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Some generalizations for $(\alpha - \psi, \phi)$-contractions in b-metric-like spaces and an application

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Abstract
In this paper, we introduce a new class of α_{qs^p}-admissible mappings and provide some fixed point theorems involving this class of mappings satisfying some new conditions of contractivity in the setting of b-metric-like spaces. Our results extend, unify, and generalize classical and recent fixed point results for contractive mappings.

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1 Introduction

In the past years extensions of a metric fixed point theory to generalized structures have received much attention. Also in these structures the concepts of fixed point theorems and contractions have appeared with a remarkable influence on applications in the theory of differential and integral equations, and giving appropriate mathematical models for solving a variety of applied problems in the mathematical sciences and engineering. Some generalizations are b-metric spaces introduced by Bakhtin [1] (and later extensively used by Czerwik [2]), partial metric spaces by Matthews [3], b-partial metric spaces by Shukla [4], metric-like spaces by Harandi [5], and b-metric-like spaces by Alghmandi et al. [6]. Later, Hussain [7] discussed the topological structure of b-metric-like spaces.

Also these generalizations have been associated with new and generalized classes of contractive mappings. In this direction, Samet et al. [8] introduced the concept of α-admissible, α-contractive, and $\alpha - \psi$-contractive mappings, further extended to the (α, β)-contractive mappings. Many papers dealing with these notions have been considered to prove fixed point results (for example, see [8–23]).

In this paper, working in this direction, we introduce the concept of an α_{qs^p}-admissible mapping and provide some fixed point results involving $\alpha_{qs^p} - \lambda$ contractions and generalized $(\alpha_{qs^p} - \psi, \phi)$ contractive mappings in the larger framework of b-spaces, precisely, in the setting of b-metric-like spaces. The presented theorems improve, extend, generalize, and unify a number of existing results in the literature.

2 Preliminaries

Definition 2.1 ([2]) Let X be a nonempty set. A mapping $d : X \times X \to [0, \infty)$ is called a b-metric if the following conditions hold for all $x, y, z \in X$ and for some $s \geq 1$:

$$d(x,y) = 0 \quad \text{if and only if} \quad x = y;$$
$$d(x,y) = d(y,x);$$
$$d(x,y) \leq s\bigl[d(x,z) + d(z,y)\bigr].$$

The pair (X, d) is called a b-metric space with parameter s.

Definition 2.2 ([3]) Let X be a nonempty set. A mapping $p : X \times X \to [0, \infty)$ is called a partial metric if the following conditions hold for all $x, y, z \in X$ and $s \geq 1$: $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;

$$p(x,x) \leq p(x,y);$$
$$p(x,y) = p(y,x);$$
$$p(x,y) \leq p(x,z) + p(z,y) - p(z,z).$$

The pair (X, p) is called a partial metric space.

Definition 2.3 ([4]) Let X be a nonempty set. A mapping $p_b : X \times X \to [0, \infty)$ is called a partial b-metric if, for any real number $s \geq 1$ and for all $x, y, z \in X$:

$$x = y \quad \Leftrightarrow \quad p_b(x,x) = p_b(x,y) = p_b(y,y);$$
$$p_b(x,x) \leq p_b(x,y);$$
$$p_b(x,y) = p_b(y,x);$$
$$p_b(x,y) \leq s\bigl[p_b(x,z) + p_b(z,y)\bigr] - p_b(z,z).$$

The pair (X, p_b) is called a partial b-metric space.

Definition 2.4 ([5]) Let X be a nonempty set. A mapping $\sigma : X \times X \to [0, \infty)$ is called metric-like if the following conditions hold for all $x, y, z \in X$:

$$\sigma(x,y) = 0 \quad \text{implies} \quad x = y;$$
$$\sigma(x,y) = \sigma(y,x);$$
$$\sigma(x,y) \leq \sigma(x,z) + \sigma(z,y).$$

The pair (X, σ) is called a metric-like space.

Definition 2.5 ([6]) Let X be a nonempty set. A mapping $\sigma_b : X \times X \to [0, \infty)$ is called b-metric-like if the following conditions hold for all $x, y, z \in X$ and for some $s \geq 1$:

$$\sigma_b(x,y) = 0 \quad \text{implies} \quad x = y;$$

$$\sigma_b(x,y) = \sigma_b(y,x);$$

$$\sigma_b(x,y) \leq s\bigl[\sigma_b(x,z) + \sigma_b(z,y)\bigr].$$

The pair (X, σ_b) is called a b-metric-like space.

In a b-metric-like space (X, σ_b), if $x, y \in X$ and $\sigma_b(x,y) = 0$, then $x = y$, but the converse need not be true, and $\sigma_b(x,x)$ may be positive for $x \in X$.

