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Nadler's fixed point theorem in ν-generalized metric spaces

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Abstract

We extend Nadler's fixed point theorem to ν-generalized metric spaces. Through the proof of the above extension, we understand more deeply the mathematical structure of a ν-generalized metric space. In particular, we study the completeness of the space. We also improve Caristi's and Subrahmanyam's fixed point theorems in the space.

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1 Introduction and preliminaries

In 1969, Nadler proved the following; the splendid fixed point theorem for set-valued contractions, which is one of generalizations of the Banach contraction principle [1, 2]. See also, *e.g.*, [3–8].

Theorem 1 (Theorem 5 in Nadler [9]) *Let (X,d) be a complete metric space and let T be a mapping from X into $\mathrm{CB}(X)$, where $\mathrm{CB}(X)$ is the set of all nonempty bounded closed subsets of X. Assume that there exists $r \in [0,1)$ such that*

$$\delta(Tx, Ty) \le rd(x,y) \tag{1}$$

for all $x, y \in X$, where δ is a function from $\mathrm{CB}(X)^2$ into $[0,\infty)$ defined by

$$\delta(A,B) = \sup_{a\in A} \inf_{b\in B} d(a,b). \tag{2}$$

Then there exists $z \in X$ such that $z \in Tz$.

Remark It is obvious that (1) is equivalent to the following:

$$H(Tx, Ty) \le rd(x,y),$$

where H is the Hausdorff metric, that is,

$$H(Tx, Ty) = \max\{\delta(Tx, Ty), \delta(Ty, Tx)\}.$$

In 2000, Branciari introduced the following, very interesting concept.

Definition 2 (Branciari [10]) Let X be a set, let d be a function from $X \times X$ into $[0, \infty)$ and let $\nu \in \mathbb{N}$. Then (X, d) is said to be a *ν-generalized metric space* if the following hold:

(N1) $d(x, y) = 0$ iff $x = y$ for any $x, y \in X$.

(N2) $d(x, y) = d(y, x)$ for any $x, y \in X$.

(N3) $d(x, y) \le D(x, u_1, u_2, \dots, u_\nu, y)$ for any $x, u_1, u_2, \dots, u_\nu, y \in X$ such that $x, u_1, u_2, \dots, u_\nu, y$ are all different, where

$$D(x, u_1, u_2, \dots, u_\nu, y) = d(x, u_1) + d(u_1, u_2) + \cdots + d(u_\nu, y).$$

We have studied the topological structure of this space. Indeed, recent studies tell that 1- and 3-generalized metric spaces have the compatible topology and that all ν-generalized metric spaces have the strongly compatible topology. Also we have proved several fixed point theorems in this space. See, *e.g.*, [11–25]. However, we have not generalized Theorem 1. Motivated by this fact, in this paper, we generalize Theorem 1. Another purpose of this paper is to understand more deeply the mathematical structure of this space. In particular, we study the completeness of this space. We also improve Caristi's and Subrahmanyam's fixed point theorems in this space.

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. For an arbitrary set A, we also denote by $\#A$ the cardinal number of A.

2 Completeness

In this section, we begin with definitions. Some of them are new.

Definition 3 Let (X, d) be a ν-generalized metric space and let $\{x_n\}$ be a sequence in X. Let $\kappa \in \mathbb{N}$.

(i) $\{x_n\}$ is said to be *Cauchy* [10] if $\lim_n \sup_{m>n} d(x_n, x_m) = 0$ holds.

(ii) $\{x_n\}$ is said to be *κ-Cauchy* [11] if

$$\lim_{n\to\infty} \sup\bigl\{d(x_n, x_{n+1+j\kappa}) : j = 0, 1, 2, \dots\bigr\} = 0$$

holds.

(iii) $\{x_n\}$ is said to be *$(\sum, \neq)$-Cauchy* if x_n $(n \in \mathbb{N})$ are all different and

$$\sum_{j=1}^{\infty} d(x_j, x_{j+1}) < \infty$$

holds.

