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Relaxed iterative algorithms for a system of generalized mixed equilibrium problems and a countable family of totally quasi-Phi-asymptotically nonexpansive multi-valued maps, with applications

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Abstract

In this article, a Krasnoselskii-type and a Halpern-type algorithm for approximating a common fixed point of a countable family of totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued maps and a solution of a system of generalized mixed equilibrium problem are constructed. Strong convergence of the sequences generated by these algorithms is proved in uniformly smooth and strictly convex real Banach spaces with the Kadec-Klee property. Several applications of our theorems are also presented. Finally, our theorems are a significant improvement of several important recent results.

MSC: 47H10; 47H04; 47J25; 47J20

Keywords: generalized mixed equilibrium problems; totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued maps; strong convergence; equally continuous maps

1 Introduction

In what follows, we assume that X is a real Banach space with dual space X^*, K is a nonempty, closed, and convex subset of X, and $\to$ and $\rightharpoonup$ will, respectively, denote strong and weak convergence.

(See, *e.g.*, Wang and Zang [1] for a similar definition for self maps.) Let $G : K \to 2^X$ be any map. A point $u \in K$ is called a *fixed point* of G if and only if $u \in Gu$ and it is called an *asymptotic fixed point* of G if there exists a sequence $\{u_n\}$ in K that converges weakly to u and $\lim_{n\to\infty} d(u_n, Gu_n) := \lim_{n\to\infty} \inf_{\eta_n \in Gu_n} \|u_n - \eta_n\| = 0$ (see Chang *et al.* [2]). We denote the set of fixed points and asymptotic fixed points of G by $F(G)$ and $\hat{F}(G)$, respectively.

A subset K of X is said to be a retract of X, if there exists a continuous map $P : X \to K$ such that $Pu = u$, for all $u \in X$. It is well known that every nonempty, closed, convex subset of a uniformly convex Banach space X is a retract of X. A map $P : X \to K$ is said to be a retraction if $P^2 = P$. A map $P : X \to K$ is said to be a nonexpansive retraction, if it is nonexpansive and it is a retraction from X to K.

Define the Lyapunov functional $\phi : X \times X \to \mathbb{R}$ by

$$\phi(u,v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2 \quad \forall u,v \in X. \tag{1.1}$$

From the definition of ϕ it is obvious that

$$\big(\|u\| - \|v\|\big)^2 \leq \phi(u,v) \leq \big(\|u\| + \|v\|\big)^2 \quad \forall u,v \in X. \tag{1.2}$$

In what follows, we assume that $P : X \to K$ is a nonexpansive retraction.

Definition 1.1 A nonself multi-valued map $G : K \to 2^X$ is said to be *relatively asymptotically nonexpansive* if $F(G) \neq \emptyset$, $\hat{F}(G) = F(G)$, and there exists a real sequence $\{\beta_n\} \subset [1,\infty)$, $\beta_n \downarrow 1$ such that $\phi(p,\eta_n) \leq \beta_n \phi(p,u)$ $\forall u \in K$, $p \in F(G)$, $\eta_n \in G(PG)^{n-1}u$, $n \geq 1$ (see, *e.g.*, Wang and Zang [1] for a similar definition for self maps).

The following definitions appear in Bo and Yi [3].

Definition 1.2 A nonself multi-valued map $G : K \to 2^X$ is said to be

- *quasi-ϕ-nonexpansive* if $F(G) \neq \emptyset$ and $\phi(p,\eta_n) \leq \phi(p,u)$ $\forall u \in K, p \in F(G)$, $\eta_n \in G(PG)^{n-1}u$, $n \geq 1$;
- *quasi-ϕ-asymptotically nonexpansive* if $F(G) \neq \emptyset$ and there exists a real sequence $\{\beta_n\} \subset [1,\infty)$, $\beta_n \downarrow 1$ such that $\phi(p,\eta_n) \leq \beta_n\phi(p,u)$ $\forall u \in K, p \in F(G)$, $\eta_n \in G(PG)^{n-1}u$, $n \geq 1$;
- *totally quasi-ϕ-asymptotically nonexpansive* if $F(G) \neq \emptyset$ and there exist nonnegative real sequences $\{\gamma_n\}$, $\{\delta_n\}$ with $\gamma_n \to 0$, $\delta_n \to 0$ $(n \to \infty)$ and a strictly increasing and continuous function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ with $\rho(0) = 0$ such that

$$\begin{aligned}&\phi(p,\eta_n) \leq \phi(p,u) + \gamma_n \rho\big[\phi(p,u)\big] + \delta_n \\ &\quad \forall u \in K, p \in F(G), \eta_n \in G(PG)^{n-1}u, n \geq 1.\end{aligned} \tag{1.3}$$

Remark 1 From the definitions, it is easy to see that the class of relatively asymptotically nonexpansive multi-valued nonself maps and the class of quasi-ϕ-nonexpansive multi-valued nonself maps are proper subclasses of the class of quasi-ϕ-asymptotically nonexpansive multi-valued nonself maps and that the class of quasi-ϕ-asymptotically nonexpansive multi-valued nonself maps is a proper subclass of the class of totally quasi-ϕ-asymptotically nonexpansive multi-valued nonself maps, but the converse may not be true.

Definition 1.3 A countable family of multi-valued nonself maps, $G_i : K \to 2^X$, $i = 1,2,3,\ldots$, is said to be

- *uniformly quasi-ϕ-asymptotically nonexpansive* if $\bigcap_{i=1}^{\infty} F(G_i) \neq \emptyset$ and there exists a sequence $\{\beta_n\} \subset [1,\infty)$, $\beta_n \downarrow 1$ such that, for each $i \geq 1$,

$$\phi(p,\eta_n) \leq \beta_n\phi(p,u) \quad \forall u \in K, p \in \bigcap_{i=1}^{\infty} F(G_i), \eta_n \in G_i(PG_i)^{n-1}u, n \geq 1$$

  (see, *e.g.*, Chang *et al.* [4]);
- *uniformly totally quasi-ϕ-asymptotically nonexpansive* if $\bigcap_{i=1}^{\infty} F(G_i) \neq \emptyset$ and there exist nonnegative real sequences $\{\gamma_n\}$, $\{\delta_n\}$ with $\gamma_n \to 0$, $\delta_n \to 0$ $(n \to \infty)$ and a

strictly increasing and continuous function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ with $\rho(0) = 0$ such that, for each $i \geq 1$,

$$\phi(p, \eta_n) \leq \phi(p, u) + \gamma_n \rho\big[\phi(p, u)\big] + \delta_n$$
$$\forall u \in K, p \in \bigcap_{i=1}^{\infty} F(G_i), \eta_n \in G_i(PG_i)^{n-1}u, n \geq 1$$

(see, *e.g.*, Yi [5]).

Remark 2 From the definitions, it is easy to see that a countable family of uniformly quasi-ϕ-asymptotically nonexpansive multi-valued nonself maps is a countable family of uniformly totally quasi-ϕ-asymptotically nonexpansive multi-valued nonself maps.

