfied:

$$p(fy, fx) > 0 \quad \text{implies} \quad p(fy, fx) < p(y, x), \quad x, y \in X. \tag{7}$$

If $f : X \to X$ is a contractive mapping, then (5) holds for each $x_0 \in X$. Now, from Lemma 2.2 we obtain the following:

Corollary 2.4 *Let* (X, p) *be a d-metric space, and let* f *be a contractive selfmapping on* X. *Then* $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$ *iff* (6) *holds.*

Let us consider

$$c_f(y,x) = \max\{p(y,x), p(fy,y), p(fx,x)\}$$

and

$$p(fy,fx) > 0 \quad \text{implies} \quad p(fy,fx) < c_f(y,x), \quad x,y \in X. \tag{8}$$

Then we obtain

$$p(x_{n+2},x_{n+1}) > 0 \quad \text{implies}$$
$$p(x_{n+2},x_{n+1}) < c_f(x_{n+1},x_n) = \max\{p(x_{n+1},x_n), p(x_{n+2},x_{n+1})\} = p(x_{n+1},x_n), \quad n \in \mathbb{N}$$

(otherwise, a contradiction). Consequently, (8) yields (5), and we have the following:

Corollary 2.5 *Let (X,p) be a d-metric space, and let f be a selfmapping on X satisfying* (8) (*or* (7) *or* (5)). *Then* $\lim_{n\to\infty} p(x_{n+1},x_n) = 0$ *iff* (6) *holds* ($\alpha < p(\cdots) < \alpha + \epsilon$ *can be also replaced by* $\alpha < c_f(\cdots) < \alpha + \epsilon$ *in* (6)).

Let us recall the notion of partial metric from Matthews [10], Definition 3.1. A *partial metric* is a mapping $p\colon X \times X \to [0,\infty)$ such that

$$x = y \quad \text{iff} \quad p(x,x) = p(x,y) = p(y,y), \quad x,y \in X, \tag{9a}$$

$$p(x,x) \le p(x,y), \quad x,y \in X, \tag{9b}$$

$$p(x,y) = p(y,x), \quad x,y \in X, \tag{9c}$$

$$p(x,z) \le p(x,y) + p(y,z) - p(y,y), \quad x,y,z \in X. \tag{9d}$$

From (9b) it follows that $p(x,y) = 0$ yields $p(x,x) = p(y,y) = 0$, that is, $x = y$ (see (9a)), and consequently, each partial metric is a d-metric. Therefore, all results of the present paper for d-metric spaces remain valid also for partial metric spaces, though the partial metric topology (see [10]) differs from the d-metric one.

Let us consider

$$m_f(y,x) = \max\{p(y,x), p(fy,y), p(fx,x), [p(fy,x) + p(fx,y)]/2\}$$

and

$$p(fy,fx) > 0 \quad \text{implies} \quad p(fy,fx) < m_f(y,x), \quad x,y \in X. \tag{10}$$

We have (see (9d))

$$\begin{aligned}
&[p(x_{n+2},x_n) + p(x_{n+1},x_{n+1})]/2 \\
&\quad \le [p(x_{n+2},x_{n+1}) + p(x_{n+1},x_n) - p(x_{n+1},x_{n+1}) + p(x_{n+1},x_{n+1})]/2 \\
&\quad = [p(x_{n+2},x_{n+1}) + p(x_{n+1},x_n)]/2 \le \max\{p(x_{n+2},x_{n+1}), p(x_{n+1},x_n)\} \\
&\quad = c_f(x_{n+1},x_n).
\end{aligned}$$

Consequently, $m_f(x_{n+1}, x_n) = c_f(x_{n+1}, x_n)$, and the reasoning for c_f applies. Each partial metric is a d-metric, and the previous reasoning, together with Corollary 2.5, yields the following:

Corollary 2.6 *Let (X, p) be a partial metric space, and let f be a selfmapping on X satisfying* (10) (*or* (8) *or* (7) *or* (5)). *Then* $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$ *iff* (6) *holds* ($\alpha < p(\cdots) < \alpha + \epsilon$ *can be also replaced by* $\alpha < m_f(\cdots) < \alpha + \epsilon$ *or by* $\alpha < c_f(\cdots) < \alpha + \epsilon$ *in* (6)).

