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Coincidence theory for compact morphisms

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Dedicated with much admiration to Ravi P. Agarwal on his seventieth birthday

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Abstract
In this paper we present several coincidence type results for morphisms (fractions) in the sense of Gorniewicz and Granas.

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1 Introduction

Morphisms (Vietoris fractions) in the sense of Gorniewicz and Granas were introduced in 1981 and coincidence spaces were discussed. In this paper, using compact morphisms, we present a variety of coincidence (and fixed point) results on particular Hausdorff topological spaces. These spaces include ES(compact), AES(compact), general admissible and general dominated spaces. Our theory is motivated partly by ideas in [1–4].

Now we present some ideas needed in Section 2. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q-dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic,

(ii) p is a perfect map, *i.e.*, p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $D(X, Y)$ be the set of all pairs $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p, q). Given two diagrams (p, q) and (p', q'), where $X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\to} Y$, we write $(p, q) \sim (p', q')$ if there are continuous maps $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p,q) \in D(X,Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

or $\phi = [(p,q)]$ and is called a morphism from X to Y. We let $M(X,Y)$ be the set of all such morphisms. Note that if $(p,q),(p_1,q_1) \in D(X,Y)$ (where $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ and $X \overset{p_1}{\Leftarrow} \Gamma' \overset{q_1}{\to} Y$) and $(p,q) \sim (p_1,q_1)$, then it is easy to see (use $q \circ g = q_1$ and $p \circ g = p_1$ where $g : \Gamma' \to \Gamma$) that for $x \in X$ we have $q_1(p_1^{-1}(x)) = q(p^{-1}(x))$. For any $\phi \in M(X,Y)$ a set $\phi(x) = qp^{-1}(x)$ where $\phi = [(p,q)]$ is called an image of x under the morphism ϕ. Let $\phi \in M(X,Y)$ and let (p,q) be a representative of ϕ. We define $\phi(X) \subseteq Y$ by $\phi(X) = q(p^{-1}(X))$. Note $\phi(X)$ does not depend on the representative of ϕ. Now $\phi \in M(X,Y)$ is called compact, provided the set $\phi(X)$ is relatively compact in Y. Note we will identify a map $f : X \to Y$ with the morphism $f = \{X \overset{Id_X}{\Leftarrow} X \overset{f}{\to} Y\} : X \to Y$. Let $X \subseteq Y$. A point $x \in X$ is called a fixed point of a morphism $\phi \in M(X,Y)$ if $x \in \phi(x)$.

Let $\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$ be a morphism. We define the coincidence set

$$\mathrm{Coin}(p,q) = \{y \in \Gamma : p(y) = q(y)\}.$$

We say ϕ has a coincidence provided the set $C(\phi) = p(\mathrm{Coin}(p,q))$ is nonempty (*i.e.*, there exists $x \in p(\mathrm{Coin}(p,q))$, *i.e.*, there exists $y \in \Gamma$ with $x = p(y) = q(y)$). Let (p',q') be another representation of ϕ, say $\phi = \{X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\to} Y\}$. Note $p(\mathrm{Coin}(p,q)) = p'(\mathrm{Coin}(p',q'))$; to see this, note that if $x \in p(\mathrm{Coin}(p,q))$, then $x = p(y) = q(y)$ for some $y \in \Gamma$. Now since $(p,q) \sim (p',q')$, with $f : \Gamma \to \Gamma'$ we have $x = q(y) = q'(f(y))$ and $x = p(y) = p'(f(y))$ so $f(y) \in \Gamma'$ and $x = q'(f(y)) = p'(f(y))$, *i.e.*, $x \in p'(\mathrm{Coin}(p',q'))$. Thus the above definition does not depend on the choice of a representation (p,q). Also $C(\phi) \neq \emptyset$ iff $\mathrm{Coin}(p,q) \neq \emptyset$ for any representation (p,q) of ϕ.