Remark 3 We also remark that a collection of countable families of uniformly totally quasi-ϕ-asymptotically nonexpansive multi-valued nonself maps is a subcollection of a collection of countable families of totally quasi-ϕ-asymptotically nonexpansive multi-valued nonself maps.

A motivation for the study of the class of totally quasi-ϕ-asymptotically nonexpansive self or nonself maps is the objective to unify various definitions of classes of maps, associated with the class of relatively nonexpansive self or nonself maps, which are extensions to arbitrary real Banach spaces of nonexpansive nonself maps, with nonempty fixed point sets in Hilbert spaces. Our objective is to prove general convergence theorems applicable to all these classes.

Definition 1.4 (See, *e.g.*, Feng *et al.* [6] for a similar definition for self maps) A multi-valued nonself map $G : K \to 2^X$ is said to be

- *equally continuous* if for $u_n, v_n \in K$ we have

$$\lim_{n\to\infty} \|u_n - v_n\| = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \|\eta_{n_u} - \eta_{n_v}\| = 0$$
$$\forall \eta_{n_u} \in G(PG)^{n-1}u_n, \eta_{n_v} \in G(PG)^{n-1}v_n;$$

- *uniformly continuous* if for $u_n, v_n \in K$ we have

$$\lim_{n\to\infty} \|u_n - v_n\| = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \|\eta_{n_u} - \eta_{n_v}\| = 0 \quad \forall \eta_{n_u} \in Gu_n, \eta_{n_v} \in Gv_n;$$

- *uniformly L-Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|\eta_u - \eta_v\| \leq L\|u - v\| \quad \forall \eta_u \in G(PG)^{n-1}u, \eta_v \in G(PG)^{n-1}v, n \geq 1.$$

Remark 4 It is easy to see that the class of uniformly L-Lipschitz multi-valued nonself maps is a proper subclass of the class of uniformly continuous multi-valued nonself maps and the class of uniformly continuous multi-valued nonself maps is a proper subclass of the class of equally continuous multi-valued nonself maps.

Let $\psi : K \to \mathbb{R}$ be a real-valued function, $A : K \to X^*$ be a nonlinear map, and $f : K \times K \to \mathbb{R}$ be a bifunction. The *generalized mixed equilibrium problem* is to find $u^* \in K$ such that

$$f(u^*,v) + \psi(v) - \psi(u^*) + \langle v - u^*, Au^*\rangle \geq 0, \quad \forall v \in K. \tag{1.4}$$

The set of solutions of the generalized mixed equilibrium problem is denoted by ***GMEP***(f, A, ψ).

The class of generalized mixed equilibrium problems includes, as special cases, the class of *mixed equilibrium problems* ($A \equiv 0$; see, *e.g.*, Ceng and Yao [7] and the references contained therein); the class of *generalized equilibrium problems* ($\zeta \equiv 0$; see, *e.g.*, Takahashi and Takahashi [8]); the class of *equilibrium problems* ($A \equiv 0$, $\zeta \equiv 0$; see, *e.g.*, Fan [9], Blum and Oettli [10], and the references contained therein); the class of *variational inequality problems* ($h \equiv 0$, $\zeta \equiv 0$; see, *e.g.*, Stampacchia [11]); and the class of *convex minimization problems* ($A \equiv 0$, $h \equiv 0$).

The generalized mixed equilibrium problem has applications in physics, economics, finance, transportation, network and structural analysis, ecology, image reconstruction, and elasticity. It includes, as special cases, fixed point problems, variational inequality problems, complementarity problems, equilibrium problems, optimization problems, Nash equilibrium problems in noncooperative games, etc. (see, *e.g.*, Blum and Otelli [10], Dafermos and Nagurney [12], Su [13], Barbagallo [14], Moudafi [15], and the references contained therein). In other words, the *GMEP*(f, A, ψ) is a unifying model for several problems arising in physics, engineering, science, optimization, finance, economics, etc. The projection method, which was first introduced by Haugazeau [16], has been utilized to solve the mixed equilibrium problem, the generalized equilibrium problem, and equilibrium problems in Banach spaces (see, *e.g.*, Qin *et al.* [17], Cholamjiak *et al.* [18], Cho *et al.* [19], Ceng and Yao [7], and the references therein). The advantage of projection methods is that strong convergence of iterative sequences can be guaranteed without any compactness assumptions imposed on maps or subsets of spaces.

Several strong and weak convergence theorems for asymptotically nonexpansive, relatively nonexpansive, quasi-ϕ-nonexpansive and quasi-ϕ-asymptotically nonexpansive self or nonself maps have been established by various authors in the setting of Banach spaces (see, *e.g.*, Thianwan [20], Nilsrakoo *et al.* [21], Wang [22], Ma and Wang [23], Chidume *et al.* [24, 25], and the references contained therein).

In 2012, Chang *et al.* [4] considered the class of uniformly quasi-ϕ-asymptotically nonexpansive *nonself maps* and studied, in a uniformly convex and uniformly smooth real Banach space, the following Halpern-type algorithm:

$$\begin{cases} u_1 \in X, \quad \text{chosen arbitrarily}, \qquad K_1 = K, \\ y_{n,i} = J^{-1}(\alpha_n Ju_1 + (1-\alpha_n)(\beta_n Ju_n + (1-\beta_n)JG_i(PG_i)^{n-1}u_n)), \quad i \geq 1, \\ K_{n+1} = \{z \in K_n : \sup_{i\geq 1} \phi(z, y_{n,i}) \leq \alpha_n \phi(z, u_1) + (1-\alpha_n)\phi(z, u_n) + \theta_n\}, \\ u_{n+1} = \Pi_{K_{n+1}} u_1, \quad n \geq 1, \end{cases} \tag{1.5}$$

where $\theta_n = (\beta_n - 1)\sup_{u\in F} \phi(u, u_n)$, $F := \bigcap_{i=1}^{\infty} F(G_i)$, and $\{G_i\}_{i=1}^{\infty}$ is a countable family of uniformly L-Lipschitz continuous and uniformly quasi-ϕ-asymptotically nonexpansive nonself maps. The authors prove that the sequence $\{u_n\}$, generated by the above iterative scheme, converges strongly to $\Pi_F u_1$, under the following conditions:

$(C1)$ $\lim_{n\to\infty}\alpha_n = 0$;
$(C2)$ $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;
$(C3)$ F is a bounded and convex subset of K,

where $\alpha_n \in [0,1]$ and $\beta_n \in (0,1)$.

In the same year, Zhao and Chang [26] proved that the sequence $\{u_n\}$, generated by algorithm (1.5), converges strongly to $\Pi_F u_1$ under conditions $(C1)$, $(C2)$, and the following condition:

$(C3^*)$ F is a nonempty bounded subset of K,

where $\{G_i\}_{i=1}^{\infty}$ is a countable family of uniformly L-Lipschitz continuous, *closed and uniformly totally* quasi-ϕ-asymptotically nonexpansive *nonself maps*.