Lemma 2.7 *Let (X, p) be a d-metric space, and let f be a selfmapping on X satisfying the following conditions*:

$$\begin{aligned} &p(x_{n+k+1}, x_{k+1}) > 0 \quad \textit{implies} \\ &p(x_{n+k+1}, x_{k+1}) < c_f(x_{n+k}, x_k), \quad k, n \in \mathbb{N}, \end{aligned} \tag{11}$$

$$\begin{aligned} &\textit{for each } \alpha > 0, \textit{ there exists } \epsilon > 0 \textit{ such that} \\ &\alpha < c_f(x_{n+k}, x_k) < \alpha + \epsilon \quad \textit{implies} \quad p(x_{n+k+1}, x_{k+1}) \le \alpha, \quad k, n \in \mathbb{N}. \end{aligned} \tag{12}$$

Then $\lim_{m,n\to\infty} p(x_n, x_m) = 0$. *In addition*, c_f *can be replaced by* p *in* (11) *or* (12) (*so also in both of them*). *Similarly, if p is a partial metric, then c_f can be replaced by m_f in* (11) *or* (12).

Proof From the triangle inequality it follows that it is sufficient to consider $m \neq n$. Clearly, if (12) and (11) hold, then, in particular, (6) and (5) are satisfied. Therefore, we have $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$ (see Corollary 2.5 or Corollary 2.6). Suppose that $\lim_{m,n\to\infty} p(x_n, x_m) = 0$ is false. Then, for an infinite subset $\mathbb{K}$ of $\mathbb{N}$ and each $k \in \mathbb{K}$, there exists $n \in \mathbb{N}$ such that $0 < \alpha < p(x_{n+k+1}, x_{k+1})$. Let $n = n(k)$ be the smallest such number. For large k, we obtain (see (11))

$$\begin{aligned} \alpha &< p(x_{n+k+1}, x_{k+1}) < c_f(x_{n+k}, x_k) = p(x_{n+k}, x_k) \\ &\le p(x_{n+k}, x_{k+1}) + p(x_{k+1}, x_k) \le \alpha + p(x_{k+1}, x_k). \end{aligned}$$

The inequality

$$c_f(x_{n+k}, x_k) \le m_f(x_{n+k}, x_k) < p(x_{n+k}, x_{k+1}) + \epsilon \le \alpha + \epsilon$$

for large k when p is a partial metric is a consequence of

$$\begin{aligned} &\bigl[p(x_{n+k+1}, x_k) + p(x_{n+k}, x_{k+1})\bigr]/2 \\ &\quad \le \bigl[p(x_{n+k+1}, x_{n+k}) + p(x_{n+k}, x_k) - p(x_{n+k}, x_{n+k}) + p(x_{n+k}, x_{k+1})\bigr]/2 \\ &\quad \le \bigl[p(x_{n+k+1}, x_{n+k}) + p(x_{n+k}, x_{k+1}) + p(x_{k+1}, x_k) - p(x_{k+1}, x_{k+1}) \\ &\qquad - p(x_{n+k}, x_{n+k}) + p(x_{n+k}, x_{k+1})\bigr]/2 \\ &\quad = p(x_{n+k}, x_{k+1}) + \bigl[p(x_{n+k+1}, x_{n+k}) - p(x_{n+k}, x_{n+k}) + p(x_{k+1}, x_k) - p(x_{k+1}, x_{k+1})\bigr]/2 \end{aligned}$$

and (9b). Now, $\alpha < c_f(x_{n+k}, x_k) < \alpha + \epsilon$ for large k and (12) yield

$$\alpha < p(x_{n+k+1}, x_{k+1}) \le \alpha,$$

a contradiction. □

In a similar way, we prove the following lemma suitable for cyclic mappings.