(iv) $\{x_n\}$ is said to *converge* to x [10] if $\lim_n d(x_n, x) = 0$ holds.

(v) $\{x_n\}$ is said to *converge only* to x [11] if

$$\lim_{n\to\infty} d(x_n, x) = 0 \quad \text{and} \quad \liminf_{n\to\infty} d(x_n, y) > 0$$

hold for any $y \in X \setminus \{x\}$.

(vi) $\{x_n\}$ is said to *converge exclusively* to x [25] if

$$\lim_{n\to\infty} d(x_n, x) = 0 \quad \text{and} \quad \liminf_{n\to\infty} d(x_n, y) > 0$$

hold for any $y \in X \setminus \{x\}$.

(vii) $\{x_n\}$ is said to *converge to x in the strong sense* [25] if $\{x_n\}$ is Cauchy and $\{x_n\}$ converges to x.

Remark We know the following.
- $\{x_n\}$ is Cauchy iff $\{x_n\}$ is 1-Cauchy.
- If $\{x_n\}$ is Cauchy, then $\{x_n\}$ is 2-Cauchy; see Proposition 6(i) in [11].
- (vii) $\Longrightarrow$ (vi) $\Longrightarrow$ (v) $\Longrightarrow$ (iv) holds; see Proposition 2.3(ii) in [25].

Definition 4 Let (X, d) be a ν-generalized metric space. Let $\kappa \in \mathbb{N}$.
- X is said to be *complete* [10] if every Cauchy sequence converges.
- X is κ-*complete* [11] if every κ-Cauchy sequence converges.
- X is $(\sum, \neq)$-*complete* if every $(\sum, \neq)$-Cauchy sequence converges.

Remark We know the following.
- X is complete iff X is 1-complete.
- If X is 2-complete, then X is complete; see Proposition 6(ii) in [11].

We next study $(\sum, \neq)$-completeness.

Lemma 5 (Proposition 7 in [11]) *Let (X, d) be a ν-generalized metric space where ν is odd. Let $\{x_n\}$ be a ν-Cauchy sequence such that x_n are all different. Then $\{x_n\}$ is Cauchy.*

Lemma 6 (Proposition 8 in [11]) *Let (X, d) be a ν-generalized metric space where ν is even. Let $\{x_n\}$ be a ν-Cauchy sequence such that x_n are all different. Then $\{x_n\}$ is 2-Cauchy.*

Lemma 7 (Lemma 9 in [11]) *Let (X, d) be a ν-generalized metric space. Then every $(\sum, \neq)$-Cauchy sequence is ν-Cauchy.*

Lemma 8 *Let (X, d) be a ν-generalized metric space and let $\kappa \in \mathbb{N}$. Let $\{x_n\}$ be a κ-Cauchy sequence converging to some $z \in X$. Assume that x_n are all different. Then $\{x_n\}$ is Cauchy.*

Remark We need the difference of x_n. See Example 28(v) below.

Proof Fix $\varepsilon > 0$. Then from the assumption, there exists some $\mu \in \mathbb{N}$ satisfying

$$\begin{aligned}
&\sup\bigl\{d(x_n, x_{n+1+j\kappa}) : j = 0, 1, 2, \ldots\bigr\} < \varepsilon \quad \text{and} \\
&0 < d(x_n, z) < \varepsilon
\end{aligned}$$

for any $n \in \mathbb{N}$ with $n \geq \mu$. Fix $m, n \in \mathbb{N}$ with $\mu \leq n < m$. Then we have

$$d(x_n, x_m) \leq D(x_n, z, x_{m+\nu-1}, \ldots, x_{m+1}, x_m) < (\nu + 1)\varepsilon.$$

Thus, we obtain the desired result. □

Lemma 9 *Let (X,d) be a ν-generalized metric space and let $\{x_n\}$ be a $(\sum,\neq)$-Cauchy sequence in X. Then the following hold*:

(i) *If ν is odd, then $\{x_n\}$ is Cauchy.*
(ii) *$\{x_n\}$ is 2-Cauchy.*
(iii) *If $\{x_n\}$ converges, then $\{x_n\}$ is Cauchy, that is, $\{x_n\}$ converges in the strong sense.*

Proof (i) follows from Lemmas 5 and 7. Similarly, (ii) follows from (i), Lemmas 6 and 7. (iii) follows from (ii) and Lemma 8. □

Lemma 10 *Let (X,d) be a ν-generalized metric space satisfying either of the following*:

- *ν is odd and X is complete.*
- *X is 2-complete.*

Then X is $(\sum,\neq)$-complete.