Suppose $\phi \in M(X,X)$ (here $\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} X\}$) has a coincidence point for (p,q), *i.e.*, suppose there exists $y \in \Gamma$ with $p(y) = q(y)$. Now since p is surjective, there exists $w \in X$ with $y \in p^{-1}(w)$ (note $p(w) = y$) and so $w \in q(p^{-1}(w)) = \phi(w)$ (note $pp^{-1}(w) = w$ and the set $q(p^{-1}(w))$ is the image of w under ϕ), *i.e.*, ϕ has a fixed point. As a result

$$p(y) = q(y), \quad y \in \Gamma \ (\text{and let } w = p(y)) \quad \Leftrightarrow \quad w \in q(p^{-1}(w)).$$

Note that if $w \in q(p^{-1}(w))$, then there exists $y \in p^{-1}(w)$ with $w = q(y)$ so $p(y) \in pp^{-1}(w) = w$ (*i.e.*, $p(y) = w$) and so $p(y) = q(y)$. In particular if the morphism $\phi \in M(X,X)$ (here (p,q) is a representation of ϕ) has a fixed point (say w, *i.e.*, $w \in q(p^{-1}(w))$), then there exists $y \in p^{-1}(w)$ with $q(y) = p(y)$, so ϕ has a coincidence point for (p,q). We can apply this argument for any representation (p,q) of ϕ (recall that if (p_1,q_1) is another representation of ϕ, then $(p,q) \sim (p_1,q_1)$ and as above $q(p^{-1}(w)) = q_1(p_1^{-1}(w))$, so $w \in q_1(p_1^{-1}(w))$, so there exists $y_1 \in p_1^{-1}(w)$ with $q_1(y_1) = p_1(y_1)$), thus $\mathrm{Coin}(p,q) \neq \emptyset$ for any representation (p,q) of ϕ, *i.e.*, ϕ has a coincidence.

For a subset K of a topological space X, we denote by $\mathrm{Cov}_X(K)$ the set of all coverings of K by open sets of X (usually we write $\mathrm{Cov}(K) = \mathrm{Cov}_X(K)$). Given a morphism $\phi \in M(X,X)$ and $\alpha \in \mathrm{Cov}(X)$, a point $x \in X$ is said to be an α-fixed point of ϕ if there exists a member $U \in \alpha$ such that $x \in U$ and $\phi(x) \cap U \neq \emptyset$. Given a morphism $\phi \in M(X,X)$ (here $\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} X\}$) and $\alpha \in \mathrm{Cov}(X)$, a point $y \in \Gamma$ is said to be an α-coincidence point for (p,q) if there exists a member $U \in \alpha$ with $p(y) \in U$ and $q(y) \in U$. We say ϕ has an α-coincidence if ϕ has an α-coincidence point for each representation (p, q) of ϕ.

Let X and Y be topological spaces. Given two morphisms $\phi \in M(X, Y)$ and $\psi \in M(X, Y)$ and $\alpha \in \text{Cov}(Y)$, ϕ and ψ are said to be α-close if for any $x \in X$, there exists $U_x \in \alpha$ with $\phi(x) \cap U_x \neq \emptyset$ and $\psi(x) \cap U_x \neq \emptyset$. Recall that, given two single valued maps $f, g : X \to Y$ and $\alpha \in \text{Cov}(Y)$, f and g are said to be α-close if for any $x \in X$ there exists $U_x \in \alpha$ containing both $f(x)$ and $g(x)$. Given a morphism $\phi \in M(X, Y)$ and a single valued map $g : X \to Y$ and $\alpha \in \text{Cov}(Y)$, ϕ and g are said to be strongly α-close if for any $x \in X$ there exists $U_x \in \alpha$ with $\phi(x) \subseteq U_x$ and $g(x) \in U_x$.

Let T be the Tychonoff cube (*i.e.*, Cartesian product of copies of the unit interval). Finally we recall the following result from the literature; see [3] (see Theorem 7.6 and the proof of Theorem 5.5) or alternatively see [5] (see Corollary 6.5 and if we take the Hausdorff locally convex topological vector space E containing T we just need to note that T is a retract of E [6]).

Theorem 1.1 *Let $\phi \in M(T, T)$ be compact. Then ϕ has a coincidence.*

2 Coincidence theory

By a space we mean a Hausdorff topological space. Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in \text{ES}(Q)$) if, $\forall X \in Q$ and $\forall K \subseteq X$ closed in X, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$.