Lemma 2.8 *Let (X,p) be a d-metric space, and let f be a selfmapping on X satisfying the following conditions for a fixed $t \in \mathbb{N}$:*

$$\begin{aligned} &p(x_{nt+k+2},x_{k+1}) > 0 \quad \textit{implies} \\ &p(x_{nt+k+2},x_{k+1}) < c_f(x_{nt+k+1},x_k), \quad k \in \mathbb{N}, n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{13}$$

$$\begin{aligned} &\textit{for each } \alpha > 0, \textit{ there exists } \epsilon > 0 \textit{ such that} \\ &\alpha < c_f(x_{nt+k+1},x_k) < \alpha + \epsilon \quad \textit{implies} \\ &p(x_{nt+k+2},x_{k+1}) \le \alpha, \quad k \in \mathbb{N}, n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{14}$$

Then $\lim_{m,n\to\infty} p(x_n,x_m) = 0$. *In addition, c_f can be replaced by p in* (13) *or* (14) (*so also in both of them*). *Similarly, if p is a partial metric, then c_f can be replaced by m_f in* (13) *or* (14).

Proof Clearly, if (14) and (13) hold, then, in particular, (6) and (5) are satisfied. Therefore, we have $\lim_{n\to\infty} p(x_{n+1},x_n) = 0$ (see Corollary 2.5 or Corollary 2.6). Suppose that $\lim_{k,n\to\infty} p(x_{nt+k+2},x_{k+1}) = 0$ is false. Then, for an infinite subset $\mathbb{K}$ of $\mathbb{N}$ and each $k \in \mathbb{K}$, there exists $n \in \mathbb{N}$ such that $0 < \alpha < p(x_{(n+1)t+k+2},x_{k+1})$. Let $n = n(k)$ be the smallest such number. For large k, we obtain (see (13))

$$\begin{aligned} \alpha &< p(x_{(n+1)t+k+2},x_{k+1}) < c_f(x_{(n+1)t+k+1},x_k) = p(x_{(n+1)t+k+1},x_k) \\ &\le p(x_{(n+1)t+k+1},x_{(n+1)t+k}) + \cdots + p(x_{nt+k+2},x_{k+1}) + p(x_{k+1},x_k) \\ &\le p(x_{(n+1)t+k+1},x_{(n+1)t+k}) + \cdots + \alpha + p(x_{k+1},x_k). \end{aligned}$$

Therefore (see Corollary 2.5), we have

$$\alpha < p(x_{(n+1)t+k+2},x_{k+1}) < c_f(x_{(n+1)t+k+1},x_k) < \alpha + \epsilon$$

for large k. Now, condition (14) yields

$$\alpha < p(x_{(n+1)t+k+2},x_{k+1}) \le \alpha,$$

a contradiction. Therefore, $\lim_{k,n\to\infty} p(x_{nt+k+2},x_{k+1}) = 0$. For any $s \in \{3,\dots,t\}$, we have

$$p(x_{nt+k+s},x_{k+1}) \le p(x_{nt+k+s},x_{nt+k+s-1}) + \cdots + p(x_{nt+k+2},x_{k+1}),$$

and Lemma 2.2 yields $\lim_{k,n\to\infty} p(x_{nt+k+s},x_{k+1}) = 0$, that is, the proof of our lemma is completed. □

It can be seen that Lemma 2.7 is a consequence of Lemma 2.8 for $t = 1$.

For a mapping $\beta\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ continuous at $(0,0)$ and such that $\beta(0,0) = 0$, let us consider

$$D_f(y,x) = p(y,x) + \beta\bigl(p(fy,y),p(fx,x)\bigr) \tag{15}$$

and

$$p(fy,fx) > 0 \quad \text{implies} \quad p(fy,fx) < D_f(y,x), \quad x,y \in X, \tag{16}$$