Later in the same year, Yi [5] established the results in the paper of Zhao and Chang [26], when $\{G_i\}_{i=1}^{\infty}$ is a countable family of uniformly L-Lipschitz continuous and *uniformly totally* quasi-ϕ-asymptotically nonexpansive *nonself maps*, under conditions $(C1)$, $(C2)$, and the following condition:

$(C3^{**})$ F is a nonempty subset of K.

In 2014, Bo and Yi [3] proved that the sequence $\{u_n\}$, generated by the iterative algorithm

$$\begin{cases} u_1 \in X, \quad \text{chosen arbitrarily}, \qquad K_1 = K, \\ y_n = J^{-1}(\alpha_n J u_1 + (1-\alpha_n)(\beta_n J u_n + (1-\beta_n) J\eta_n)), \quad \eta_n \in G(PG)^{n-1}u_n, \\ K_{n+1} = \{z \in K_n : \phi(z, y_n) \le \alpha_n \phi(z, u_1) + (1-\alpha_n)\phi(z, u_n) + \theta_n\}, \\ u_{n+1} = \Pi_{K_{n+1}} u_1, \quad n \ge 1, \end{cases}$$

converges strongly to $\Pi_F u_1$ under conditions $(C1)$, $(C2)$, and $(C3^{**})$, where $F := F(G)$ and G is a uniformly L-Lipschitz continuous and totally quasi-ϕ-asymptotically nonexpansive *nonself multi-valued* map, while $\theta_n = \gamma_n \sup_{u\in F}\phi(u, u_n) + \delta_n$.

The results of Bo and Yi [3], Yi [5], Zhao and Chang [26], and Chang *et al.* [4] are important generalizations and improvements of important known results.

Motivated by these authors, it is our purpose in this paper to study the following Krasnoselskii-type and Halpern-type algorithms:

$$\begin{cases} u_0 \in X, \quad \text{chosen arbitrarily}, \qquad K_1 = K, \qquad u_1 = \Pi_{K_1} u_0, \\ y_n = J^{-1}(\sigma J u_n + (1-\sigma) J\eta_{m_n}^{(i_n)}), \quad (\eta_{m_n}^{(i_n)} \in G_{i_n}(PG_{i_n})^{m_n-1}u_n), \\ z_n = \Lambda_{r_{i_n}} y_n, \\ K_{n+1} = \{v \in K_n : \phi(v, z_n) \le \phi(v, u_n) + \omega_n\}, \\ u_{n+1} = \Pi_{K_{n+1}} u_0, \quad n \ge 1, \end{cases} \tag{1.6}$$

$$\begin{cases} u_0 \in X, \quad \text{chosen arbitrarily}, \qquad K_1 = K, \qquad u_1 = \Pi_{K_1} u_0, \\ y_n = J^{-1}(\sigma_n J u_0 + (1-\sigma_n) J\eta_{m_n}^{(i_n)}), \quad (\eta_{m_n}^{(i_n)} \in G_{i_n}(PG_{i_n})^{m_n-1}u_n), \\ z_n = \Lambda_{r_{i_n}} y_n, \\ K_{n+1} = \{v \in K_n : \phi(v, z_n) \le \sigma_n \phi(v, u_0) + (1-\sigma_n)\phi(v, u_n) + \omega_n\}, \\ u_{n+1} = \Pi_{K_{n+1}} u_0, \quad n \ge 1. \end{cases} \tag{1.7}$$

We also aim to prove, in a uniformly smooth and strictly convex real Banach space X with Kadec-Klee property, that the sequences generated by these algorithms converge strongly to an element in $W := (\bigcap_{i=1}^{\infty} F(G_i)) \cap (\bigcap_{i=1}^{\infty} GMEP(h_i, A_i, \zeta_i))$, where $\{G_i\}_{i=1}^{\infty}$ is a countable family of *equally continuous and totally* quasi-ϕ-asymptotically nonexpansive nonself *multi-valued* maps; $\{A_i\}_{i=1}^{\infty}$, $A_i : K \to X^*$ is a sequence of continuous and monotone maps; $\{h_i\}_{i=1}^{\infty}$, $h_i : K \times K \to \mathbb{R}$ is a sequence of bifunctions satisfying appropriate conditions and $\{\zeta_i\}_{i=1}^{\infty}$, $\zeta_i : K \to \mathbb{R}$ is a sequence of lower-semicontinuous and convex functions. Our theorems are significant improvements and generalizations of numerous results for this class of nonlinear problems (in particular, the results of Bo and Yi [3], Yi [5], Zhao and Chang [26], Chang *et al.* [4], Lv [27], Wang and Zhang [28], Dadashi and Postolache [29], Yao and Postolache [30], and the results of a host of other authors [see Remark 6 below]).

2 Preliminaries

A map $J : X \to 2^{X^*}$ defined by $Ju := \{u^* \in X^* : \langle u, u^* \rangle = \|u\|\|u^*\|, \|u\| = \|u^*\|\}$ is called a *normalized duality map* on X, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between elements of X and X^*.

We now present some lemmas that will be used in the sequel.

Lemma 2.1 (See Bo and Yi [3]) *Let X be a smooth, strictly convex and reflexive Banach space and K be a nonempty, closed, convex subset of X. Let $G : K \to X$ be a total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with $\delta_1 = 0$. Then $F(G)$ is a closed and convex subset of K.*

Lemma 2.2 (See Chang *et al.* [2]) *Let X be a uniformly smooth and strictly convex real Banach space with Kadec-Klee property and let K be a nonempty closed convex subset of X. Let $\{u_n\}$ and $\{y_n\}$ be two sequences in K such that $u_n \to u^*$ and $\phi(u_n, y_n) \to 0$, where ϕ is the function defined by* (1.1). *Then $y_n \to u^*$.*

Lemma 2.3 (See Alber [31]) *Let K be a nonempty closed and convex subset of a reflexive strictly convex and smooth Banach space X. Then*

$$\phi(u, \Pi_K y) + \phi(\Pi_K y, y) \le \phi(u, v), \quad \forall u \in K, v \in X. \tag{2.1}$$

Let K be a nonempty closed and convex subset of a Banach space X. For solving the generalized mixed equilibrium problem (1.4), we assume that a bifunction $h : K \times K \to \mathbb{R}$ satisfies the following conditions:

($B1$) $h(u, u) = 0, \forall u \in X$,
($B2$) h is monotone, that is, $h(u, v) + h(v, u) \le 0, \forall u, v \in X$,
($B3$) for all $u, y, z \in X$, $\limsup_{t \downarrow 0} h(tz + (1 - t)u, v) \le h(u, v)$,
($B4$) for all $u \in K$, $h(u, \cdot) : K \to \mathbb{R}$ is convex and lower-semicontinuous.