Proof Let $\{x_n\}$ be a $(\sum,\neq)$-Cauchy sequence. Then from the assumption and Lemma 9(i) and (ii), $\{x_n\}$ converges. □

Lemma 11 *Let (X,d) be a ν-generalized metric space and let $\{x_n\}$ be a Cauchy sequence in X converging to some $z \in X$. Let $\{y_n\}$ be a sequence in X satisfying $\lim_n d(x_n,y_n) = 0$. Then $\{y_n\}$ also converges to z.*

Proof We consider the following two cases:

(i) $\#\{x_n : n \in \mathbb{N}\} < \infty$,
(ii) $\#\{x_n : n \in \mathbb{N}\} = \infty$.

In the first case, there exists $\mu \in \mathbb{N}$ satisfying $x_n = z$ for any $n \geq \mu$. Therefore $\lim_n d(z,y_n) = \lim_n d(x_n,y_n) = 0$ holds. In the second case, we fix $\varepsilon > 0$. Then from the assumption, there exists some $\mu \in \mathbb{N}$ satisfying

$$\sup\bigl\{d(x_n,x_m) : m > n\bigr\} < \varepsilon,$$
$$d(x_n,z) < \varepsilon \quad \text{and} \quad d(x_n,y_n) < \varepsilon,$$

for any $n \geq \mu$. Fix $n \in \mathbb{N}$ with $n \geq \mu$. We further consider the following two cases:

(ii-1) $x_n = z$ or $y_n = z$ or $x_n = y_n$,
(ii-2) $x_n \neq z$, $y_n \neq z$, $x_n \neq y_n$.

In the case of (ii-1), $d(y_n,z) < \varepsilon$ obviously holds. In the case of (ii-2), we choose $n_1,\dots,n_{\nu-1} \in \mathbb{N}$ such that $n_j \geq \mu$ holds and $x_n,y_n,z,x_{n_1},\dots,x_{n_{\nu-1}}$ are all different. Then we have

$$d(y_n,z) \leq D(y_n,x_n,x_{n_1},\dots,x_{n_{\nu-1}},z) < (\nu+1)\varepsilon.$$

Thus, we obtain the desired result. □

Lemma 12 *Let (X,d) be a ν-generalized metric space and let $\{x_n\}$ be a Cauchy sequence in X satisfying $\liminf_n d(x_n,z) = 0$ for some $z \in X$. Then $\{x_n\}$ converges to z.*

Proof There exists a subsequence $\{f(n)\}$ of the sequence $\{n\}$ in $\mathbb{N}$ such that $\{x_{f(n)}\}$ converges to z. We note that $\{x_{f(n)}\}$ is Cauchy and that $\lim_n d(x_{f(n)},x_n) = 0$ holds. So by Lemma 11, we obtain the desired result. □

Lemma 13 *Let (X,d) be a $(\sum,\neq)$-complete, ν-generalized metric space. Then X is complete.*

Proof Let $\{x_n\}$ be a Cauchy sequence in X. We consider the following two cases:

- $\#\{x_n : n \in \mathbb{N}\} < \infty$,
- $\#\{x_n : n \in \mathbb{N}\} = \infty$.

