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Common best proximity point theorem for multivalued mappings in partially ordered metric spaces

V Pragadeeswarar[1], G Poonguzali[2], M Marudai[2] and Stojan Radenović[3,4*]

*Correspondence:
stojan.radenovic@tdt.edu.vn
[3]Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam
[4]Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
Full list of author information is available at the end of the article

Abstract
In this paper, we prove the existence of a common best proximity point for a pair of multivalued non-self mappings in partially ordered metric spaces. Also, we provide some interesting examples to illustrate our main results.

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Keywords: partially ordered set; proximal relation; *P*-property; altering distance function; best proximity point; common best proximity point

1 Introduction

The study of multivalued mappings plays a vital role in pure and applied mathematics because of its many applications, for instance, in real and complex analysis. In the literature, there are many researchers focusing on the study of abstract and practical problems which involve multivalued mappings. As a matter of fact, amongst the various approaches to develop this theory, one of the most interesting is based on best proximity point theory.

One can find the existence and convergence of best proximity points in [1–8]. For the existence of a best proximity point in the setting of partially ordered metric spaces, see [9–13]. Also, for more results on a best proximity point of multivalued non-self mappings, we suggest [14–16].

2 Preliminaries

In this section, we give some basic definitions and notions that will be used frequently.

Let X be a nonempty set such that $(X,d,\preceq)$ is a partially ordered metric space. Consider A and B to be nonempty subsets of the metric space (X,d). We denote by $CB(X)$ the class of all nonempty closed and bounded subsets of X.

$$\delta(A,B) := \sup\bigl\{d(a,b) : a \in A \text{ and } b \in B\bigr\},$$

$$D(A,B) := \inf\bigl\{d(a,b) : a \in A \text{ and } b \in B\bigr\},$$

$$A_0 = \bigl\{a \in A : d(a,b) = D(A,B) \text{ for some } b \in B\bigr\},$$

$$B_0 = \bigl\{b \in B : d(a,b) = D(A,B) \text{ for some } a \in A\bigr\}.$$

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Definition 2.1 Let $T : A \to 2^B$ be any multivalued mapping. Then an element $x \in A$ is said to be a best proximity point if $D(x, Tx) = D(A, B)$.

Definition 2.2 Given multivalued non-self mappings $S : A \to 2^B$ and $T : A \to 2^B$, an element $a_0 \in A$ is called a *common best proximity point* of the mappings if they satisfy the condition that $D(a_0, Sa_0) = D(a_0, Ta_0) = D(A, B)$.

Definition 2.3 ([17]) A function $\psi : [0, \infty) \to [0, \infty)$ is said to be an altering distance function if it satisfies the following conditions:

(i) ψ is continuous and nondecreasing.
(ii) $\psi(t) = 0$ if and only if $t = 0$.

Example 2.4 Define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = kt$, where $k < 1$. Then ψ is an altering distance function.

Definition 2.5 ([8]) Let (A, B) be a pair of nonempty subsets of a metric space X with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if and only if

$$\left.\begin{aligned} d(a_1, b_1) = D(A, B) \\ d(a_2, b_2) = D(A, B) \end{aligned}\right\} \quad \Longrightarrow \quad d(a_1, a_2) = d(b_1, b_2),$$

where $a_1, a_2 \in A_0$ and $b_1, b_2 \in B_0$.

The existence of fixed points in partially ordered metric spaces was first established by Nieto and Rodriguez-Lopez [18]. In this direction, Choudhury and Metiya [19] proved the existence of a fixed point for multivalued self mappings in partially ordered metric spaces.

In this paper, our main objective is to establish the existence of best proximity points and common best proximity points of multivalued mappings in partially ordered metric spaces. Also, our results generalize the corresponding results of [19]. In particular, the aim of this paper is to initiate the study of common best proximity points of multivalued mappings in partially ordered metric spaces.

Here we define the notion of proximal relation between two subsets of X.

Definition 2.6 ([16]) Let A and B be two nonempty subsets of a partially ordered metric space $(X, d, \preceq)$ such that $A_0 \neq \emptyset$. Let B_1 and B_2 be two nonempty subsets of B_0. The proximal relations between B_1 and B_2 are denoted and defined as follows:

(i) $B_1 \prec_{(1)} B_2$ if, for every $b_1 \in B_1$ with $d(a_1, b_1) = D(A, B)$, there exists $b_2 \in B_2$ with $d(a_2, b_2) = D(A, B)$ such that $a_1 \preceq a_2$.
(ii) $B_1 \prec_{(2)} B_2$ if, for every $b_2 \in B_2$ with $d(a_2, b_2) = D(A, B)$, there exists $b_1 \in B_1$ with $d(a_1, b_1) = D(A, B)$ such that $a_1 \preceq a_2$.
(iii) $B_1 \prec_{(3)} B_2$ if $B_1 \prec_{(1)} B_2$ and $B_1 \prec_{(2)} B_2$.

3 Main results

Now, we state our first main result in this section.