Theorem 2.1 *Let $X \in$ ES(compact) and $\phi \in M(X, X)$ is compact. Then ϕ has a coincidence.*

Proof Let $\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} X\} : X \to X$. We know [4] that every compact space is homeomorphic to a closed subset of the Tychonoff cube T, so as a result $K = \overline{\phi(X)}$ can be embedded as a closed subset $K^\star$ of T; let $s : K \to K^\star$ be a homeomorphism. Also let $i : K \hookrightarrow X$ and $j : K^\star \hookrightarrow T$ be inclusions. Now since $X \in$ ES(compact) and $is^{-1} : K^\star \to X$, is^{-1} extends to a continuous function $h : T \to X$. Let $\psi = js\phi h$ and note (see [3], see (4.2)) $\psi \in M(T, T)$ is compact. Now Theorem 1.1 guarantees that $js\phi h$ has a coincidence and therefore (see [3] (Lemma 6.3)) $hjs\phi$ has a coincidence. Thus there exists a $y \in \Gamma$ with $hjsq(y) = p(y)$; note $hjs\phi = \{X \overset{\overline{p}}{\Leftarrow} \Gamma \boxtimes K \overset{\overline{q}}{\to} X\}$ where $\Gamma \boxtimes K = \{(z_1, z_2) \in \Gamma \times K : q(z_1) = z_2\}$, $\overline{p}(z_1, z_2) = pf_1(z_1, z_2)$, $\overline{q}(z_1, z_2) = hjsf_2(z_1, z_2)$, $f_1(z_1, z_2) = z_1$ and $f_2(z_1, z_2) = z_2$ so $\overline{p}(z_1, z_2) = p(z_1)$ and $\overline{q}(z_1, z_2) = hjs(z_2) = hjsq(z_1)$. Also note $hj(z) = is^{-1}(z)$ for $z \in K^\star$ so $hjs(w) = i(w) = w$ for $w \in K$. Consequently $q(y) = p(y)$, so ϕ has a coincidence point (for (p, q)). We can apply the above argument for any representation (p, q) of ϕ, so $C(\phi) \neq \emptyset$. □

A space Y is an approximate extension space for Q (written $Y \in \text{AES}(Q)$) if, $\forall \alpha \in \text{Cov}(Y)$, $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and any continuous function $f_0 : K \to Y$, there exists a continuous function $f : X \to Y$ such that $f|_K$ is α-close to f_0.

Theorem 2.2 *Let $X \in$ AES(compact) and $\phi \in M(X, X)$ is compact. Then for any $\alpha \in \text{Cov}_X(\overline{\phi(X)})$, ϕ has an α-coincidence.*

Proof Let $\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} X\} : X \to X$ and let $\alpha \in \text{Cov}_X(K)$, where $K = \overline{\phi(X)}$. Now K can be embedded as a closed subset $K^\star$ of T; let $s : K \to K^\star$ be a homeomorphism. Also let

$i : K \hookrightarrow X$ and $j : K^\star \hookrightarrow T$ be inclusions. Now let $\alpha' = \alpha \cup \{X \backslash K\}$ and note α' is an open covering of X. Let the continuous map $h : T \to X$ be such that $h|_{K^\star}$ and s^{-1} are α'-close (guaranteed since $X \in \mathrm{AES}(\text{compact})$). Thus (note $\alpha' = \alpha \cup \{X\backslash K\}$) $hs : K \to X$ and $i : K \to X$ are α-close. Let $\psi = js\phi h$ and note $\psi \in M(T, T)$ is compact so Theorem 1.1 guarantees that $js\phi h$ has a coincidence. Then from [3] (Lemma 6.3), $hjs\phi$ has a coincidence, *i.e.*, there exists a $y \in \Gamma$ with $hjsq(y) = p(y)$. Now since $hs : K \to X$ and $i : K \to X$ are α-close there exists $U \in \alpha$ with

$$hs\big(q(y)\big) \in U \quad \text{and} \quad i\big(q(y)\big) \in U \quad \textit{i.e.} \quad p(y) = hs\big(q(y)\big) \in U \quad \text{and} \quad q(y) \in U.$$

Thus ϕ has an α-coincidence (for (p, q)). We can apply the above argument for any representation (p, q) of ϕ. □

Remark 2.3 One can put conditions on the space X and the morphism ϕ so that ϕ has an α-coincidence for each $\alpha \in \mathrm{Cov}_X(\overline{\phi(X)})$ would guarantee that ϕ has a coincidence; for examples we refer the reader to [7] (Lemma 1.2), [3] (Lemma 6.1), [8] (Theorem 1.4 and Remark 1.6). We say (X, ϕ) has the α-coincidence property if ϕ, having an α-coincidence for each $\alpha \in \mathrm{Cov}_X(\overline{\phi(X)})$, guarantees that ϕ has a coincidence. Thus we have: Suppose $X \in \mathrm{AES}(\text{compact})$, $\phi \in M(X, X)$ is compact, and (X, ϕ) has the α-coincidence property. Then ϕ has a coincidence.