In the first case, we can prove that $\{x_n\}$ converges as in the proof of Lemma 11. In the second case, we can choose a subsequence $\{f(n)\}$ of $\{n\}$ such that $x_{f(n)}$ are all different and

$$\sup\bigl\{d(x_{f(n)},x_m) : m > f(n)\bigr\} < 2^{-n}$$

holds for any $n \in \mathbb{N}$. We have

$$\sum_{j=1}^{\infty} d(x_{f(j)},x_{f(j+1)}) < \sum_{j=1}^{\infty} 2^{-j} = 1 < \infty.$$

Since X is $(\sum,\neq)$-complete, $\{x_{f(n)}\}$ converges to some $z \in X$. By Lemma 12, $\{x_n\}$ itself converges to z. We have shown that X is complete. □

Definition 14 (see Example 1.1 in [15]) Let (X,d) be a ν-generalized metric space. X is said to be *Hausdorff* if $\lim_n d(x_n,x) = \lim_n d(x_n,y) = 0$ implies $x = y$.

Lemma 15 *Let (X,d) be a 2-complete, ν-generalized metric space. Then X is Hausdorff.*

Proof Arguing by contradiction, we assume that X is not Hausdorff, that is, there exists a sequence $\{x_n\}$ in X converging to some u and v, where $u \neq v$ holds. Define a sequence $\{y_n\}$ in X by

$$y_n = \begin{cases} x_n & \text{if } n \in \{2k-1 : k \in \mathbb{N}\}, \\ u & \text{if } n \in \{4k-2 : k \in \mathbb{N}\}, \\ v & \text{if } n \in \{4k : k \in \mathbb{N}\}. \end{cases}$$

We note that $\{y_n\}$ is as follows:

$$x_1, u, x_3, v, x_5, u, x_7, v, x_9, \ldots.$$

It is obvious that $\{y_n\}$ is 2-Cauchy. Since X is 2-complete, $\{y_n\}$ converges to some z. However, we have

$$\limsup_{n\to\infty} d(y_n,z) = 0 < \max\bigl\{d(u,z),d(v,z)\bigr\} \le \limsup_{n\to\infty} d(y_n,z),$$

which implies a contradiction. Therefore X is Hausdorff. □

Lemma 16 *Let (X,d) be a $(\sum,\neq)$-complete, Hausdorff, ν-generalized metric space. Then X is 2-complete.*

Proof Let $\{x_n\}$ be a 2-Cauchy sequence. Define two subsets A_1 and A_2 of X by

$$A_1 = \{x_{2n-1} : n \in \mathbb{N}\} \quad \text{and} \quad A_2 = \{x_{2n} : n \in \mathbb{N}\}.$$

We consider the following two cases:
- $\#A_1 < \infty$ or $\#A_2 < \infty$,
- $\#A_1 = \infty$ and $\#A_2 = \infty$.

In the first case, without loss of generality, we may assume $\#A_2 < \infty$. Define a subset B_2 of A_2 by

$$B_2 = \bigl\{x \in X : \#\{n \in \mathbb{N} : x_{2n} = x\} = \infty\bigr\}.$$

We note $0 < \#B_2 < \#A_2 < \infty$. Since $\{x_n\}$ is 2-Cauchy,

$$\lim_{n\to\infty} d(x_{2n-1}, x) = \lim_{n\to\infty} d(x_{2n-1}, y) = 0$$

holds for any $x, y \in B_2$. Since X is Hausdorff, we obtain $x = y$. Therefore we have shown $\#B_2 = 1$. We let $x \in X$ satisfy $B_2 = \{x\}$. Then we obtain

$$\lim_{n\to\infty} d(x_{2n}, x) = d(x, x) = 0.$$

Therefore $\{x_n\}$ converges to x. In the second case, we can choose a subsequence $\{f(n)\}$ of $\{n\}$ such that $x_{f(n)}$ are all different, $f(2n-1)$ is odd, $f(2n)$ is even and

$$\sup\bigl\{d(x_j, x_{j+2k-1}) : j > f(n), k \in \mathbb{N}\bigr\} < 2^{-n}$$

holds for any $n \in \mathbb{N}$. We have

$$\sum_{j=1}^{\infty} d(x_{f(j)}, x_{f(j+1)}) < \sum_{j=1}^{\infty} 2^{-j} = 1 < \infty.$$