Theorem 3.1 *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let A and B be nonempty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$ and (A, B) satisfies the*

P-property. Let $T : A \to CB(B)$ be a multivalued mapping such that the following conditions are satisfied:

(i) *There exist elements a_0, a_1 in A_0 and $b_0 \in Ta_0$ such that*

$$d(a_1, b_0) = D(A,B) \quad \text{and} \quad a_0 \preceq a_1.$$

(ii) *Ta_0 is included in B_0 for all $a_0 \in A_0$ and*

$$\delta(Ta, Tb) \le \psi\bigl(M(a,b)\bigr) + LN(a,b) \quad \text{for all comparable } a, b \in A, \tag{1}$$

where $M(a,b) = \max\{d(a,b), D(a,Ta) - D(A,B), D(b,Tb) - D(A,B), \frac{D(a,Tb)+D(b,Ta)}{2} - D(A,B)\}$, $L \ge 0$, $N(a,b) = \min\{D(a,Ta) - D(A,B), D(b,Tb) - D(A,B), D(a,Tb) - D(A,B), D(b,Ta) - D(A,B)\}$ and $\psi : [0,\infty) \to [0,\infty)$ is a nondecreasing and upper-semicontinuous function with $\psi(t) < t$ for each $t > 0$.

(iii) *For $a, b \in A_0$, $a \preceq b$ implies $Ta \prec_{(1)} Tb$.*

(iv) *If $\{a_n\}$ is a nondecreasing sequence in A such that $a_n \to a$, then $a_n \preceq a$ for all n.*

Then there exists an element a in A such that

$$D(a, Ta) = D(A,B).$$

Proof By assumption (i), there exist two elements a_0, a_1 in A_0 and $b_0 \in Ta_0$ such that $d(a_1, b_0) = D(A,B)$ and $a_0 \preceq a_1$. By assumption (iii), $Ta_0 \prec_{(1)} Ta_1$, there exists $b_1 \in Ta_1$ with $d(a_2, b_1) = D(A,B)$ such that $a_1 \preceq a_2$. In general, for each $n \in \mathbb{N}$, there exist $a_{n+1} \in A_0$ and $b_n \in Ta_n$ such that $d(a_{n+1}, b_n) = D(A,B)$. Hence, we obtain

$$\begin{aligned} &d(a_{n+1}, b_n) = D(a_{n+1}, Ta_n) = D(A,B) \\ &\quad \text{for all } n \in \mathbb{N} \text{ with } a_0 \preceq a_1 \preceq a_2 \preceq \cdots \preceq a_n \preceq a_{n+1} \preceq \cdots. \end{aligned} \tag{2}$$

If there exists n_0 such that $a_{n_0} = a_{n_0+1}$, then $D(a_{n_0+1}, Ta_{n_0}) = D(a_{n_0}, Ta_{n_0}) = D(A,B)$. This means that a_{n_0} is a best proximity point of T and hence the proof. Thus, we can suppose that $a_n \neq a_{n+1}$ for all n. Since $d(a_{n+1}, b_n) = D(A,B)$ and $d(a_n, b_{n-1}) = D(A,B)$ and (A,B) has the P-property,

$$d(a_n, a_{n+1}) = d(b_{n-1}, b_n) \quad \text{for all } n \in \mathbb{N}. \tag{3}$$

Since $a_{n-1} \prec a_n$,

$$d(a_n, a_{n+1}) = d(b_{n-1}, b_n) \le \delta(Ta_{n-1}, Ta_n) \le \psi\bigl(M(a_{n-1}, a_n)\bigr) + LN(a_{n-1}, a_n). \tag{4}$$

By the triangle inequality of d, we have

$$\begin{aligned} &M(a_{n-1}, a_n) \\ &\quad = \max\biggl\{ d(a_{n-1}, a_n), D(a_{n-1}, Ta_{n-1}) - D(A,B), D(a_n, Ta_n) - D(A,B), \\ &\qquad \frac{D(a_{n-1}, Ta_n) + D(a_n, Ta_{n-1})}{2} - D(A,B) \biggr\} \end{aligned}$$

$$
\begin{aligned}
&\leq \max\bigg\{d(a_{n-1},a_n), d(a_{n-1},b_{n-1}) - D(A,B), d(a_n,b_n) - D(A,B),\\
&\quad \frac{d(a_{n-1},b_n)+d(a_n,b_{n-1})}{2} - D(A,B)\bigg\}\\
&\leq \max\bigg\{d(a_{n-1},a_n), d(a_{n-1},b_{n-2}) + d(b_{n-2},b_{n-1}) - D(A,B), d(a_n,b_{n-1})\\
&\quad + d(b_{n-1},b_n) - D(A,B), \frac{d(a_{n-1},b_{n-2}) + d(b_{n-2},b_{n-1}) + d(b_{n-1},b_n) + d(a_n,b_{n-1})}{2}\\
&\quad - D(A,B)\bigg\}\\
&\leq \max\bigg\{d(a_{n-1},a_n), D(A,B) + d(a_{n-1},a_n) - D(A,B), D(A,B) + d(a_n,a_{n+1}) - D(A,B),\\
&\quad \frac{D(A,B) + d(a_{n-1},a_n) + d(a_n,a_{n+1}) + D(A,B)}{2} - D(A,B)\bigg\}\\
&= \max\big\{d(a_{n-1},a_n), d(a_n,a_{n+1})\big\}. \qquad (5)
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
N(a_{n-1},a_n) = {}& \min\big\{D(a_{n-1},Ta_{n-1}) - D(A,B), D(a_n,Ta_n)\\
& - D(A,B), D(a_{n-1},Ta_n) - D(A,B),\\
& D(a_n,Ta_{n-1}) - D(A,B)\big\}\\
\leq {}& \min\big\{d(a_{n-1},b_{n-1}) - D(A,B), d(a_n,b_n) - D(A,B),\\
& d(a_{n-1},b_n) - D(A,B), d(a_n,b_{n-1}) - D(A,B)\big\}.
\end{aligned}
$$

Since $d(a_n,b_{n-1}) = D(A,B)$, hence $N(a_{n-1},a_n) = 0$ for all $n \in \mathbb{N}$.