Next we generalise the above results motivated in part from Schauder projections [4]. Let W be a space. We say W is admissible if, for all compact subsets K of W, all $\alpha \in \mathrm{Cov}_W(K)$, there exists a single valued continuous map $g_\alpha : K \to W$ such that:

(i) g_α and $i : K \hookrightarrow W$ are α-close,

(ii) $g_\alpha(K)$ is contained in a subset $C_\alpha \subseteq W$ and C_α has the coincidence property (*i.e.*, any compact $\theta \in M(C_\alpha, C_\alpha)$ has a coincidence).

Theorem 2.4 *Let W be admissible and $\phi \in M(W, W)$ is compact.*

(i) *Then, for any $\alpha \in \mathrm{Cov}_W(\overline{\phi(W)})$, ϕ has an α-coincidence.*

(ii) *If (W, ϕ) has the α-coincidence property, then ϕ has a coincidence.*

Proof (i) Let $\phi = \{W \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} W\} : W \to W$ and let $\alpha \in \mathrm{Cov}_W(K)$ where $K = \overline{\phi(W)}$. Now there exists a single valued continuous map $g_\alpha : K \to W$ and C_α as described in the definition of admissible. Let $j_\alpha : C_\alpha \hookrightarrow W$ be the inclusion and note $g_\alpha \phi j_\alpha \in M(C_\alpha, C_\alpha)$ is compact. Since C_α has the coincidence property, there exists $y \in p^{-1}(C_\alpha) \subseteq \Gamma$ with $g_\alpha q(y) = p(y)$; note $g_\alpha \phi j_\alpha = \{C_\alpha \overset{\overline{p}}{\Leftarrow} p^{-1}(C_\alpha) \boxtimes K \overset{\overline{q}}{\to} C_\alpha\}$, where $p^{-1}(C_\alpha) \boxtimes K = \{(z_1, z_2) \in p^{-1}(C_\alpha) \times K : q(z_1) = z_2\}$, $\overline{p}(z_1, z_2) = pf_1(z_1, z_2)$, $\overline{q}(z_1, z_2) = g_\alpha f_2(z_1, z_2)$, $f_1(z_1, z_2) = z_1$ and $f_2(z_1, z_2) = z_2$ so $\overline{p}(z_1, z_2) = p(z_1)$ and $\overline{q}(z_1, z_2) = g_\alpha(z_2) = g_\alpha q(z_1)$. Since g_α and $i : K \hookrightarrow W$ are α-close, there exists $U \in \alpha$ with

$$g_\alpha\big(q(y)\big) \in U \quad \text{and} \quad i\big(q(y)\big) \in U \quad \textit{i.e.} \quad p(y) = g_\alpha\big(q(y)\big) \in U \quad \text{and} \quad q(y) \in U.$$

Thus ϕ has an α-coincidence (for (p, q)). We can apply the above argument for any representation (p, q) of ϕ.

(ii). Immediate from the definition and part (i). □

Let W be a space. We say W is general admissible if for all compact subsets K of W and all $\alpha \in \mathrm{Cov}_W(K)$, there exists a $\psi_\alpha \in M(K,W)$ such that:

(i) ψ_α and $i : K \hookrightarrow W$ are strongly α-close,

(ii) $\psi_\alpha(K)$ is contained in a subset $C_\alpha \subseteq W$ and C_α has the coincidence property.

In our first result we will phrase it as a fixed point result but immediately after the proof we will rephrase it as a coincidence result (see Theorem 2.6).