Lemma 2.9 *Let (X,p) be a d-metric space, and let f be such a selfmapping on X that $\lim_{n\to\infty} p(x_{n+1},x_n) = 0$, and the following conditions hold for a fixed $t \in \mathbb{N}$:*

$$\begin{aligned} &p(x_{nt+k+2},x_{k+1}) > 0 \quad \textit{implies} \\ &p(x_{nt+k+2},x_{k+1}) < D_f(x_{nt+k+1},x_k), \quad k \in \mathbb{N}, n \in \mathbb{N}\cup\{0\}, \end{aligned} \tag{17}$$

$$\begin{aligned} &\textit{for each } \alpha > 0, \textit{ there exists } \epsilon > 0 \textit{ such that} \\ &\alpha < D_f(x_{nt+k+1},x_k) < \alpha + \epsilon \quad \textit{implies} \\ &p(x_{nt+k+2},x_{k+1}) \le \alpha, \quad k \in \mathbb{N}, n \in \mathbb{N}\cup\{0\}. \end{aligned} \tag{18}$$

Then $\lim_{m,n\to\infty} p(x_n,x_m) = 0$. In addition, $p(y,x)$ can be replaced by $c_f(y,x)$ (or by $m_f(y,x)$ if p is a partial metric) in (15) *for* (17) *or* (18).

Proof Suppose that $\lim_{k,n\to\infty} p(x_{nt+k+2},x_{k+1}) = 0$ is false. Then for an infinite subset $\mathbb{K}$ of $\mathbb{N}$ and each $k \in \mathbb{K}$, there exists $n \in \mathbb{N}$ such that $0 < \alpha < p(x_{(n+1)t+k+2},x_{k+1})$. Let $n = n(k)$ be the smallest such number. We obtain (see (17))

$$\begin{aligned} \alpha &< p(x_{(n+1)t+k+2},x_{k+1}) < D_f(x_{(n+1)t+k+1},x_k) \\ &= p(x_{(n+1)t+k+1},x_k) + \beta\bigl(p(x_{(n+1)t+k+2},x_{(n+1)t+k+1}),p(x_{k+1},x_k)\bigr) \\ &\le p(x_{(n+1)t+k+1},x_{(n+1)t+k}) + \cdots + p(x_{nt+k+2},x_{k+1}) + p(x_{k+1},x_k) \\ &\quad + \beta(\ldots,\ldots) \\ &\le p(x_{(n+1)t+k+1},x_{(n+1)t+k}) + \cdots + \alpha + p(x_{k+1},x_k) + \beta(\ldots,\ldots) \end{aligned}$$

for large k. Therefore (β is continuous at $(0,0)$, and $\beta(0,0) = 0$), we have

$$\alpha < D_f(x_{(n+1)t+k+1},x_k) < \alpha + \epsilon$$

for large k. Our condition (18) yields

$$\alpha < p(x_{(n+1)t+k+2},x_{k+1}) \le \alpha,$$

a contradiction. Therefore, $\lim_{k,n\to\infty} p(x_{nt+k+2},x_{k+1}) = 0$. Now, we follow the final part of the proof of Lemma 2.8. □

Definition 2.10 A selfmapping f on a d-metric space (X,p) is 0-continuous at x if $\lim_{n\to\infty} p(x,x_n) = 0$ implies $\lim_{n\to\infty} p(fx,fx_n) = 0$ for each sequence $(x_n)_{n\in\mathbb{N}}$ in X; f is 0-continuous if it is 0-continuous at each point $x \in X$.

Lemma 2.11 *Let (X,p) be a d-metric space, and let f be a selfmapping on X. If f is contractive, then f has at most one fixed point; the same holds if f satisfies* (8) *or* (10) *and p satisfies*

(9b) *or if* p *is a metric and* (16) *holds. If* f *is* 0*-continuous at* x (*e.g., if* f *is contractive*) *and* $\lim_{n\to\infty} p(x, f^n x_0) = 0$, *then* $x = fx$ *and* $p(x,x) = 0$.