Since X is $(\sum, \neq)$-complete, $\{x_{f(n)}\}$ converges to some $z \in X$. Since $\{x_{f(n)}\}$ is still 2-Cauchy, $\{x_{f(n)}\}$ is Cauchy by Lemma 8. Noting that $f(n+1) - n$ is odd for any $n \in \mathbb{N}$, we have $\lim_n d(x_{f(n+1)}, x_n) = 0$. By Lemma 11, we obtain $\lim_n d(x_n, z) = 0$. We have shown that X is 2-complete. □

Proposition 17 *Let (X, d) be a ν-generalized metric space where ν is odd. Then the following are equivalent*:
- *X is complete.*
- *X is $(\sum, \neq)$-complete.*

Proof The conclusion follows from Lemmas 10 and 13. □

Proposition 18 *Let (X, d) be a ν-generalized metric space. Then the following are equivalent*:
- *X is 2-complete.*
- *X is $(\sum, \neq)$-complete and Hausdorff.*

Proof The conclusion follows from Lemmas 10, 15 and 16. □

Proposition 19 *Let (X,d) be a Hausdorff, ν-generalized metric space where ν is odd. Then the following are equivalent*:

- *X is complete.*
- *X is $(\sum, \neq)$-complete.*
- *X is 2-complete.*

Proof The conclusion follows from Propositions 17 and 18. □

3 Fixed point theorems

In this section, we first generalize Theorem 1.

Theorem 20 *Let (X,d) be a $(\sum, \neq)$-complete, ν-generalized metric space. Let T be a set-valued mapping on X satisfying the following*:

- *For any $x \in X$, Tx is a nonempty subset of X.*
- *If a sequence $\{y_n\}$ in Tx converges to y, then $y \in Tx$ holds.*
- *There exists $r \in [0,1)$ satisfying $\delta(Tx,Ty) \leq rd(x,y)$ for all $x,y \in X$, where δ is defined by* (2).

Then there exists $z \in X$ satisfying $z \in Tz$.

Proof Replace the value of r by $r := (1 + r)/2 \in (0,1)$. We note $r > 0$ and the following:

- For any $x,y \in X$ and $u \in Tx$ with $x \neq y$, there exists $v \in Ty$ satisfying $d(u,v) < rd(x,y)$.

Define a function f from X into $[0,\infty)$ by

$$f(x) = \inf\bigl\{d(x,b) : b \in Tx\bigr\}.$$

Arguing by contradiction, we assume $f(x) > 0$ for any $x \in X$. Fix $u_1 \in X$ and choose $u_2 \in Tu_1$ satisfying $d(u_1,u_2) < (1/r)f(u_1)$. Since $f(u_2) < rd(u_1,u_2)$, we can choose $u_3 \in Tu_2$ satisfying

$$d(u_2,u_3) < \min\bigl\{rd(u_1,u_2),(1/r)f(u_2)\bigr\}.$$

Then we have

$$rd(u_2,u_3) < f(u_2) \leq d(u_2,u_3) < rd(u_1,u_2) < f(u_1).$$

Continuing this argument, we can define a sequence $\{u_n\}$ in X satisfying

$$f(u_{n+1}) \leq d(u_{n+1},u_{n+2}) < rd(u_n,u_{n+1}) < f(u_n),$$

for any $n \in \mathbb{N}$. Since $\{f(u_n)\}$ is strictly decreasing, u_n ($n \in \mathbb{N}$) are all different. We also have

$$\sum_{j=1}^{\infty} d(u_j,u_{j+1}) \leq \sum_{j=1}^{\infty} r^{j-1} d(u_1,u_2) = \frac{d(u_1,u_2)}{1-r} < \infty.$$

Since X is $(\sum, \neq)$-complete, $\{u_n\}$ converges to some $y \in X$. We note that $\{u_n\}$ is Cauchy by Lemma 9(iii). From the assumption, we can choose a sequence $\{v_n\}$ in Ty satisfying

$$d(u_{n+1},v_n) \leq rd(u_n,y)$$

for any $n \in \mathbb{N}$. Then $\lim_n d(u_{n+1}, v_n) = 0$ holds. By Lemma 11, we have $\lim_n d(v_n, y) = 0$. Hence $f(y) = 0$ holds, which implies a contradiction. Therefore we have shown that there exists $z \in X$ satisfying $f(z) = 0$. From the assumption, $z \in Tz$ holds. □

As a direct consequence of Theorem 20, we obtain the following.