Lemma 2.4 (See Zhang [32]) *Let X be a smooth, strictly convex and reflexive Banach space and let K be a nonempty closed convex subset of X. Let $A : K \to X^*$ be a continuous and monotone mapping, $\zeta : K \to \mathbb{R}$ be a lower-semicontinuous and convex function, and $h : K \times K \to \mathbb{R}$ be a bifunction satisfying the conditions* ($B1$)-($B4$). *Let $r > 0$ be any given number and $u \in X$ be any given point. Then the following observations hold*:

(1) *There exists* $z \in K$ *such that*

$$h(z,v) + \zeta(v) - \zeta(z) + \langle v - z, Az \rangle + \frac{1}{r}\langle v - z, Jz - Ju \rangle \geq 0, \quad \forall v \in K.$$

(2) *If we define a mapping* $\Lambda_r : K \to K$ *by*

$$\Lambda_r(u) = \biggl\{ z \in K : h(z,v) + \zeta(v) - \zeta(z) + \langle v - z, Az \rangle + \frac{1}{r}\langle v - z, Jz - Ju \rangle \geq 0, \\ \forall v \in K \biggr\}, \quad u \in K,$$

then the mapping G_r *has the following properties*:

(a) G_r *is single-valued*;

(b) $F(\Lambda_r) = GMEP(h, A, \zeta) = \hat{F}(\Lambda_r)$;

(c) $GMEP(h, A, \zeta)$ *is a closed convex set of* K;

(d) $\phi(q, \Lambda_r u) + \phi(\Lambda_r u, u) \leq \phi(q, u)$ $\forall q \in F(\Lambda_r)$, $u \in X$.

3 Main results

In what follows, i_n and m_n are the unique solutions to the positive integer equation $n = i + \frac{(m-1)m}{2}$ ($m \geq i$, $n = 1, 2, \ldots$). That is, for each $n \geq 1$, there exist unique i_n and m_n such that

$$i_1 = 1, \quad i_2 = 1, \quad i_3 = 2, \quad i_4 = 1,$$
$$i_5 = 2, \quad i_6 = 3, \quad i_7 = 1, \quad i_8 = 2, \ldots;$$
$$m_1 = 1, \quad m_2 = 2, \quad m_3 = 2, \quad m_4 = 3,$$
$$m_5 = 3, \quad m_6 = 3, \quad m_7 = 4, \quad m_8 = 4, \ldots.$$

See Deng [33]. We now prove the following strong convergence theorem using a *Krasnoselskii-type* algorithm.

Theorem 3.1 *Let* X *be a uniformly smooth and strictly convex real Banach space with Kadec-Klee property and let* X^* *be its dual space. Let* K *be a nonempty closed and convex subset of* X *and* $h_i : K \times K \to \mathbb{R}$, $i = 1, 2, 3, \ldots$, *be a sequence of bifunctions satisfying* ($B1$)-($B4$). *Let* $A_i : K \to X^*$, $i = 1, 2, 3, \ldots$, *be a sequence of continuous monotone maps and* $G_i : K \to 2^X$, $i = 1, 2, \ldots$, *be an infinite family of equally continuous and totally quasi-*ϕ*-asymptotically nonexpansive nonself multi-valued maps with nonnegative real sequences* $\{\gamma_n^{(i)}\}$, $\{\delta_n^{(i)}\}$ *and a sequence of strictly increasing and continuous functions* $\{\rho_i\}$, $\rho_i : \mathbb{R}^+ \to \mathbb{R}^+$, *such that* $\gamma_n^{(i)} \to 0$, $\delta_n^{(i)} \to 0$ *and* $\rho_i(0) = 0$. *Let* $\zeta_i : K \to \mathbb{R}$, $i = 1, 2, 3, \ldots$, *be a sequence of convex and lower-semicontinuous functions. Suppose for each* i, $\delta_1^{(i)} = 0$ *and* $\Omega := (\bigcap_{i=1}^{\infty} F(G_i)) \cap (\bigcap_{i=1}^{\infty} GMEP(h_i, A_i, \zeta_i))$ *is a nonempty subset of* K. *Then the sequence* $\{u_n\}$ *generated by algorithm* (1.6) *converges strongly to* $\Pi_\Omega u_0$, *where* $\sigma \in (0,1)$, $r_{i_n} \in [a, \infty)$ *for some* $a > 0$, *and* $\omega_n = \gamma_{m_n}^{(i_n)} \rho_{i_n}[\phi(p, u_n)] + \delta_{m_n}^{(i_n)}$, $p \in \Omega$.

Proof The proof is presented in a number of steps.

Step 1: K_n is closed and convex for all $n \geq 1$.

Clearly, $K_1 = K$ is closed and convex. Assume K_n is closed and convex for some $n \geq 1$. It is easy to see that $K_{n+1} = \{v \in K_n : 2\langle v, Ju_n - Jz_n\rangle \leq \|u_n\| - \|z_n\| + \omega_n\}$. Consequently, it is closed and convex. Hence, Step 1 is completed.

Step 2: $\Omega \subset K_n$ for all $n \geq 1$.

Again, we proceed by induction. Clearly, $\Omega \subset K_1$. Suppose $\Omega \subset K_n$ for some $n \geq 1$. Let $q \in \Omega$. Then, by applying Lemma 2.4(d), the definition of ϕ, the convexity of $\|\cdot\|^2$, and the totally quasi-ϕ-asymptotically nonexpansiveness of G_i, we have

$$\begin{aligned}
\phi(q, z_n) &= \phi(q, \Lambda_{r_{i_n}} y_n) \leq \phi(q, y_n) = \phi\big(q, J^{-1}\big(\sigma Ju_n + (1-\sigma)J\eta_{m_n}^{(i_n)}\big)\big) \\
&= \|q\|^2 - 2\big\langle q, \sigma Ju_n + (1-\sigma)J\eta_{m_n}^{(i_n)}\big\rangle + \big\|\sigma Ju_n + (1-\sigma)J\eta_{m_n}^{(i_n)}\big\|^2 \\
&\leq \|q\|^2 - 2\sigma\langle q, Ju_n\rangle - 2(1-\sigma)\big\langle q, J\eta_{m_n}^{(i_n)}\big\rangle + \sigma\|u_n\|^2 + (1-\sigma)\big\|\eta_{m_n}^{(i_n)}\big\|^2 \\
&= \sigma\phi(q, u_n) + (1-\sigma)\phi\big(q, \eta_{m_n}^{(i_n)}\big) \\
&\leq \sigma\phi(q, u_n) + (1-\sigma)\big[\phi(q, u_n) + \gamma_{m_n}^{(i_n)}\rho_{i_n}\big(\phi(q, u_n)\big) + \delta_{m_n}^{(i_n)}\big] \\
&\leq \phi(q, u_n) + \gamma_{m_n}^{(i_n)}\rho_{i_n}\big(\phi(q, u_n)\big) + \delta_{m_n}^{(i_n)} = \phi(q, u_n) + \omega_n,
\end{aligned}$$

which implies that $q \in K_{n+1}$. Therefore, $\Omega \subset K_n$ for all $n \geq 1$.

Step 3: $\{\phi(u_n, u_0)\}$ is convergent and $\lim_{n\to\infty}\omega_n = 0$.