Theorem 2.5 *Let W be general admissible and $\phi \in M(W,W)$ is compact.*

(i) *Then for any $\alpha \in \mathrm{Cov}_W(\overline{\phi(W)})$, ϕ has an α-fixed point.*

(ii) *If (W,ϕ) has the α-fixed point property* (i.e., *ϕ, having an α-fixed point for each $\alpha \in \mathrm{Cov}_X(\overline{\phi(X)})$, guarantees that ϕ has a fixed point*), *then ϕ has a coincidence.*

Proof (i) Let $\phi = \{W \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} W\} : W \to W$ and let $\alpha \in \mathrm{Cov}_W(K)$ where $K = \overline{\phi(W)}$. Since W is general admissible, there exists a $\psi_\alpha \in M(K,W)$ (here $\psi_\alpha = \{K \overset{p_\alpha}{\Leftarrow} \Gamma' \overset{q_\alpha}{\rightarrow} W\}$, *i.e.*, (p_α, q_α) is a representation of ψ_α) and C_α as described in the definition of general admissible. Let $j_\alpha : C_\alpha \hookrightarrow W$ be the inclusion and note $\psi_\alpha \phi j_\alpha \in M(C_\alpha, C_\alpha)$ is compact (note ϕ is compact and the map $x \mapsto q_\alpha(p_\alpha^{-1}(x))$ is upper semicontinuous with nonempty compact values [2]). Since C_α has the coincidence property, $\psi_\alpha\phi$ has a coincidence so (see Section 1) $\psi_\alpha\phi$ has a fixed point, *i.e.*, there exists a $w \in C_\alpha$ with $w \in \psi_\alpha(\phi(w))$. Now $\psi_\alpha\phi = \{C_\alpha \overset{\overline{p}}{\Leftarrow} p^{-1}(C_\alpha) \boxtimes \Gamma' \overset{\overline{q}}{\rightarrow} C_\alpha\}$ where $p^{-1}(C_\alpha) \boxtimes \Gamma' = \{(z_1,z_2) \in p^{-1}(C_\alpha) \times \Gamma' : q(z_1) = p_\alpha(z_2)\}$, $\overline{p}(z_1,z_2) = pf_1(z_1,z_2)$, $\overline{q}(z_1,z_2) = q_\alpha f_2(z_1,z_2)$, $f_1(z_1,z_2) = z_1$ and $f_2(z_1,z_2) = z_2$. Now [9] (2.1), [2] (Section 40) guarantee that

$$w \in \overline{q}\bigl((\overline{p})^{-1}(w)\bigr) = q_\alpha\bigl(p_\alpha^{-1}\bigl(q\bigl(p^{-1}(w)\bigr)\bigr)\bigr). \tag{1}$$

Thus there exists a $y \in q(p^{-1}(w)) = \phi(w)$ with $w \in q_\alpha(p_\alpha^{-1}(y)) = \psi_\alpha(y)$. Now since ψ_α and $i : K \hookrightarrow W$ are strongly α-close, there exists $U \in \alpha$ with

$$\psi_\alpha(y) \subseteq U \quad \text{and} \quad i(y) \in U,$$

and since $w \in \psi_\alpha(y)$, we have $w \in U$ and $y \in U$, *i.e.*,

$$w \in U \quad \text{and} \quad \phi(w) \cap U \neq \emptyset \quad \bigl(\text{since } y \in \phi(w) \text{ and } y \in U\bigr).$$

Thus ϕ has an α-fixed point (for (p,q)). We can apply the above argument for any representation (p,q) of ϕ.

(ii) From part (i) we know that ϕ has an α-fixed point for any $\alpha \in \mathrm{Cov}_W(\overline{\phi(W)})$. Now since (W,ϕ) has the α-fixed point property, ϕ has a fixed point and, as in Section 1, ϕ has a coincidence. □

Theorem 2.6 *Let W be general admissible and $\phi \in M(W,W)$ is compact.*

(i) *Then for any $\alpha \in \mathrm{Cov}_W(\overline{\phi(W)})$, ϕ has an α-coincidence.*

(ii) *If (W,ϕ) has the α-coincidence property, then ϕ has a coincidence.*

Proof (i) Follow the proof in Theorem 2.5 (i) to obtain (1). Then there exists a $y \in q(p^{-1}(w))$ with $w \in q_\alpha(p_\alpha^{-1}(y)) = \psi_\alpha(y)$ and since ψ_α and $i : K \hookrightarrow W$ are strongly α-close, there exists

$U \in \alpha$ with $\psi_\alpha(y) \in U$ and $i(y) \in U$. Thus $w \in U$ and $y \in U$. Also, there exists $a \in p^{-1}(w)$ with $y = q(a)$ and note $p(a) = w$, *i.e.*, $y = q(a)$ and $p(a) = w$. As a result

$$p(a) \in U \quad \text{and} \quad q(a) \in U,$$

so ϕ has an α-coincidence (for (p,q)). We can apply the above argument for any representation (p,q) of ϕ.