Corollary 21 (Branciari [10]) *Let (X, d) be a complete, ν-generalized metric space and let T be a contraction on X, that is, there exists $r \in [0, 1)$ such that*

$$d(Tx, Ty) \le rd(x, y)$$

for any $x, y \in X$. Then T has a fixed point.

Remark See also [15, 16, 18].

We improve Caristi's fixed point theorem; see [26, 27].

Definition 22 Let (X, d) be a ν-generalized metric space.

- A function f from X into $(-\infty, +\infty]$ is *proper* if $\{x \in X : f(x) \in \mathbb{R}\}$ is nonempty.
- A function f from X into $(-\infty, +\infty]$ is said to be *sequentially lower semicontinuous* if $f(x) \le \liminf_n f(x_n)$ holds whenever $\{x_n\}$ converges to x.
- A mapping T on X is said to be *sequentially continuous* if $\{Tx_n\}$ converges to Tx whenever $\{x_n\}$ converges to x.

Theorem 23 (Theorem 2 in [28], Theorem 14 in [11]) *Let (X, d) be a $(\sum, \neq)$-complete, ν-generalized metric space and let T be a mapping on X. Let f be a proper, sequentially lower semicontinuous function from X into $(-\infty, +\infty]$ bounded from below. Assume that*

$$f(Tx) + d(x, Tx) \le f(x)$$

for all $x \in X$. Then T has a fixed point.

Proof We use Lemma 9(iii) in this paper instead of Lemma 12 in [11]. Then we can prove the conclusion as in the proof of Theorem 14 in [11]. □

Remark We can weaken the assumption on the continuity of f as follows:

- $f(x) \le \liminf_n f(x_n)$ holds whenever $\{x_n\}$ converges to x in the strong sense.

We next improve Subrahmanyam's fixed point theorem; see [29–32].

Theorem 24 (Theorem 13 in [11]) *Let (X, d) be a $(\sum, \neq)$-complete, ν-generalized metric space and let T be a sequentially continuous mapping on X. Assume that there exists $r \in [0, 1)$ satisfying*

$$d(Tx, T^2x) \le rd(x, Tx)$$

for all $x \in X$. Then for any $x \in X$, $\{T^n x\}$ converges to a fixed point of T in the strong sense.

Proof We use Lemma 9(iii) in this paper instead of Lemma 12 in [11]. Then we can prove the conclusion as in the proof of Theorem 13 in [11]. □

Remark We can weaken the assumption on the continuity of T as follows:

- $\{Tx_n\}$ converges to Tx whenever $\{x_n\}$ converges to x in the strong sense.

4 Counterexamples

In this section, we give counterexamples on some results in Sections 2 and 3. The following example is a counterexample on Proposition 17 and Theorem 20.

Example 25 (see Example 1 in [14]) Put $X = \mathbb{N}$ and define a function d from $X \times X$ into $[0, \infty)$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } |x-y| \in 2\mathbb{N}, \\ |2^{1-x} - 2^{1-y}| & \text{otherwise.} \end{cases}$$

Define a set-valued mapping T by

$$Tx = \{x+1, x+2, \ldots\}.$$

Then the following hold:

(i) (X,d) is a 2-generalized metric space.
(ii) X is complete.
(iii) $\sum_{j=1}^{\infty} d(j,j+1) = 1 < \infty$ holds, however, $\{j\}$ is not Cauchy.
(iv) X is not $(\sum, \neq)$-complete. Hence X is not 2-complete.
(v) T satisfies the assumption of Theorem 20. However, T does not have a fixed point.