Since $u_n = \Pi_{K_n}u_0$ and $K_{n+1} \subset K_n$ for all $n \geq 1$, we have $\phi(u_n, u_0) \leq \phi(u_{n+1}, u_0)$, which implies that $\{\phi(u_n, u_0)\}$ is nondecreasing. Furthermore, by Lemma 2.3, we have

$$\phi(u_n, u_0) = \phi(\Pi_{K_n}u_0, u_0) \leq \phi(q, u_0) - \phi(q, u_n) \leq \phi(q, u_0),$$

for all $n \geq 1$ and $q \in \Omega$. Thus, $\{\phi(u_n, u_0)\}$ is bounded. Hence, $\{\phi(u_n, u_0)\}$ is convergent. Furthermore, by inequality (1.2), $\{u_n\}$ is bounded. Now, for each $i \geq 1$, define $K_i := \{k \geq 1 : k = i + \frac{(m-1)m}{2}, m \geq i, m \in \mathbb{N}\}$. Observe that if, for each $i \geq 1$, $k \in K_i$, then $\gamma_{m_k}^{(i_k)} = \gamma_{m_k}^{(i)}$, $\delta_{m_k}^{(i_k)} = \delta_{m_k}^{(i)}$, and $\rho_{i_k} = \rho_i$. Also, $m_k \to \infty$ as $k \to \infty$, $k \in K_i$. Therefore, $\lim_{n\to\infty}\omega_n = 0$.

Step 4: $u_n \to u^*$, $z_n \to u^*$, and $y_n \to u^*$ as $n \to \infty$, for some $u^* \in K$.

By the boundedness of $\{u_n\}$ and reflexivity of X, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \rightharpoonup u^*$ as $k \to \infty$. Since K_{n_k} is weakly closed, $u^* \in K_{n_k} \subset K$. Thus, $u_{n_k} = \Pi_{K_{n_k}}u_0$ implies that $\phi(u_{n_k}, u_0) \leq \phi(u^*, u_0)$ $\forall k \geq 1$. By the weak lower-semicontinuity of $\|\cdot\|$, we have

$$\begin{aligned}
\liminf_{k\to\infty}\phi(u_{n_k}, u_0) &= \liminf_{k\to\infty}\big\{\|u_{n_k}\|^2 - 2\langle u_{n_k}, Ju_0\rangle + \|u_0\|^2\big\} \\
&\geq \big\|u^*\big\|^2 - 2\big\langle u^*, Ju_0\big\rangle + \|u_0\|^2 = \phi\big(u^*, u_0\big),
\end{aligned}$$

which implies that $\phi(u^*, u_0) \leq \liminf_{k\to\infty}\phi(u_{n_k}, u_0) \leq \limsup_{k\to\infty}\phi(u_{n_k}, u_0) \leq \phi(u^*, u_0)$. Hence, $\lim_{k\to\infty}\phi(u_{n_k}, u_0) = \phi(u^*, u_0)$. Therefore, $\lim_{k\to\infty}\|u_{n_k}\| = \|u^*\|$. By the Kadec-Klee property of X, we have $\lim_{k\to\infty}u_{n_k} = u^*$. Since $\{\phi(u_n, u_0)\}$ is convergent and $\lim_{k\to\infty}\phi(u_{n_k}, u_0) = \phi(u^*, u_0)$, we have $\lim_{n\to\infty}\phi(u_n, u_0) = \phi(u^*, u_0)$.

Claim: $u_n \to u^*$.

Suppose there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \to p$ as $j \to \infty$. By applying Lemma 2.3, we have

$$\phi\big(u^*, p\big) = \lim_{j,k\to\infty}\phi(u_{n_k}, u_{n_j}) = \lim_{j,k\to\infty}\phi(u_{n_k}, \Pi_{K_{n_j}}u_0)$$

$$\begin{aligned}&\le \lim_{j,k\to\infty}\big(\phi(u_{n_k},u_0)-\phi(\Pi_{K_{n_j}}u_0,u_0)\big)\\&=\lim_{j,k\to\infty}\big(\phi(u_{n_k},u_0)-\phi(u_{n_j},u_0)\big)=\phi\big(u^*,u_0\big)-\phi\big(u^*,u_0\big)=0,\end{aligned}$$

which implies $u^*=p$. Hence, the claim holds. Again, by Lemma 2.3 we have

$$\phi(u_{n+1},u_n)=\phi(u_{n+1},\Pi_{K_n}u_0)\le\phi(u_{n+1},u_0)-\phi(\Pi_{K_n}u_0,u_0)=\phi(u_{n+1},u_0)-\phi(u_n,u_0),$$

which implies that $\phi(u_{n+1},u_n)\to 0$ as $n\to\infty$. Since $u_{n+1}\in K_{n+1}$, we have $\phi(u_{n+1},z_n)\le\phi(u_{n+1},u_n)+\omega_n$. Consequently, $\phi(u_{n+1},z_n)\to 0$ as $n\to\infty$. Therefore, by Lemma 2.2, $z_n\to u^*$. From $\phi(q,z_n)\le\phi(q,y_n)\le\phi(q,u_n)+\omega_n$, we have $\phi(q,y_n)\le\phi(q,u_n)+\omega_n$. Combining this with the fact that $z_n=\Lambda_{r_{i_n}}y_n$ and Lemma 2.4(d), we have

$$\begin{aligned}\phi(z_n,y_n)&=\phi(\Lambda_{r_{i_n}}y_n,y_n)\le\phi(q,y_n)-\phi(q,\Lambda_{r_{i_n}}y_n)\\&\le\phi(q,u_n)-\phi(q,\Lambda_{r_{i_n}}y_n)+\omega_n=\phi(q,u_n)-\phi(q,z_n)+\omega_n,\end{aligned}$$

for any $q\in\Omega$. This implies that $\lim_{n\to\infty}\phi(z_n,y_n)=0$. Again, by Lemma 2.2 we have $y_n\to u^*$.

Step 5: $u^*\in\Omega$.