(ii) Immediate from the definition and part (i). □

Let W be a space and C a space with the coincidence property, (*i.e.*, any compact $\theta \in M(C,C)$ has a coincidence). We say C dominates W if, for every compact subset K of W and every $\alpha \in \mathrm{Cov}_W(K)$, there exist single valued continuous maps $s_\alpha : W \to C$, $r_\alpha : C \to W$ with $r_\alpha s_\alpha : K \to W$ and $i : K \hookrightarrow W$ α-close.

Theorem 2.7 *Let W be a space and C a space with the coincidence property. Suppose $\phi \in M(W,W)$ is compact and C dominates W. Then, for any $\alpha \in \mathrm{Cov}_W(\overline{\phi(W)})$, ϕ has an α-coincidence.*

Proof Let $\phi = \{W \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} W\} : W \to W$ and let $\alpha \in \mathrm{Cov}_W(K)$, where $K = \overline{\phi(W)}$. Since C dominates W there exist single valued continuous maps $s_\alpha : W \to C$, $r_\alpha : C \to W$ as described in the definition. Note $s_\alpha \phi r_\alpha \in M(C,C)$ is compact. Since C has the coincidence property, $s_\alpha \phi r_\alpha$ has a coincidence and therefore (see [3] (Lemma 6.3)) $r_\alpha s_\alpha \phi$ has a coincidence. Thus there exists $y \in \Gamma$ with $r_\alpha s_\alpha q(y) = p(y)$; note $r_\alpha s_\alpha \phi = \{W \overset{\overline{p}}{\Leftarrow} \Gamma \boxtimes K \overset{\overline{q}}{\to} W\}$ where $\Gamma \boxtimes K = \{(z_1,z_2) \in \Gamma \times K : q(z_1) = z_2\}$, $\overline{p}(z_1,z_2) = pf_1(z_1,z_2)$, $\overline{q}(z_1,z_2) = r_\alpha s_\alpha f_2(z_1,z_2)$, $f_1(z_1,z_2) = z_1$ and $f_2(z_1,z_2) = z_2$, so $\overline{p}(z_1,z_2) = p(z_1)$ and $\overline{q}(z_1,z_2) = r_\alpha s_\alpha(z_2) = r_\alpha s_\alpha q(z_1)$. Since $r_\alpha s_\alpha : K \to W$ and $i : K \hookrightarrow W$ are α-close, there exists $U \in \alpha$ with

$$r_\alpha s_\alpha q(y) \in U \quad \text{and} \quad i(q) \in U \quad \textit{i.e.} \quad p(y) = r_\alpha s_\alpha q(y) \in U \quad \text{and} \quad q(y) \in U.$$

Thus ϕ has an α-coincidence (for (p,q)). We can apply the above argument for any representation (p,q) of ϕ. □

Let W be a space and C a space with the coincidence property. We say C generally dominates W if, for every compact subset K of W and every $\alpha \in \mathrm{Cov}_W(K)$, there exist $S_\alpha \in M(W,C)$ and $R_\alpha \in M(C,W)$ with $R_\alpha S_\alpha \in M(K,W)$ and $i : K \hookrightarrow W$ strongly α-close.

In our first result we will phrase it as a fixed point result but immediately after the proof we will rephrase it as a coincidence result (see Theorem 2.9).

Theorem 2.8 *Let W be a space and C a space with the coincidence property. Suppose $\phi \in M(W,W)$ is compact and C generally dominates W. Then for any $\alpha \in \mathrm{Cov}_W(\overline{\phi(W)})$, ϕ has an α-fixed point.*

Proof Let $\phi = \{W \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} W\} : W \to W$ and let $\alpha \in \mathrm{Cov}_W(K)$, where $K = \overline{\phi(W)}$. Since C generally dominates W, there exist $S_\alpha \in M(W,C)$ and $R_\alpha \in M(C,W)$ as described in the definition. Note with $S_\alpha = \{W \overset{p_1}{\Leftarrow} \Gamma_1 \overset{q_1}{\to} C\}$, $R_\alpha = \{C \overset{p_2}{\Leftarrow} \Gamma_2 \overset{q_2}{\to} W\}$ then [3], (4.2), guarantee we have $R_\alpha S_\alpha = \{W \overset{p_\alpha}{\Leftarrow} \Gamma_1 \boxtimes \Gamma_2 \equiv \Gamma_3 \overset{q_\alpha}{\to} W\}$, where $\Gamma_1 \boxtimes \Gamma_2 = \{(z_1,z_2) \in \Gamma_1 \times \Gamma_2 : q_1(z_1) = p_2(z_2)\}$, $p_\alpha(z_1,z_2) = p_1 f_1(z_1,z_2)$, $q_\alpha(z_1,z_2) = q_2 f_2(z_1,z_2)$, $f_1(z_1,z_2) = z_1$ and $f_2(z_1,z_2) = z_2$. Note