Proof If x, y are fixed points of f and (9b) holds, then we obtain

$$\begin{aligned} m_f(y,x) &= \max\{p(y,x), p(y,y), p(x,x), [p(y,x) + p(x,y)]/2\} \\ &= c_f(y,x) = p(y,x). \end{aligned}$$

In addition, if $x \neq y$, then each of conditions (7), (8), and(10) yields

$$0 < p(y,x) = p(fy,fx) < p(y,x),$$

a contradiction. If p is a metric and (16) holds, then we have

$$0 < p(y,x) = p(fy,fx) < D_f(y,x) = p(y,x) + \beta(0,0) = p(y,x),$$

also a contradiction. Let us consider $x \in X$ with $\lim_{n\to\infty} p(x,x_n) = 0$. Then we have

$$p(fx,x) \leq p(fx,x_{n+1}) + p(x_{n+1},x),$$

and in view of the 0-continuity of f at x, we also obtain $\lim_{n\to\infty} p(fx,x_{n+1}) = 0$, that is, $p(fx,x) = 0$. □

3 Theorems

Let us recall ([11], Definition 2.3) that a d-metric space (X,p) is 0*-complete* if for each sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\lim_{m,n\to\infty} p(x_n,x_m) = 0$, there exists $x \in X$ such that $\lim_{n\to\infty} p(x,x_n) = 0$.

Now, we are ready to extend the Ćirić theorem [3] and the Matkowski Theorem 1.5.1 in [2] (here Theorem 1.2) to the case of d-metric spaces (and c_f in place of p).

Theorem 3.1 *Let* f *be a* 0*-continuous selfmapping on a* 0*-complete d-metric space* (X,p). *Assume that* (8) *or* (7) *holds and the following condition is satisfied*:

$$\begin{aligned} &\textit{for each } \alpha > 0, \textit{ there exists } \epsilon > 0 \textit{ such that} \\ &\alpha < c_f(y,x) < \alpha + \epsilon \quad \textit{implies} \quad p(fy,fx) \leq \alpha, \quad x,y \in X. \end{aligned} \tag{19}$$

Then f *has a unique fixed point, say* x, *and* $\lim_{n\to\infty} p(x, f^n x_0) = p(x,x) = 0$, $x_0 \in X$.

Proof Our space is 0-complete, and, therefore, the sequence $(f^n x_0)_{n\in\mathbb{N}}$ converges (Lemma 2.7) to a unique fixed point of f (Lemma 2.11). □

Lemma 29 from [12] and the previous theorem yield the following result.

Theorem 3.2 *Let* h *be a selfmapping on a* 0*-complete d-metric space* (X,p) *such that* $f = h^s$ (*for some* $s \in \mathbb{N}$) *satisfies the assumptions of Theorem* 3.1. *Then* h *has a unique fixed point, say* x, *and* $\lim_{n\to\infty} p(x, h^n x_0) = p(x,x) = 0$, $x_0 \in X$.

Lemma 2.8 enables us to extend the previous theorems to the case of cyclic mappings. The idea was introduced by Kirk *et al.* [13], and we apply Definition 2.5 from [5]. For a fixed $t \in \mathbb{N}$, we put $t++ = 1$ and $j++ = j+1$ for $j \in \{1,\dots,t-1\}$. Then $f: X \to X$ is *cyclic* if $X = X_1 \cup \cdots \cup X_t$ and $f(X_j) \subset X_{j++}$, $j = 1,\dots,t$.

Theorem 3.3 *Let f be a* 0*-continuous cyclic selfmapping on a* 0*-complete d-metric space* (X,p), *and let the following conditions be satisfied*:

$$\begin{aligned} &p(fy,fx) > 0 \quad \textit{implies} \\ &p(fy,fx) < c_f(y,x), \quad x \in X_j, y \in X_{j++}, j = 1,\dots,t, \end{aligned} \tag{20}$$

$$\begin{aligned} &\textit{for each } \alpha > 0, \textit{ there exists } \epsilon > 0 \textit{ such that} \\ &\alpha < c_f(y,x) < \alpha + \epsilon \quad \textit{implies} \\ &p(fy,fx) \le \alpha, \quad x \in X_j, y \in X_{j++}, j = 1,\dots,t. \end{aligned} \tag{21}$$

Then f has a unique fixed point, say x, and $\lim_{n\to\infty} p(x,f^n x_0) = p(x,x) = 0$, $x_0 \in X$.

Proof Our space is 0-complete, and, therefore, the sequence $(f^n x_0)_{n\in\mathbb{N}}$ converges (Lemma 2.8) to a unique fixed point of f (Lemma 2.11). □

An analogue of Theorem 3.2 for cyclic mappings is the following consequence of Theorem 3.3 and of [12], Lemma 29.