We first show that $u^*\in\bigcap_{i=1}^{\infty}F(G_i)$. Note that $G_{i_k}(PG_{i_k})^{m_k-1}=G_i(PG_i)^{m_k-1}$ whenever $k\in K_i$ for each $i\ge 1$. From Step 4 and by the uniform continuity of J on bounded subsets of X, we have $\|Jy_k-Ju^*\|\to 0$, $\|Ju_k-Ju^*\|\to 0$ as $k\to\infty$. Thus, for each $i\ge 1$, $\eta_{m_k}^{(i)}\in G_{i_k}(PG_{i_k})^{m_k-1}u_k$, we have

$$\begin{aligned}\big\|Jy_k-Ju^*\big\|&=\big\|\sigma Ju_k+(1-\sigma)J\eta_{m_k}^{(i)}-Ju^*\big\|\\&=\big\|(1-\sigma)\big(J\eta_{m_k}^{(i)}-Ju^*\big)-\sigma\big(Ju^*-Ju_k\big)\big\|\\&\ge(1-\sigma)\big\|J\eta_{m_k}^{(i)}-Ju^*\big\|-\sigma\big\|Ju^*-Ju_k\big\|,\end{aligned}$$

which implies that $\lim_{k\to\infty}\|J\eta_{m_k}^{(i)}-Ju^*\|=0$ for each $i\ge 1$. By the norm-to-weak continuity of J^{-1}, we have, for each $i\ge 1$, $\eta_{m_k}^{(i)}\rightharpoonup u^*$ as $k\to\infty$. Furthermore,

$$\big|\big\|\eta_{m_k}^{(i)}\big\|-\big\|u^*\big\|\big|=\big|\big\|J\eta_{m_k}^{(i)}\big\|-\big\|Ju^*\big\|\big|\le\big\|J\eta_{m_k}^{(i)}-Ju^*\big\|\to 0.$$

Thus, for each $i\ge 1$, $\lim_{k\to\infty}\|\eta_{m_k}^{(i)}\|=\|u^*\|$. Hence, by the Kadec-Klee property of X, we have

$$\lim_{k\to\infty}\eta_{m_k}^{(i)}=u^*\quad\forall i\ge 1.\tag{3.1}$$

We now consider the sequence $\{w_{m_k}^{(i)}\}_{k\in K_i}$, generated by

$$w_{m_{k+1}}^{(i)}\in G_iP\eta_{m_k}^{(i)}\subset G_{i_k}(PG_{i_k})^{m_k}u_k,\quad k\in K_i\ \forall i\ge 1.$$

By the continuity of G_iP, we have, for each $i \geq 1$, $\lim_{k\to\infty} w^{(i)}_{m_{k+1}} = w^*$, $w^* \in G_iPu^* = G_iu^*$, since $u^* \in K$. Using the continuity of both G_i and (3.1), we obtain for each $i \geq 1$

$$\begin{aligned}\left\| w^{(i)}_{m_{k+1}} - \eta^{(i)}_{m_k} \right\| &\leq \left\| w^{(i)}_{m_{k+1}} - \eta^{(i)}_{m_{k+1}} \right\| + \left\| \eta^{(i)}_{m_{k+1}} - u_{k+1} \right\| + \| u_{k+1} - u_k \| \\ &\quad + \left\| u_k - \eta^{(i)}_{m_k} \right\| \to 0, \quad k \to \infty.\end{aligned}$$

Therefore, for each $i \geq 1$, $w^{(i)}_{m_{k+1}} \to u^*$ as $k \to \infty$. Hence, by the uniqueness of the limit, we have $w^* = u^*$. Thus, $u^* \in \bigcap_{i=1}^\infty F(G_i)$.

We now show that $u^* \in \bigcap_{i=1}^\infty GMEP(h_i, A_i, \zeta_i)$. Let $i \geq 1$. Define a function $\Psi_i : K \times K \to \mathbb{R}$ by

$$\Psi_i(u,v) = h_i(u,v) + \zeta_i(v) - \zeta_i(u) + \langle v - u, A_i u\rangle \quad \forall u, v \in K.$$

Then, as shown by Zhang [32], for each i, Ψ_i satisfies $(B1)$-$(B4)$. Note that if $k \in K_i$, for each $i \geq 1$, then $\Psi_{i_k} = \Psi_i$ and $\Lambda_{r_{i_k}} = \Lambda_{r_i}$. Now, the equation $z_k = \Lambda_{r_{i_k}} y_k$ implies that, for each $i \geq 1$,

$$\Psi_i(z_k, v) + \frac{1}{r_i}\langle v - z_k, Jz_k - Jy_k\rangle \geq 0 \quad \forall v \in K. \tag{3.2}$$

By applying $(B2)$, we have, for each $i \geq 1$,

$$\frac{1}{r_i}\langle v - z_k, Jz_k - Jy_k\rangle \geq -\Psi_i(z_k, v) \geq \Psi_i(v, z_k) \quad \forall v \in K, \tag{3.3}$$

which implies that

$$\Psi_i(v, z_k) \leq \frac{1}{r_i}\langle v - z_k, Jz_k - Jy_k\rangle \leq \frac{1}{r_i}\|v - z_k\|\|Jz_k - Jy_k\| \leq \frac{1}{r_i}\big(\|v\| + M\big)\|Jz_k - Jy_k\|,$$

for all $v \in K$, $i \geq 1$, and some $M > 0$. This implies that $\liminf_{k\to\infty} \Psi_i(v, z_k) \leq 0$ for all $v \in K$, $i \geq 1$. From $(B4)$, we obtain, for $i \geq 1$, $\Psi_i(v, u^*) \leq \liminf_{k\to\infty} \Psi_i(v, z_k) \leq 0$ $\forall v \in K$. Let $t \in (0,1)$ and $v \in K$. Then $v_t = tv + (1-t)u^* \in K$. Therefore, for each $i \geq 1$, $\Psi_i(v_t, u^*) \leq 0$. From conditions $(B1)$ and $(B4)$ we have, for each $i \geq 1$,

$$0 = \Psi_i(v_t, v_t) \leq t\Psi_i(v_t, v) + (1-t)\Psi_i\big(v_t, u^*\big) \leq t\Psi_i(v_t, v) \quad \Longrightarrow \quad \Psi_i(v_t, v) \geq 0 \quad \forall v \in K.$$

By $(B3)$, we have, for each $i \geq 1$, $\Psi_i(u^*, v) \geq \limsup_{t\downarrow 0} \Psi_i(v_t, v) \geq 0$ $\forall v \in K$. Therefore, $u^* \in GMEP(h_i, \zeta_i, A_i)$ for each $i \geq 1$. Hence, $u^* \in \bigcap_{i=1}^\infty GMEP(h_i, \zeta_i, A_i)$.

Step 6: $u^* = \Pi_\Omega u_0$.

Let $v = \Pi_\Omega u_0$. Since $u^* \in \Omega$, we have

$$\phi(v, u_0) \leq \phi\big(u^*, u_0\big). \tag{3.4}$$

Also, since $u_n = \Pi_{K_n} u_0$ and $v \in \Omega \subset K_n$, we have

$$\phi(u_n, u_0) \leq \phi(v, u_0).$$

Since $u_n \to u^*$, we have

$$\phi(u^*, u_0) \le \phi(v, u_0). \tag{3.5}$$

From inequalities (3.4) and (3.5) we obtain $\phi(u^*, u_0) = \phi(v, u_0)$. Thus, $u^* = v = \Pi_\Omega u_0$. This completes the proof. □

We now prove the following strong convergence theorem using a *Halpern-type algorithm*.