Using (5) and the inequality in (4), we get

$$
d(a_n,a_{n+1}) \leq \psi\big(\max\big\{d(a_{n-1},a_n), d(a_n,a_{n+1})\big\}\big). \qquad (6)
$$

If $d(a_n,a_{n+1}) > d(a_{n-1},a_n)$. From (6) we obtain

$$
d(a_n,a_{n+1}) \leq \psi\big(d(a_n,a_{n+1})\big) < d(a_n,a_{n+1}),
$$

which is a contradiction. So, we have

$$
d(a_n,a_{n+1}) \leq d(a_{n-1},a_n). \qquad (7)
$$

Hence, the sequence $\{d(a_n,a_{n+1})\}$ is monotone nonincreasing and bounded below. Thus, there exists $k \geq 0$ such that

$$
\lim_{n\to\infty} d(a_n,a_{n+1}) = k \geq 0. \qquad (8)
$$

Suppose that $\lim_{n\to\infty} d(a_n,a_{n+1}) = k > 0$. Using (7), inequality (6) becomes

$$
d(a_n,a_{n+1}) \leq \psi\big(d(a_{n-1},a_n)\big).
$$

Taking $n \to \infty$ in the above inequality and using the properties of ψ, we have

$$k \leq \limsup_{n\to\infty} \psi\bigl(d(a_{n-1}, a_n)\bigr) \leq \psi(k),$$

which is a contradiction unless $k = 0$. Hence,

$$\lim_{n\to\infty} d(a_n, a_{n+1}) = 0. \tag{9}$$

Now, we claim that the sequence $\{a_n\}$ is a Cauchy sequence. Suppose that $\{a_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ with subsequences $\{a_{m(r)}\}$ and $\{a_{n(r)}\}$ of $\{a_n\}$ such that $n(r)$ is the smallest index for which $n(r) > m(r) > r$, $d(a_{m(r)}, a_{n(r)}) \geq \epsilon$. This means that

$$d(a_{m(r)}, a_{n(r)-1}) < \epsilon. \tag{10}$$

Now, we have

$$\begin{aligned} \epsilon &\leq d(a_{m(r)}, a_{n(r)}) \\ &\leq d(a_{m(r)}, a_{n(r)-1}) + d(a_{n(r)-1}, a_{n(r)}) \\ &< \epsilon + d(a_{n(r)-1}, a_{n(r)}). \end{aligned}$$

Letting $r \to \infty$ and using (9), we can conclude that

$$\lim_{r\to\infty} d(a_{m(r)}, a_{n(r)}) = \epsilon. \tag{11}$$

Again,

$$d(a_{m(r)}, a_{n(r)-1}) \leq d(a_{m(r)}, a_{n(r)}) + d(a_{n(r)}, a_{n(r)-1})$$

and

$$d(a_{m(r)}, a_{n(r)}) \leq d(a_{m(r)}, a_{n(r)-1}) + d(a_{n(r)}, a_{n(r)-1}).$$

Therefore,

$$\bigl|d(a_{m(r)}, a_{n(r)-1}) - d(a_{m(r)}, a_{n(r)})\bigr| \leq d(a_{n(r)}, a_{n(r)-1}).$$

Taking $r \to \infty$ and using (11) and (9), we get

$$\lim_{r\to\infty} d(a_{m(r)}, a_{n(r)-1}) = \epsilon. \tag{12}$$

Similarly, we can prove that

$$\begin{aligned} \lim_{r\to\infty} d(a_{m(r)-1}, a_{n(r)}) &= \lim_{r\to\infty} d(a_{m(r)-1}, a_{n(r)-1}) = \lim_{r\to\infty} d(a_{m(r)+1}, a_{n(r)}) \\ &= \lim_{r\to\infty} d(a_{m(r)}, a_{n(r)+1}) = \epsilon. \end{aligned} \tag{13}$$

Since $m(r) < n(r)$, $a_{m(r)-1} \preceq a_{n(r)-1}$, from (3) and (1), we have

$$d(a_{m(r)}, a_{n(r)}) \leq \delta(Ta_{m(r)-1}, Ta_{n(r)-1}) \leq \psi\bigl(M(a_{m(r)-1}, a_{n(r)-1})\bigr) + LN(a_{m(r)-1}, a_{n(r)-1}), \quad (14)$$