Proof We have proved (i)-(iii) in [14]. (iv) follows from (iii) and Lemma 10. Let us prove (v). Fix $x, y \in X$ with $x < y$. Since $Ty \subset Tx$, we have

$$\delta(Ty, Tx) = 0 \leq 2^{-1} d(y,x).$$

In the case where $y - x$ is odd, we have

$$\delta(Tx, Ty) = d(x+1, y+1) = 2^{-x} - 2^{-y} = 2^{-1} d(x,y).$$

In the other case, where $y - x$ is even, we have

$$\delta(Tx, Ty) = d(x+1, y+2) = 2^{-x} - 2^{-1-y} < 2^{-1} = 2^{-1} d(x,y).$$

It is clear that T satisfies the other assumption of Theorem 20. It is obvious that T does not have a fixed point. We have shown (v). □

Lemma 26 (Proposition 4.1 in [20]) *Let (X,d) be a ν-generalized metric space and let $\lambda \in \mathbb{N}$ such that λ is divisible by ν. Then (X,d) is a λ-generalized metric space.*

The following is a slight generalization of Lemma 4.1 in [19].

Lemma 27 *Let $\nu \in \mathbb{N}$. Let X be a nonempty set and let A and B be two subsets of X with $A \cap B = \varnothing$. Assume that A consists of at most $(\nu - 1)/2$ elements in the case where ν is odd. Let (Y, ρ) be a metric space and let S be a mapping from $A \cup B$ into Y such that $S(A) \cap S(B) = \varnothing$ holds and there exists some positive real number M satisfying*

$$\rho(Sx, Sy) \leq M$$

for all $x \in A$ and $y \in B$. Define a function d from $X \times X$ into $[0, \infty)$ by

$$\begin{aligned}
&d(x,x) = 0, \\
&d(x,y) = d(y,x) = \rho(Sx, Sy) \quad \text{if } x \in A \text{ and } y \in B, \\
&d(x,y) = M \quad \text{otherwise.}
\end{aligned}$$

Then (X, d) is a ν-generalized metric space.

Remark The proof below employs the methods in the proofs of Lemma 4 in [18] and Lemmas 4.2 and 4.3 in [20].

Proof It is obvious that (N1) and (N2) hold. In order to show (N3), we consider the following three cases:

(a) $\nu = 2$.
(b) ν is odd.
(c) ν is even.

In the case of (a), we let $x, y, u, v \in X$ be all different. Put

$$t = d(x,u) + d(u,v) + d(v,y).$$

In the case where $t \geq M$, (N3) holds because $d(x,y) \leq M$. In the other case, where $t < M$, without loss of generality, we may assume $x \in A$. Then we have $v \in A$ and $u, y \in B$ from the definition of d. Hence we obtain

$$\begin{aligned}
d(x,y) = \rho(Sx, Sy) &\leq \rho(Sx, Su) + \rho(Su, Sv) + \rho(Sv, Sy) \\
&= d(x,u) + d(u,v) + d(v,y).
\end{aligned}$$

In the case of (b), we let $x_1, \ldots, x_{\nu+2} \in X$ be all different. Then we have

$$d(x_1, x_{\nu+2}) \leq M \leq D(x_1, \ldots, x_{\nu+2}).$$

In the case of (c), from (a) and Lemma 26, we obtain the desired result. □

The following example is a counterexample on Lemma 8 and Proposition 18. Also this example tells that Theorem 20 is a true generalization of Theorem 1.

Example 28 (see Example 1.1 in [15]) Put $A = \{0, 2, 3\}$, $B = \{2^{-n} : n \in \mathbb{N}\}$ and $X = A \cup B$. Define a function d from $X \times X$ into $[0, \infty)$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \text{ and either } \{x,y\} \subset A \text{ or } \{x,y\} \subset B, \\ y & \text{if } x \in A, y \in B, \\ d(y,x) & \text{otherwise.} \end{cases}$$