Theorem 3.2 *Let X be a uniformly smooth and strictly convex real Banach space with Kadec-Klee property and let X^* be its dual space. Let K be a nonempty closed and convex subset of X and $h_i : K \times K \to \mathbb{R}$, $i = 1,2,3,\ldots$, be a sequence of bifunctions satisfying* (B1)-(B4). *Let $A_i : K \to X^*$, $i = 1,2,3,\ldots$, be a sequence of continuous monotone maps and $G_i : K \to 2^X$, $i = 1,2,\ldots$, be an infinite family of equally continuous and totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued maps with nonnegative real sequences $\{\gamma_n^{(i)}\}$, $\{\delta_n^{(i)}\}$ and a sequence of strictly increasing and continuous functions $\{\rho_i\}$, $\rho_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\gamma_n^{(i)} \to 0$, $\delta_n^{(i)} \to 0$, and $\rho_i(0) = 0$. Let $\zeta_i : K \to \mathbb{R}$, $i = 1,2,3,\ldots$, be a sequence of convex and lower-semicontinuous functions. Suppose, for each i, $\delta_1^{(i)} = 0$ and $\Omega := (\bigcap_{i=1}^\infty F(G_i)) \cap (\bigcap_{i=1}^\infty GMEP(h_i, A_i, \zeta_i))$ is a nonempty subset of K. Then the sequence $\{u_n\}$, generated by algorithm* (1.7), *converges strongly to $\Pi_\Omega u_0$, where $\{\sigma_n\} \subset (0,1)$ with $\lim_{n\to\infty} \sigma_n = 0$, $r_{i_n} \in [a,\infty)$ for some $a > 0$, and $\omega_n = \gamma_{m_n}^{(i_n)} \rho_{i_n}[\phi(p,u_n)] + \delta_{m_n}^{(i_n)}$, $p \in \Omega$.*

Proof As in the proof of Theorem 3.1, the proof of this theorem is presented in six steps.

Step 1: K_n is closed and convex for all $n \ge 1$.

This follows easily by induction, just as in the proof of Theorem 3.1.

Step 2: $\Omega \subset K_n$ for all $n \ge 1$. Clearly, $\Omega \subset K_1$. Suppose $\Omega \subset K_n$ for some $n \ge 1$. Let $q \in \Omega$. Then by applying Lemma 2.4(d), the definition of ϕ, the convexity of $\|\cdot\|^2$, and the totally quasi-ϕ-asymptotically nonexpansiveness of G_i, as in Step 2 of the proof of Theorem 3.1, we have

$$\begin{aligned}
\phi(q, z_n) &= \phi(q, \Lambda_{r_{i_n}} y_n) \\
&\le \phi(q, y_n) \\
&= \phi\bigl(q, J^{-1}\bigl(\sigma_n Ju_0 + (1-\sigma_n) J\eta_{m_n}^{(i_n)}\bigr)\bigr) \\
&= \|q\|^2 - 2\bigl\langle q, \sigma_n Ju_0 + (1-\sigma_n) J\eta_{m_n}^{(i_n)}\bigr\rangle + \bigl\|\sigma_n Ju_0 + (1-\sigma_n) J\eta_{m_n}^{(i_n)}\bigr\|^2 \\
&\le \|q\|^2 - 2\sigma_n \langle q, Ju_0\rangle - 2(1-\sigma_n)\bigl\langle q, J\eta_{m_n}^{(i_n)}\bigr\rangle + \sigma_n \|u_0\|^2 + (1-\sigma_n)\bigl\|\eta_{m_n}^{(i_n)}\bigr\|^2 \\
&= \sigma_n \phi(q, u_0) + (1-\sigma_n)\phi\bigl(q, \eta_{m_n}^{(i_n)}\bigr) \\
&\le \sigma_n \phi(q, u_0) + (1-\sigma_n)\bigl[\phi(q, u_n) + \gamma_{m_n}^{(i_n)} \rho_{i_n}\bigl(\phi(q, u_n)\bigr) + \delta_{m_n}^{(i_n)}\bigr] \\
&\le \sigma_n \phi(q, u_0) + (1-\sigma_n)\phi(q, u_n) + \gamma_{m_n}^{(i_n)} \rho_{i_n}\bigl(\phi(q, u_n)\bigr) + \delta_{m_n}^{(i_n)} \\
&= \sigma_n \phi(q, u_0) + (1-\sigma_n)\phi(q, u_n) + \omega_n,
\end{aligned}$$

which implies that $q \in K_{n+1}$. Therefore, by induction, $\Omega \subset K_n$ for all $n \ge 1$.

Step 3: $\{\phi(u_n, u_0)\}$ is convergent and $\lim_{n\to\infty} \omega_n = 0$.

This follows just as in Step 3 of the proof of Theorem 3.1.

Step 4: $u_n \to u^*$, $z_n \to u^*$, and $y_n \to u^*$ as $n \to \infty$, for some $u^* \in K$.

The verification of this step follows the same pattern as in the verification of Step 4 in the proof of Theorem 3.1.

Step 5: $u^* \in \Omega$.

We first show that $u^* \in \bigcap_{i=1}^{\infty} F(G_i)$. Note that $G_{i_k}(PG_{i_k})^{m_k-1} = G_i(PG_i)^{m_k-1}$ whenever $k \in K_i$ for each $i \geq 1$. From Step 4 and the uniform continuity of J on bounded subsets of X, we have $\|Jy_k - Ju^*\| \to 0$ as $k \to \infty$. Thus, for each $i \geq 1$, $\eta_{m_k}^{(i)} \in G_{i_k}(PG_{i_k})^{m_k-1}u_k$, we have

$$\begin{aligned} \left\|Jy_k - Ju^*\right\| &= \left\|\sigma_n Ju_0 + (1-\sigma_n)J\eta_{m_k}^{(i)} - Ju^*\right\| \\ &= \left\|(1-\sigma_n)\left(J\eta_{m_k}^{(i)} - Ju^*\right) - \sigma_n\left(Ju^* - Ju_0\right)\right\| \\ &\geq (1-\sigma_n)\left\|J\eta_{m_k}^{(i)} - Ju^*\right\| - \sigma_n\left\|Ju^* - Ju_0\right\|, \end{aligned}$$

which implies that $\lim_{k\to\infty}\|J\eta_{m_k}^{(i)} - Ju^*\| = 0$ for each $i \geq 1$. The rest of the justification of this step follows the same pattern as in the justification of Step 5 in the proof of Theorem 3.1.

Step 6: $u^* = \Pi_\Omega u_0$.

This is the same as Step 6 of the proof of Theorem 3.1. Hence, the proof is completed. □

A prototype for the control parameter in Theorem 3.2 is the canonical choice, $\sigma_n = \frac{1}{n}$.

4 Applications

In this section, we present some applications of Theorem 3.1. Similar applications of Theorem 3.2 also follow.

4.1 Countable family of totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued maps and system of equilibrium problems

By setting $A \equiv 0$, $\zeta \equiv 0$ in Theorem 3.1, the sequence $\{u_n\}$, defined in Theorem 3.1, converges strongly to $\Pi_\Omega u_0$, where $\Omega := (\bigcap_{i=1}^{\infty} F(G_i)) \cap (\bigcap_{i=1}^{\infty} EP(h_i))$ and $EP(h)$ is the set of solutions of the equilibrium problem for h.