where

$$\begin{aligned}
&M(a_{m(r)-1}, a_{n(r)-1}) \\
&\quad = \max\Biggl\{d(a_{m(r)-1}, a_{n(r)-1}), D(a_{m(r)-1}, Ta_{m(r)-1}) - D(A,B), \\
&\qquad D(a_{n(r)-1}, Ta_{n(r)-1}) - D(A,B), \frac{D(a_{m(r)-1}, Ta_{n(r)-1}) + D(a_{n(r)-1}, Ta_{m(r)-1})}{2} \\
&\qquad - D(A,B)\Biggr\} \\
&\quad \leq \max\Biggl\{d(a_{m(r)-1}, a_{n(r)-1}), d(a_{m(r)-1}, b_{m(r)-1}) - D(A,B), d(a_{n(r)-1}, b_{n(r)-1}) \\
&\qquad - D(A,B), \frac{d(a_{m(r)-1}, b_{n(r)-1}) + d(a_{n(r)-1}, b_{m(r)-1}) - 2D(A,B)}{2}\Biggr\} \\
&\quad \leq \max\Biggl\{d(a_{m(r)-1}, a_{n(r)-1}), d(a_{m(r)-1}, a_{m(r)}) + d(a_{m(r)}, b_{m(r)-1}) - D(A,B), \\
&\qquad d(a_{n(r)-1}, a_{n(r)}) + d(a_{n(r)}, b_{n(r)-1}) - D(A,B), \biggl(\frac{1}{2}d(a_{m(r)-1}, a_{n(r)}) \\
&\qquad + d(a_{n(r)}, b_{n(r)-1}) + d(a_{n(r)-1}, a_{m(r)}) + d(a_{m(r)}, b_{m(r)-1}) - 2D(A,B)\biggr)\Biggr\}.
\end{aligned}$$

Using $d(a_{n+1}, b_n) = D(A,B)$ in the above inequality, we get

$$\begin{aligned}
M(a_{m(r)-1}, a_{n(r)-1}) \leq \max\Biggl\{&d(a_{m(r)-1}, a_{n(r)-1}), d(a_{m(r)-1}, a_{m(r)}), d(a_{n(r)-1}, a_{n(r)}), \\
&\frac{d(a_{m(r)-1}, a_{n(r)}) + d(a_{n(r)-1}, a_{m(r)})}{2}\Biggr\}
\end{aligned} \quad (15)$$

and

$$\begin{aligned}
&N(a_{m(r)-1}, a_{n(r)-1}) \\
&\quad = \min\bigl\{D(a_{m(r)-1}, Ta_{m(r)-1}) - D(A,B), D(a_{n(r)-1}, Ta_{n(r)-1}) - D(A,B), \\
&\qquad D(a_{m(r)-1}, Ta_{n(r)-1}) - D(A,B), D(a_{n(r)-1}, Ta_{m(r)-1}) - D(A,B)\bigr\} \\
&\quad \leq \min\bigl\{d(a_{m(r)-1}, b_{m(r)-1}) - D(A,B), d(a_{n(r)-1}, b_{n(r)-1}) - D(A,B), \\
&\qquad d(a_{m(r)-1}, b_{n(r)-1}) - D(A,B), d(a_{n(r)-1}, b_{m(r)-1}) - D(A,B)\bigr\} \\
&\quad \leq \min\bigl\{d(a_{m(r)-1}, a_{m(r)}) + d(a_{m(r)}, b_{m(r)-1}) - D(A,B), d(a_{n(r)-1}, a_{n(r)}) \\
&\qquad + d(a_{n(r)}, b_{n(r)-1}) - D(A,B), d(a_{m(r)-1}, a_{n(r)}) + d(a_{n(r)}, b_{n(r)-1}) - D(A,B), \\
&\qquad d(a_{n(r)-1}, a_{m(r)}) + d(a_{m(r)}, b_{m(r)-1}) - D(A,B)\bigr\}.
\end{aligned}$$

Using $d(a_{n+1}, b_n) = D(A,B)$ in the above inequality, we get

$$\begin{aligned} N(a_{m(r)-1}, a_{n(r)-1}) \leq \min\big\{&d(a_{m(r)-1}, a_{m(r)}), d(a_{n(r)-1}, a_{n(r)}), \\ &d(a_{m(r)-1}, a_{n(r)}), d(a_{n(r)-1}, a_{m(r)})\big\}. \end{aligned} \tag{16}$$

Using (15) and (16) in (14) and taking $r \to \infty$, from (9), (11), (12) and (13), we get

$$\epsilon \leq \psi\big(\max\{\epsilon, 0, 0, \epsilon\}\big) + L\min\{0, 0, \epsilon, \epsilon\} \tag{17}$$

$$= \psi(\epsilon) < \epsilon, \tag{18}$$

which is a contradiction to the property of ψ. Thus, $\{a_n\}$ is a Cauchy sequence in A and hence it converges to some element a in A. Since $d(a_n, a_{n+1}) = d(b_{n-1}, b_n)$, the sequence $\{b_n\}$ in B is Cauchy and hence it is convergent. Suppose that $b_n \to b$. By the relation $d(a_{n+1}, b_n) = D(A,B)$, for all n, we conclude that $d(a,b) = D(A,B)$. We now claim that $b \in Ta$.

Since $\{a_n\}$ is an increasing sequence in A and $a_n \to a$, by hypothesis (iv), $a_n \preceq a$, $\forall n$.

$$\begin{aligned} &D(b_n, Ta) \\ &\quad \leq \delta(Ta_n, Ta) \\ &\quad \leq \psi\left(\max\left\{d(a_n, a), D(a_n, Ta_n) - D(A,B), D(a, Ta) - D(A,B),\right.\right. \\ &\qquad \left.\left.\frac{D(a_n, Ta) + D(a, Ta_n)}{2} - D(A,B)\right\}\right) + L\min\big\{D(a_n, Ta_n) - D(A,B), \\ &\qquad D(a, Ta) - D(A,B), D(a_n, Ta) - D(A,B), D(a, Ta_n) - D(A,B)\big\} \\ &\quad \leq \psi(\max\left\{d(a_n, a), d(a_n, b_n) - D(A,B), D(a, Ta) - D(A,B), \frac{D(a_n, Ta) + d(a, b_n)}{2}\right. \\ &\qquad \left. - D(A,B)\right\} + L\min\big\{d(a_n, b_n) - D(A,B), D(a, Ta) - D(A,B), D(a_n, Ta) - D(A,B), \\ &\qquad d(a, b_n) - D(A,B)\big\}. \end{aligned}$$