Remark 2.6 The class of b-metric-like spaces is larger than either metric-like spaces or b-metric-spaces, since a b-metric-like space is a metric-like space when $s = 1$ and since every b-metric space is a b-metric-like space with the same parameter s. However, the converse implications do not hold.

Example 2.7 ([6]) Let $X = R^+ \cup \{0\}$. Define the function $\sigma_b : X^2 \to [0, \infty)$ by $\sigma_b(x,y) = (x+y)^2$ for all $x, y \in X$. Then (X, σ_b) is a b-metric-like space with parameter $s = 2$.

Example 2.8 ([24]) Let $X = R^+ \cup \{0\}$. Define the function $\sigma_b : X^2 \to [0, \infty)$ by $\sigma_b(x,y) = (\max\{x,y\})^2$ for all $x, y \in X$. Then (X, σ_b) is a b-metric-like space with parameter $s = 2$. Clearly, (X, σ_b) is not a b-metric or metric-like space.

Definition 2.9 ([6]) Let (X, σ_b) be a b-metric-like space with parameter s, let $\{x_n\}$ be any sequence in X, and let $x \in X$. Then

(a) The sequence $\{x_n\}$ is said to converge to x if $\lim_{n\to\infty} \sigma_b(x_n, x) = \sigma_b(x,x)$;

(b) The sequence $\{x_n\}$ is said to be a Cauchy sequence in (X, σ_b) if $\lim_{n,m\to\infty} \sigma_b(x_n, x_m)$ exists and is finite;

(c) (X, σ_b) is said to be a complete b-metric-like space if, for every Cauchy sequence $\{x_n\}$ in X, there exists $x \in X$ such that $\lim_{n,m\to\infty} \sigma_b(x_n, x_m) = \lim_{n\to\infty} \sigma_b(x_n, x) = \sigma_b(x,x)$.

The limit of a sequence in a b-metric-like space need not be unique.

Proposition 2.10 ([6]) *Let (X, σ_b) be a b-metric-like space with parameter s, and let $\{x_n\}$ be any sequence in X with $x \in X$ such that* $\lim_{n\to\infty} \sigma_b(x_n, x) = 0$.*Then*

(a) *x is unique*,

(b) $\sigma_b(x,y)/s \leq \lim_{n\to\infty} \sigma_b(x_n, y) \leq s\sigma_b(x,y)$ *for all $y \in X$.*

In 2012, Samet et al. [8] introduced the class of α-admissible mappings.

Definition 2.11 Let X be a nonempty set, $f : X \to X$, and $\alpha : X \times X \to R^+$. We say that f is an α-admissible mapping if $\alpha(x,y) \geq 1$ implies that $\alpha(fx, fy) \geq 1$ for all $x, y \in X$.

Since, in general, a b-metric-like space is not continuous, we quote the following lemmas about the convergence of sequences.

Lemma 2.12 ([7]) *Let (X,σ_b) be a b-metric-like space with parameter $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are σ_b-convergent to x and y, respectively. Then we have*

$$\begin{aligned}\frac{1}{s^2}\sigma_b(x,y) - \frac{1}{s}\sigma_b(x,x) - \sigma_b(y,y) &\leq \liminf_{n\to\infty}\sigma_b(x_n,y_n)\\ &\leq \limsup_{n\to\infty}\sigma_b(x_n,y_n) \leq s\sigma_b(x,x) + s^2\sigma_b(y,y) + s^2\sigma_b(x,y).\end{aligned}$$

In particular, if $\sigma_b(x,y) = 0$, then we have $\lim_{n\to\infty}\sigma_b(x_n,y_n) = 0$.

Moreover, for each $z \in X$, we have

$$\begin{aligned}\frac{1}{s}\sigma_b(x,z) - \sigma_b(x,x) &\leq \liminf_{n\to\infty}\sigma_b(x_n,z)\\ &\leq \limsup_{n\to\infty}\sigma_b(x_n,z) \leq s\sigma_b(x,z) + s\sigma_b(x,x).\end{aligned}$$

In particular, if $\sigma_b(x,x) = 0$, then

$$\begin{aligned}\frac{1}{s}\sigma_b(x,z) &\leq \liminf_{n\to\infty}\sigma_b(x_n,z)\\ &\leq \limsup_{n\to\infty}\sigma_b(x_n,z) \leq s\sigma_b(x,z).\end{aligned}$$

The following result is useful.