4.2 Countable family of totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued maps and system of convex optimization problems

By setting $A \equiv 0$, $h \equiv 0$ in Theorem 3.1, the sequence $\{u_n\}$, defined in Theorem 3.1, converges strongly to $\Pi_\Omega u_0$, where $\Omega := (\bigcap_{i=1}^{\infty} F(G_i)) \cap (\bigcap_{i=1}^{\infty} CMP(\zeta_i))$ and $CMP(\zeta)$ is the set of solutions of the convex minimization problem for ζ.

4.3 Countable family of totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued maps and system of variational inequality problems

By setting $h \equiv 0$, $\zeta \equiv 0$ in Theorem 3.1, the sequence $\{u_n\}$, defined in Theorem 3.1, converges strongly to $\Pi_\Omega u_0$, where $\Omega := (\bigcap_{i=1}^{\infty} F(G_i)) \cap (\bigcap_{i=1}^{\infty} VIP(K,A_i))$ and $VIP(K,A)$ is the set of solutions of the variational inequality problem for A over K.

4.4 Application in classical Banach spaces

Let $X = L_p, l_p$, or $W_p^m(\Omega)$, $1 < p < \infty$, where $W_p^m(\Omega)$ denotes the usual Sobolev space, and let X^* be the dual space of X. Clearly, X is uniformly convex and uniformly smooth. Consequently, Theorem 3.1 is applicable in these spaces.

Remark 5 (See, *e.g.*, Alber and Ryazantseva [34], p.36) The analytical representations of duality maps are known in l^p, $L^p(G)$, and Sobolev spaces $W^p_m(G)$, $p \in (1,\infty)$, and $p^{-1} + q^{-1} = 1$.

4.5 Application in Hilbert spaces

The following theorem follows immediately from Theorem 3.1.

Theorem 4.1 *Let H be a real Hilbert space and K be a nonempty closed and convex subset of H. Let $h_i : K \times K \to \mathbb{R}$, $i = 1,2,3,\dots$, be a sequence of bifunctions satisfying $(B1)$-$(B4)$. Let $A_i : K \to H$ be a sequence of continuous monotone maps and $G_i : K \to 2^H$, $i = 1,2,\dots$, be an infinite family of equally continuous and totally quasi-asymptotically nonexpansive nonself multi-valued maps with nonnegative real sequences $\{\gamma_n^{(i)}\}$, $\{\delta_n^{(i)}\}$ and a sequence of strictly increasing and continuous functions $\{\rho_i\}$, $\rho_i : \mathbb{R}^+ \to \mathbb{R}^+$, such that $\gamma_n^{(i)} \to 0$, $\delta_n^{(i)} \to 0$, and $\rho_i(0) = 0$. Let $\zeta_i : K \to \mathbb{R}$, $i = 1,2,3,\dots$, be a sequence of convex and lower-semicontinuous functions. Suppose, for each i, $\delta_1^{(i)} = 0$ and $\Omega := (\bigcap_{i=1}^{\infty} F(G_i)) \cap (\bigcap_{i=1}^{\infty} GMEP(h_i, A_i, \zeta_i))$ is a nonempty subset of K. Then the sequence $\{u_n\}$, generated by*

$$
\begin{cases}
u_0 \in H, \quad \text{chosen arbitrarily}, \qquad K_1 = K, \qquad u_1 = \Pi_{K_1} u_0, \\
y_n = \sigma u_n + (1-\sigma)\eta_{m_n}^{(i_n)}, \quad (\eta_{m_n}^{(i_n)} \in G_{i_n}(PG_{i_n})^{m_n-1} u_n), \\
z_n = G_{r_{i_n}} y_n, \\
K_{n+1} = \{v \in K_n : \|v - z_n\|^2 \le \|v - u_n\|^2 + \omega_n\}, \\
u_{n+1} = P_{K_{n+1}} u_0, \quad n \ge 1,
\end{cases}
$$

converges strongly to $P_\Omega u_0$, where P_K is the metric projection of H onto K, $\sigma \in (0,1)$, $r_{i_n} \in [a,\infty)$ for some $a > 0$, and $\omega_n = \gamma_{m_n}^{(i_n)} \rho_{i_n}(\|u_n - p\|^2) + \delta_{m_n}^{(i_n)}$.

Remark 6

(1) Theorem 3.2 improves the results of Bo and Yi [3] in the following ways:
- In Theorem 3.2, a countable family of nonself multi-valued maps is considered, whereas in Bo and Yi [3] a single nonself multi-valued map is considered.
- The requirement that G is uniformly L-Lipschitz continuous in Bo and Yi [3] is weakened to: for each i, G_i is equally continuous in Theorem 3.2.
- The algorithm in Theorem 3.2 involves only one control parameter $\{\sigma_n\} \subset (0,1)$ satisfying condition $(C1)$, whereas the algorithm of Bo and Yi [3] contains two control parameters $\{\beta_n\} \subset (0,1)$ and $\{\sigma_n\} \subset [0,1]$, satisfying conditions $(C1)$ and $(C2)$.
- The Banach spaces considered in Theorem 3.2 are uniformly smooth and strictly convex real Banach spaces with Kadec-Klee property, which include uniformly smooth and uniformly convex real Banach spaces studied in Bo and Yi [3].

(2) Theorem 3.2 improves and generalizes the results in Zhao and Chang [26] in a number of ways:
- The class of maps considered in Zhao and Chang [26] is extended from the class of *uniformly* quasi-ϕ-asymptotically nonexpansive *single-valued* nonself maps to the slightly more general class of countable family of totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued maps.

- The requirement that, for each i, G_i is uniformly L_i-Lipschitz continuous in Zhao and Chang [26] is weakened to the following statement: for each i, G_i is equally continuous in Theorem 3.2.
- The results of Zhao and Chang [26] are proved in uniformly smooth and uniformly convex real Banach spaces, while Theorem 3.2 is proved in the more general uniformly smooth and strictly convex real Banach spaces with the Kadec-Klee property.
- The control parameter in the algorithm considered in Theorem 3.2 is $\{\sigma_n\} \subset (0,1)$ satisfying condition $(C1)$, whereas the algorithm of Zhao and Chang [26] contains two control parameters $\{\beta_n\} \subset (0,1)$ and $\{\sigma_n\} \subset [0,1]$, satisfying conditions $(C1)$ and $(C2)$.

5 Conclusion

In this article, iterative schemes of the Krasnoselskii-type and the Halpern-type for approximating a common point in the set of common fixed points of a countable family of totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued maps and the set of solutions of a system of generalized mixed equilibrium problems are constructed. Strong convergence of the sequences generated by these algorithms is established in certain Banach spaces. Among other applications, our theorems are applied to solve convex feasibility problems, a system of convex minimization problems, a system of variational inequality problems, and a system of generalized equilibrium problems in uniformly smooth and strictly convex real Banach spaces with the Kadec-Klee property. Finally, our theorems are important improvements of several important recent results on these classes of nonlinear problems.

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