Define sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$x_n = \begin{cases} 0 & \text{if } n = 4k+1 \text{ for some } k \in \mathbb{N} \cup \{0\}, \\ 2 & \text{if } n = 4k+3 \text{ for some } k \in \mathbb{N} \cup \{0\}, \\ 2^{-k} & \text{if } n = 2k \text{ for some } k \in \mathbb{N}, \end{cases}$$

and

$$y_n = \begin{cases} 0 & \text{if } n = 2k-1 \text{ for some } k \in \mathbb{N}, \\ 2^{-k} & \text{if } n = 2k \text{ for some } k \in \mathbb{N}. \end{cases}$$

Define a set-valued mapping T on X by

$$Tx = \begin{cases} A & \text{if } x \in A, \\ \{x/2, 0\} & \text{if } x \in B. \end{cases}$$

Then the following hold:

(i) (X, d) is a ν-generalized metric space for $\nu \in \mathbb{N} \setminus \{1, 3, 5\}$.
(ii) $\{2^{-n}\}$ converges to 0, 2 and 3. Therefore X is not Hausdorff.
(iii) X is $(\sum, \neq)$-complete.
(iv) $\{x_n\}$ is 2-Cauchy, however, it does not converge. Therefore X is not 2-complete.
(v) $\{y_n\}$ is 2-Cauchy and it converges to 0. However, $\{y_n\}$ is not Cauchy.
(vi) All the assumptions of Theorem 20 are satisfied.
(vii) There does not exist a metric q on X satisfying (1) with $d := q$.

Proof (i) follows from Lemma 27. (ii) obviously holds. There does not exist a $(\sum, \neq)$-Cauchy sequence in X. So (iii) holds. We note that $\{x_n\}$ is as follows:

$$0, 2^{-1}, 2, 2^{-2}, 0, 2^{-3}, 2, 2^{-4}, 0, \ldots.$$

It is obvious that $\{x_n\}$ is 2-Cauchy. However, $\{x_n\}$ does not converge. We have shown (iv). (v) obviously holds. We can easily prove

$$\delta(Tx, Ty) \leq (1/2) d(x, y)$$

for all $x, y \in X$. So, (vi) holds. Let us prove (vii). Arguing by contradiction, we assume that there exist a metric q on X and $r \in [0, 1)$ satisfying $\delta(Tx, Ty) \leq rq(x, y)$ for all $x, y \in X$, where

δ is defined by (2) with $d := q$. Let $x \in A$ and $y \in B$ be arbitrary. We have

$$\min\{q(y/2,0), q(y/2,2), q(y/2,3)\} = \delta(Ty, Tx) \leq rq(y,x)$$

and hence

$$\min\{q(y/2,0), q(y/2,2), q(y/2,3)\} \leq r \min\{q(y,0), q(y,2), q(y,3)\}.$$

Therefore

$$\lim_{n\to\infty} \min\{q(2^{-n},0), q(2^{-n},2), q(2^{-n},3)\} = 0 \tag{3}$$

holds. We also have

$$\begin{aligned}&\max\{\min\{q(2,0), q(2,2^{-n-1})\}, \min\{q(3,0), q(3,2^{-n-1})\}\}\\ &\quad = \delta(Tx, T2^{-n}) \leq rq(x, 2^{-n})\end{aligned}$$

and hence

$$\begin{aligned}&\max\{\min\{q(2,0), q(2,2^{-n-1})\}, \min\{q(3,0), q(3,2^{-n-1})\}\}\\ &\quad \leq r \min\{q(2^{-n},0), q(2^{-n},2), q(2^{-n},3)\}.\end{aligned}$$

Combining this and (3),

$$\lim_{n\to\infty} q(2, 2^{-n-1}) = \lim_{n\to\infty} q(3, 2^{-n-1}) = 0$$

holds. So we obtain 2 = 3, which implies a contradiction. □

5 Conclusions

In this paper, we study the completeness of ν-generalized metric space (see Propositions 17-19). We extend Nadler's fixed point theorem to ν-generalized metric spaces (see Theorem 20). We also improve Caristi's and Subrahmanyam's fixed point theorems (see Theorems 23 and 24).

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The author declares that he has no competing interests.

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