As $n \to \infty$ in the above inequality, using $a_n \to a$, $b_n \to b$, $d(a,b) = D(A,B)$ and since ψ is upper-semicontinuous, we get

$$\begin{aligned} D(b, Ta) &\leq \psi\left(\max\left\{0, 0, D(a, Ta) - D(A,B), \frac{D(a, Ta) + D(A,B)}{2} - D(A,B)\right\}\right) \\ &\qquad + L\min\big\{0, D(a, Ta) - D(A,B), D(a, Ta) - D(A,B), 0\big\} \\ &= \psi\big(D(a, Ta) - D(A,B)\big) \\ &\leq \psi\big(d(a,b) + D(b, Ta) - D(A,B)\big) \\ &= \psi\big(D(b, Ta)\big) < D(b, Ta), \end{aligned}$$

which is a contradiction unless $D(b, Ta) = 0$.

This implies that $b \in Ta$, and hence $D(a, Ta) = D(A,B)$. That is, a is a best proximity point of the mapping T. □

Example 3.2 Let $X := \mathbb{R}^2$ with the order $\preceq$ defined as follows: for $(a_1, b_1), (a_2, b_2) \in X$, $(a_1, b_1) \preceq (a_2, b_2)$ if and only if $a_1 \leq a_2, b_1 \leq b_2$, where $\leq$ is the usual order in $\mathbb{R}$. Then $(X, \preceq, d)$ becomes a complete partially ordered metric space with respect to a metric $d((a_1, b_1), (a_2, b_2)) = |a_1 - a_2| + |b_1 - b_2|$ for each $(a_1, b_1), (a_2, b_2) \in X$.

Let $A = \{(0,0), (0,3), (0,6), (0,9)\}$ and $B = \{(1,-1), (1,2), (1,5), (1,8)\}$. Then (A, B) satisfies the P-property, and $D(A, B) = 2$. Let $T : A \to CB(B)$ be defined as follows:

$$Ta = \begin{cases} \{(1,8), (1,5)\} & \text{if } a = (0,0), \\ \{(1,8)\} & \text{otherwise,} \end{cases}$$

and define $\psi : [0, \infty) \to [0, \infty)$ as $\psi(t) = \frac{3t}{4}$.

It is easy to see that for all comparable $a, b \in X$ and $L \geq 0$, T satisfies the following:

$$\begin{aligned} \delta(Ta, Tb) \leq \psi \biggl(\max \biggl\{ & d(a, b), D(a, Ta) - D(A, B), D(b, Tb) - D(A, B), \\ & \frac{D(a, Tb) + D(b, Ta)}{2} - D(A, B) \biggr\} \biggr) + L \min \bigl\{ D(a, Ta) - D(A, B), \\ & D(b, Tb) - D(A, B), D(a, Tb) - D(A, B), D(b, Ta) - D(A, B) \bigr\}. \end{aligned}$$

Also, it is easy to verify that this T satisfies all the conditions in Theorem 3.1. It is clear that $(0, 9)$ is a best proximity point of T.

The following corollary is a particular case of Theorem 3.1 when $A = B$. Also, it is a partial generalization of Theorem 2.1 in [19].

Corollary 3.1 *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let A be a nonempty closed subset of X and $T : A \to CB(A)$ be a multivalued mapping such that the following conditions are satisfied*:

(i) *There exist elements a_0, a_1 in A and $b_0 \in Ta_0$ such that $d(a_1, b_0) = 0$ and $a_0 \preceq a_1 = b_0$.*

(ii) *T satisfies*

$$\begin{aligned} \delta(Ta, Tb) \leq \psi \biggl(\max \biggl\{ D(a, b), D(a, Ta), D(b, Tb), \frac{D(a, Tb) + D(b, Ta)}{2} \biggr\} \biggr) \\ + L \min \bigl\{ D(a, Ta), D(b, Tb), D(a, Tb), D(b, Ta) \bigr\} \end{aligned}$$

for all comparable $a, b \in A$, where $L \geq 0$ and ψ is an altering distance function.

(iii) *For $a, b \in A_0$, $a \preceq b$ implies $Ta \prec_{(1)} Tb$.*

(iv) *If $\{a_n\}$ is a nondecreasing sequence in A such that $a_n \to a$, then $a_n \preceq a$ for all n.*

Then there exists an element a in A such that $D(a, Ta) = 0$. That is, a is a fixed point of the mapping T.

The following corollary is a particular case of Theorem 3.1 when T is a single-valued self mapping.

Corollary 3.2 *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let A be a nonempty closed subset of X. Let $T : A \to A$ be a single-valued mapping such that the following conditions are satisfied*:

(i) *There exist elements a_0 and a_1 in A such that*

$$d(a_1, Ta_0) = 0 \quad \text{and} \quad a_0 \preceq a_1.$$

(ii) *T satisfies*

$$\delta(Ta, Tb) \leq \psi\left(\max\left\{d(a,b), d(a,Ta), d(b,Tb), \frac{d(a,Tb) + d(b,Ta)}{2}\right\}\right) + L\min\left\{d(a,Ta), d(b,Tb), d(a,Tb), d(b,Ta)\right\}$$

for all comparable $a, b \in A$, where $L \geq 0$, and ψ is an altering distance function.