Theorem 3.4 *Let h be a selfmapping on a* 0*-complete d-metric space* (X,p) *such that* $f = h^s$ (*for some* $s \in \mathbb{N}$) *satisfies the assumptions of Theorem* 3.3. *Then h has a unique fixed point, say x, and* $\lim_{n\to\infty} p(x,h^n x_0) = p(x,x) = 0$, $x_0 \in X$.

Let us note that a partial metric space (X,p) is 0-complete iff (X,p) treated as a d-metric space is 0-complete (see [12], Corollary 4, Proposition 5).

Remark 3.5 In view of Lemma 2.7, c_f can be replaced by p in any of conditions of Theorems 3.1, 3.2, 3.3, and 3.4; if p is a partial metric, then in view of Lemma 2.8, c_f can be replaced by m_f in any condition of those theorems. Theorem 3.1 for m_f becomes an extension of a theorem of Jachymski ([6], Theorem 2) to the case of partial metric spaces.

Theorem 3.6 *Let* (X,p) *be a* 0*-complete d-metric space, and let f be a* 0*-continuous cyclic selfmapping on X such that* $\lim_{n\to\infty} p(f^{n+1}x_0,f^n x_0) = 0$, $x_0 \in X$. *Assume that the following conditions hold*:

$$\begin{aligned} &p(fy,fx) > 0 \quad \textit{implies} \\ &p(fy,fx) < D_f(y,x), \quad x \in X_j, y \in X_{j++}, j = 1,\dots,t, \end{aligned} \tag{22}$$

$$\begin{aligned} &\textit{for each } \alpha > 0, \textit{ there exists } \epsilon > 0 \textit{ such that} \\ &\alpha < D_f(y,x) < \alpha + \epsilon \quad \textit{implies} \\ &p(fy,fx) \le \alpha, \quad x \in X_j, y \in X_{j++}, j = 1,\dots,t. \end{aligned} \tag{23}$$

Then f has a fixed point, say x, such that $\lim_{n\to\infty} p(x, f^n x_0) = 0$, $x_0 \in X$, *and x is unique if p is a metric. In addition, $p(y,x)$ can be replaced by $c_f(y,x)$ (or by $m_f(y,x)$ if p is a partial metric) in* (15) *for* (22) *or* (23) (*so also for both of them*).

Proof We apply Lemmas 2.9 and 2.11. □

Theorem 3.6 with $t = 1$ yields the following one.

Theorem 3.7 *Let (X,p) be a 0-complete d-metric space, and let f be a 0-continuous self-mapping on X such that* $\lim_{n\to\infty} p(f^{n+1}x_0, f^n x_0) = 0$, $x_0 \in X$. *Assume that the following conditions hold*:

$$p(fy,fx) > 0 \quad \textit{implies} \quad p(fy,fx) < D_f(y,x), \quad x,y \in X, \tag{24}$$

$$\begin{aligned} &\textit{for each } \alpha > 0, \textit{ there exists } \epsilon > 0 \textit{ such that} \\ &\alpha < D_f(y,x) < \alpha + \epsilon \quad \textit{implies} \quad p(fy,fx) \le \alpha, \quad x,y \in X. \end{aligned} \tag{25}$$

Then f has a fixed point, say x, such that $\lim_{n\to\infty} p(x, f^n x_0) = 0$, $x_0 \in X$, *and x is unique if p is a metric. In addition, $p(y,x)$ can be replaced by $c_f(y,x)$ (or by $m_f(y,x)$ if p is a partial metric) in* (15) *for* (24) *or* (25) (*so also for both of them*).

We can easily present extensions of the previous theorems for $f = h^s$ (see Theorems 3.2 and 3.4).

Theorem 3.7 is a further extension of Theorem 4.2 in [7] (here Theorem 1.5).

Remark 3.8 Clearly, the results of the present section stay valid if we assume that (X,p) is 0-complete for orbits of f because in the proofs of our lemmas only orbits were used.

Competing interests

The author declares that he has no competing interests.

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