$S_\alpha\phi R_\alpha \in M(C,C)$ is compact (note ϕ is compact and the map $x \mapsto q_1(p_1^{-1}(x))$ is upper semicontinuous with nonempty compact values [2]). Since C has the coincidence property, $S_\alpha\phi R_\alpha$ has a coincidence and therefore (see [3] (Lemma 6.3)) $R_\alpha S_\alpha\phi$ has a coincidence. From Section 1 note $R_\alpha S_\alpha\phi$ has a fixed point, *i.e.*, there exists a $w \in W$ with $w \in R_\alpha S_\alpha\phi(w)$. Next note $R_\alpha S_\alpha\phi = \{W \stackrel{\overline{p}}{\Leftarrow} \Gamma \boxtimes \Gamma_3 \stackrel{\overline{q}}{\to} W\}$, where $\Gamma \boxtimes \Gamma_3 = \{(z_1,z_2) \in \Gamma \times \Gamma_3 : q(z_1) = p_\alpha(z_2)\}$, $\overline{p}(z_1,z_2) = pf_1(z_1,z_2)$, $\overline{q}(z_1,z_2) = q_\alpha f_2(z_1,z_2)$, $f_1(z_1,z_2) = z_1$ and $f_2(z_1,z_2) = z_2$. Now it is easy to note (see [9] (2.1), [2] (Section 40)) that

$$w \in \overline{q}\big((\overline{p})^{-1}(w)\big) = q_\alpha\big(p_\alpha^{-1}\big(q\big(p^{-1}(w)\big)\big)\big). \tag{2}$$

Thus there exists a $y \in q(p^{-1}(w)) = \phi(w)$ with $w \in q_\alpha(p_\alpha^{-1}(y)) = R_\alpha S_\alpha(y)$. Now since $R_\alpha S_\alpha$ and $i : K \hookrightarrow W$ are strongly α-close, there exists $U \in \alpha$ with

$$R_\alpha S_\alpha(y) \subseteq U \quad \text{and} \quad i(y) \in U,$$

and since $w \in R_\alpha S_\alpha(y)$, we have $w \in U$ and $y \in U$, *i.e.*,

$$w \in U \quad \text{and} \quad \phi(w) \cap U \neq \emptyset \quad \big(\text{since } y \in \phi(w) \text{ and } y \in U\big).$$

Then ϕ has an α-fixed point (for (p,q)). We can apply the above argument for any representation (p,q) of ϕ. □

Theorem 2.9 *Let W be a space and C a space with the coincidence property. Suppose $\phi \in M(W,W)$ is compact and C generally dominates W. Then for any $\alpha \in \mathrm{Cov}_W(\overline{\phi(W)})$, ϕ has an α-coincidence.*

Proof Follow the proof in Theorem 2.8 to obtain (2). Then there exists a $y \in q(p^{-1}(w))$ with $w \in q_\alpha(p_\alpha^{-1}(y)) = R_\alpha S_\alpha(y)$ and, since $R_\alpha S_\alpha$ and $i : K \hookrightarrow W$ are strongly α-close, there exists $U \in \alpha$ with $R_\alpha S_\alpha(y) \in U$ and $i(y) \in U$. Thus $w \in U$ and $y \in U$. Also there exists $a \in p^{-1}(w)$ with $y = q(a)$ and note $p(a) = w$, *i.e.*, $y = q(a)$ and $p(a) = w$. As a result $p(a) \in U$ and $q(a) \in U$, so ϕ has an α-coincidence (for (p,q)). We can apply the above argument for any representation (p,q) of ϕ. □

3 Conclusions

In this paper, using new ideas, we present a number of coincidence and α-coincidence results for compact morphisms (Vietoris fractions) defined on a variety of admissible and dominating type spaces.

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author read and approved the final manuscript.

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