(iii) *For $a, b \in A$, $a \preceq b$ implies $\{Ta\} \prec_{(1)} \{Tb\}$.*

(iv) *If $\{a_n\}$ is a nondecreasing sequence in A such that $a_n \to a$, then $a_n \preceq a$ for all n.*

Then there exists an element a in A such that $d(a, Ta) = 0$. That is, a is a fixed point of the mapping T.

Now, we state our second main result in this section.

Theorem 3.3 *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let A and B be nonempty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$ and (A, B) satisfies the P-property. Let $S, T : A \to CB(B)$ be a multivalued mapping such that the following conditions are satisfied*:

(i) *$Sa_0, Ta_0 \subseteq B_0$ for all $a_0 \in A_0$.*

(ii) *There exists $a_0 \in A_0$ with $d(a_0, b_0) = D(A, B)$ for some $b_0 \in B_0$ with $\{b_0\} \prec_{(1)} Ta_0$.*

(iii) *For any $a, b \in A_0$ with $a \preceq b$ implies $Sb \prec_{(3)} Ta$.*

(iv) *If $\{a_n\}$ is any sequence in A whose consecutive terms are comparable with $a_n \to a$, then $a_n \preceq a$ for all n.*

(v) *T and S satisfy*

$$\delta(Ta, Sb) \leq \alpha M(a, b) + LN(a, b) \quad \text{for all comparable } a, b \in A, \tag{19}$$

where $M(a, b) = \max\{d(a, b), D(a, Ta) - D(A, B), D(b, Sb) - D(A, B), \frac{D(a,Sb)+D(b,Ta)}{2} - D(A, B)\}$, $L \geq 0$ and $N(a, b) = \min\{D(a, Ta) - D(A, B), D(b, Sb) - D(A, B), D(a, Sb) - D(A, B), D(b, Ta) - D(A, B)\}$, $0 < \alpha < 1$.

Then T and S have a common best proximity point.

Proof By assumption (ii), there exists $a_0 \in A_0$ with $d(a_0, b_0) = D(A, B)$ such that $\{b_0\} \prec_{(1)} Ta_0$. For this $b_0 \in B_0$, there exists $b_1 \in Ta_0$ with $d(a_1, b_1) = D(A, B)$ such that $a_0 \preceq a_1$. By assumption (iii), $Sa_1 \prec_{(3)} Ta_0$, which implies $Sa_1 \prec_{(2)} Ta_0$. So, for this $b_1 \in Ta_0$, there exists $b_2 \in Sa_1$ with $d(a_2, b_2) = D(A, B)$ such that $a_2 \preceq a_1$. Again, by assumption (iii), $Sa_1 \prec_{(3)} Ta_2$, which implies $Sa_1 \prec_{(1)} Ta_2$. Therefore, there exists $b_3 \in Ta_2$ with $d(a_3, b_3) = D(A, B)$ such that $a_2 \preceq a_3$. Continuing in this way, we can construct a sequence $\{a_n\}$ such that

(1) for each n, $a_n \in A_0$ and $b_{2n+1} \in Ta_{2n}$ and $b_{2n+2} \in Sa_{2n+1}$ with $d(a_n, b_n) = D(A, B)$;

(2) for each n, $a_{2n} \preceq a_{2n+1}$ and $a_{2n+2} \preceq a_{2n+1}$.

First we claim that any best proximity point of T is a best proximity point of S and conversely. Now suppose that p is a best proximity point of T but not a best proximity point of S. Consider

$$D(p,Sp) \leq D(p,Tp) + \delta(Tp,Sp) = D(A,B) + \delta(Tp,Sp).$$

Hence, $D(p,Sp) - D(A,B) \leq \delta(Tp,Sp)$. Then, by condition (v),

$$\begin{aligned}
D(p,Sp) - D(A,B) &\leq \delta(Tp,Sp) \\
&\leq \alpha \max\biggl\{d(p,p), D(p,Tp) - D(A,B), D(p,Sp) - D(A,B), \\
&\quad \frac{D(p,Sp) + D(p,Tp)}{2} - D(A,B)\biggr\} + L\min\bigl\{D(p,Tp) - D(A,B), \\
&\quad D(p,Sp) - D(A,B), D(p,Sp) - D(A,B), D(p,Tp) - D(A,B)\bigr\} \\
&= \alpha \max\biggl\{0, 0, D(p,Sp) - D(A,B), \frac{D(p,Sp) - D(A,B)}{2}\biggr\} \\
&\quad + L\min\bigl\{0, D(p,Sp) - D(A,B)\bigr\} \\
&= \alpha\bigl(D(p,Sp) - D(A,B)\bigr),
\end{aligned}$$

which is a contradiction, unless $D(p,Sp) = D(A,B)$. Hence p is a best proximity point to S. Using a similar argument, we can prove that any best proximity of S is a best proximity point of T.

If there exists a positive integer $2N$ such that $a_{2N} = a_{2N+1}$, then a_{2N} becomes a common best proximity point. Similarly, the same conclusion holds if $a_{2N+1} = a_{2N+2}$ for some N. Therefore, we may assume that $a_n \neq a_{n+1}$ for all $n \geq 0$. We know that $d(a_n, b_n) = d(a_{n+1}, b_{n+1}) = D(A,B)$ and so by the P-property, we have $d(a_n, a_{n+1}) = d(b_n, b_{n+1})$.

Now,

$$d(a_{2n+1}, a_{2n+2}) = d(b_{2n+1}, b_{2n+2}) \leq \delta(Ta_{2n}, Sa_{2n+1}). \tag{20}$$

Consider

$$\begin{aligned}
M(a_{2n}, a_{2n+1}) = &\max\biggl\{d(a_{2n}, a_{2n+1}), D(a_{2n}, Ta_{2n}) - D(A,B), D(a_{2n+1}, Sa_{2n+1}) - D(A,B), \\
&\frac{D(a_{2n}, Sa_{2n+1}) + D(a_{2n+1}, Ta_{2n})}{2} - D(A,B)\biggr\} \\
\leq &\max\biggl\{d(a_{2n}, a_{2n+1}), d(a_{2n}, b_{2n+1}) - D(A,B), d(a_{2n+1}, b_{2n+2}) - D(A,B), \\
&\frac{d(a_{2n}, b_{2n+2}) + d(a_{2n+1}, b_{2n+1})}{2} - D(A,B)\biggr\} \\
\leq &\max\biggl\{d(a_{2n}, a_{2n+1}), d(a_{2n}, b_{2n}) + d(b_{2n}, b_{2n+1}) - D(A,B), \\
&d(a_{2n+1}, b_{2n+1}) + d(b_{2n+1}, b_{2n+2}) - D(A,B),
\end{aligned}$$

$$\begin{aligned}&\frac{d(a_{2n},b_{2n})+d(b_{2n},b_{2n+1})+d(b_{2n+1},b_{2n+2})+D(A,B)}{2}-D(A,B)\Big\}\\&=\max\Big\{d(a_{2n},a_{2n+1}),d(a_{2n+1},a_{2n+2}),\frac{d(a_{2n},a_{2n+1})+d(a_{2n+1},a_{2n+2})}{2}\Big\}\\&=\max\big\{d(a_{2n},a_{2n+1}),d(a_{2n+1},a_{2n+2})\big\}\end{aligned}$$

and

$$\begin{aligned}N(a_{2n},a_{2n+1})&=\min\big\{D(a_{2n},Ta_{2n})-D(A,B),D(a_{2n+1},Sa_{2n+1})-D(A,B),\\&\quad D(a_{2n},Sa_{2n+1})-D(A,B),D(a_{2n+1},Ta_{2n})-D(A,B)\big\}\\&\le\min\big\{d(a_{2n},b_{2n+1})-D(A,B),d(a_{2n+1},b_{2n+2})-D(A,B),\\&\quad d(a_{2n},b_{2n+2})-D(A,B),d(a_{2n+1},b_{2n+1})-D(A,B)\big\}\\&=\min\big\{d(a_{2n},b_{2n+1})-D(A,B),d(a_{2n+1},b_{2n+2})\\&\quad -D(A,B),d(a_{2n},b_{2n+2})-D(A,B),0\big\}=0.\end{aligned}$$

Therefore, by inequality (20) and by inequality (19), we get

$$d(a_{n+1},a_{2n+2})\le\alpha\max\big\{d(a_{2n},a_{2n+1}),d(a_{2n+1},a_{2n+2})\big\}. \tag{21}$$

Suppose that $d(a_{2n},a_{2n+1})\le d(a_{2n+1},a_{2n+2})$ for some positive integer n. Then from (21) we have

$$d(a_{2n+1},a_{2n+2})\le\alpha d(a_{2n+1},a_{2n+2})<d(a_{2n+1},a_{2n+2}),$$

which is a contradiction. Hence, $d(a_{2n+1},a_{2n+2})<d(a_{2n},a_{2n+1})$ for all $n\ge 0$. In a similar way, we can prove that $d(a_{2n+2},a_{2n+3})<d(a_{2n+1},a_{2n+2})$ for all $n>0$. Then $\{d(a_n,a_{n+1})\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists $k\ge 0$ such that $\lim_{n\to\infty}d(a_n,a_{n+1})=k$. We will now claim that $k=0$.

From the above discussion, we have

$$d(a_{n+1},a_{n+2})\le\alpha d(a_n,a_{n+1})\quad\text{for all } n\ge 0.$$

Taking $n\to\infty$ in the above inequality, we get

$$k\le\alpha k,$$

which is a contradiction unless $k=0$. Therefore,

$$\lim_{n\to\infty}d(a_n,a_{n+1})=0.$$

Now we will prove that $\{a_n\}$ is a Cauchy sequence. Let $m>n$. Then

$$\begin{aligned}d(a_m,a_n)&\le d(a_m,a_{m-1})+d(a_{m-1},a_{m-2})+\cdots+d(a_{n+1},a_n)\\&\le\big[\alpha^{m-1}+\alpha^{m-2}+\cdots+\alpha^n\big]d(a_0,a_1)\le\frac{\alpha^n}{1-\alpha}d(a_0,a_1)\to 0,\quad\text{as } n\to\infty,\end{aligned}$$

which implies that $\{a_n\}$ is a Cauchy sequence in A. From the completeness of X, there exists $a \in X$ such that $a_n \to a$ as $n \to \infty$. Since A is closed, $a \in A$. Also, by assumption (iv), $a_n \preceq a$ for all n. Since $d(a_n, a_{n+1}) = d(b_n, b_{n+1})$, the sequence $\{b_n\}$ becomes Cauchy. Hence it converges to some $b \in X$. By the relation $d(a_n, b_n) = D(A,B)$ for all n, we conclude that $d(a,b) = D(A,B)$. We now claim that $b \in Ta$.

$$
\begin{aligned}
&D(Ta, b_{2n+2}) \\
&\quad \le \delta(Ta, Sa_{2n+1}) \\
&\quad \le \alpha \max\Big\{ d(a, a_{2n+1}), D(a, Ta) - D(A,B), D(a_{2n+1}, Sa_{2n+1}) - D(A,B), \\
&\qquad \frac{D(a, Sa_{2n+1}) + D(a_{2n+1}, Ta)}{2} - D(A,B) \Big\} + L \min\big\{ D(a, Ta) - D(A,B), \\
&\qquad D(a_{2n+1}, Sa_{2n+1}) - D(A,B), D(a_{2n+1}, Ta) - D(A,B), D(a, Sa_{2n+1}) - D(A,B) \big\} \\
&\quad \le \alpha \max\Big\{ d(a, a_{2n+1}), D(a, Ta) - D(A,B), d(a_{2n+1}, b_{2n+2}) - D(A,B), \\
&\qquad \frac{d(a, b_{2n+2}) + D(a_{2n+1}, Ta)}{2} - D(A,B) \Big\} + L \min\big\{ D(a, Ta) - D(A,B), \\
&\qquad d(a_{2n+1}, b_{2n+2}) - D(A,B), D(a_{2n+1}, Ta) - D(A,B), d(a, b_{2n+2}) - D(A,B) \big\} \\
&\quad \le \alpha \max\Big\{ d(a, a_{2n+1}), D(a, Ta) - D(A,B), d(b_{2n+1}, b_{2n+2}), \\
&\qquad \frac{d(a, b_{2n+2}) + D(a_{2n+1}, Ta)}{2} - D(A,B) \Big\} + L \min\big\{ D(a, Ta) - D(A,B), \\
&\qquad d(b_{2n+1}, b_{2n+2}), D(a_{2n+1}, Ta) - D(A,B), d(a, b_{2n+2}) - D(A,B) \big\}.
\end{aligned}
$$

As $n \to \infty$ in the above inequality, and using the properties of sequences $\{a_n\}$ and $\{b_n\}$, we get

$$
\begin{aligned}
D(Ta, b) &\le \alpha \max\Big\{ 0, D(a, Ta) - D(A,B), 0, \frac{d(a,b) + D(a, Ta)}{2} - D(A,B) \Big\} \\
&\quad + L \min\big\{ D(a, Ta) - D(A,B), 0, D(a, Ta) - D(A,B), d(a,b) - D(A,B) \big\} \\
&\le \alpha \max\Big\{ d(a,b) + D(b, Ta) - D(A,B), \frac{d(a,b) + d(a,b) + D(b, Ta)}{2} - D(A,B) \Big\} \\
&= \alpha D(b, Ta),
\end{aligned}
$$

which is true only if $D(b, Ta) = 0$. Hence $b \in Ta$, that is, a is the best proximity point of T. By what we have proved already, a is a common best proximity point of T and S. □

Example 3.4 Let $X := \mathbb{R}^2$ with the order $\preceq$ defined as follows: for $(a_1, b_1), (a_2, b_2) \in X$, $(a_1, b_1) \preceq (a_2, b_2)$ if and only if $a_1 \le a_2, b_1 \le b_2$, where $\le$ is the usual order in $\mathbb{R}$. Then $(X, \preceq, d)$ becomes a complete partially ordered metric space with respect to a metric $d((a_1, b_1), (a_2, b_2)) = |a_1 - a_2| + |b_1 - b_2|$ for each $(a_1, b_1), (a_2, b_2) \in X$.

Let $A = \{(0,0),(0,3),(0,6),(0,9)\}$ and $B = \{(1,-1),(1,2),(1,5),(1,8)\}$. Then (A,B) satisfies the P-property, and $D(A,B) = 2$. Let $T : A \to CB(B)$ and $S : A \to CB(B)$ be defined as follows:

$$Ta = \begin{cases} \{(1,8),(1,5)\} & \text{if } a = (0,0), \\ \{(1,8)\} & \text{otherwise,} \end{cases}$$

and $Sa = \{(1,8)\}$ for all $a \in A$ respectively. It is easy to see that for all comparable $a,b \in X$, $\alpha = \frac{3}{4}$ and $L \geq 0$, T and S satisfy the following:

$$\begin{aligned} \delta(Ta,Sb) \leq \alpha \max\biggl\{ & d(a,b), D(a,Ta) - D(A,B), D(b,Sb) - D(A,B), \\ & \frac{D(a,Sb) + D(b,Ta)}{2} - D(A,B)\biggr\} + L\min\bigl\{D(a,Ta) - D(A,B), \\ & D(b,Sb) - D(A,B), D(a,Sb) - D(A,B), D(b,Ta) - D(A,B)\bigr\}. \end{aligned}$$

Also, T and S satisfy all the other conditions in Theorem 3.3. It is easy to check that $(0,9)$ is a common best proximity point to S and T.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of the paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Amrita University, Coimbatore, India. [2]Department of Mathematics, Bharathidasan University, Tiruchirappalli, Tamil Nadu 620 024, India. [3]Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam. [4]Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam.

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