Lemma 2.13 *Let (X,σ_b) be a b-metric-like space with parameter $s \geq 1$. Then*

(a) *If $\sigma_b(x,y) = 0$, then $\sigma_b(x,x) = \sigma_b(y,y) = 0$;*

(b) *If (x_n) is a sequence such that $\lim_{n\to\infty}\sigma_b(x_n,x_{n+1}) = 0$, then we have*

$$\lim_{n\to\infty}\sigma_b(x_n,x_n) = \lim_{n\to\infty}\sigma_b(x_{n+1},x_{n+1}) = 0;$$

(c) *If $x \neq y$, then $\sigma_b(x,y) > 0$.*

Proof The proof is obvious. □

Lemma 2.14 *Let (X,σ_b) be a complete b-metric-like space with parameter $s \geq 1$, and let $\{x_n\}$ be a sequence such that*

$$\lim_{n\to\infty}\sigma_b(x_n,x_{n+1}) = 0. \tag{2.1}$$

If $\{x_n\}$ is not Cauchy, then there exist $\varepsilon > 0$ and two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ (positive integers) such that $\sigma_b(x_{m_k},x_{n_k}) \geq \varepsilon$, $\sigma_b(x_{m_k},x_{n_k-1}) < \varepsilon$, $\varepsilon/s^2 \leq \limsup_{k\to\infty}\sigma_b(x_{m_k-1},x_{n_k-1}) \leq \varepsilon s$, $\varepsilon/s \leq \limsup_{k\to\infty}\sigma_b(x_{n_k-1},x_{m_k}) \leq \varepsilon s^2$, and $\varepsilon/s \leq \limsup_{k\to\infty}\sigma_b(x_{m_k-1},x_{n_k}) \leq \varepsilon s^2$.

Proof If $\{x_n\}$ is not a σ_b-Cauchy sequence, then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that n_k is the smallest index for which

$$n_k > m_k > k, \quad \sigma_b(x_{m_k},x_{n_k}) \geq \varepsilon. \tag{2.2}$$

This means that

$$\sigma_b(x_{m_k}, x_{n_k-1}) < \varepsilon. \tag{2.3}$$

From (2.2) and property (c) of Definition 2.4 we have

$$\begin{aligned} \varepsilon &\leq \sigma_b(x_{m_k}, x_{n_k}) \leq s\sigma_b(x_{m_k}, x_{m_k-1}) + s\sigma_b(x_{m_k-1}, x_{n_k}) \\ &\leq s\sigma_b(x_{m_k}, x_{m_k-1}) + s^2\sigma_b(x_{m_k-1}, x_{n_k-1}) + s^2\sigma_b(x_{n_k-1}, x_{n_k}). \end{aligned} \tag{2.4}$$

Taking the upper limit as $k \to \infty$ in (2.4) and using (2.1), (2.2), and (2.3), we get

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} \sigma_b(x_{m_k-1}, x_{n_k-1}). \tag{2.5}$$

By the triangle inequality we have

$$\sigma_b(x_{m_k-1}, x_{n_k-1}) \leq s\sigma_b(x_{m_k-1}, x_{m_k}) + s\sigma_b(x_{m_k}, x_{n_k-1}),$$

so, taking the upper limit as $k \to \infty$ and using (2.1), we get

$$\limsup_{k\to\infty} \sigma_b(x_{m_k-1}, x_{n_k-1}) \leq \varepsilon s. \tag{2.6}$$

By (2.5) and (2.6) we have

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} \sigma_b(x_{m_k-1}, x_{n_k-1}) \leq \varepsilon s. \tag{2.7}$$

Also, we have

$$\varepsilon \leq \sigma_b(x_{m_k}, x_{n_k}) \leq s\sigma_b(x_{m_k}, x_{m_k-1}) + s\sigma_b(x_{m_k-1}, x_{n_k}),$$

and, taking the upper limit as $k \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} \sigma_b(x_{m_k-1}, x_{n_k}). \tag{2.8}$$

Again

$$\varepsilon \leq \sigma_b(x_{m_k}, x_{n_k}) \leq s\sigma_b(x_{m_k}, x_{n_k-1}) + s\sigma_b(x_{n_k-1}, x_{n_k}).$$

Taking the upper limit as $k \to \infty$ and using (2.1), we get

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} \sigma_b(x_{n_k-1}, x_{m_k}). \tag{2.9}$$

Since $\sigma_b(x_{n_k-1}, x_{m_k}) \leq s\sigma_b(x_{n_k-1}, x_{m_k-1}) + s\sigma_b(x_{m_k-1}, x_{m_k})$, from (2.1) and (2.7) we have

$$\limsup_{k\to\infty} \sigma_b(x_{n_k-1}, x_{m_k}) \leq s \limsup_{k\to\infty} \sigma_b(x_{n_k-1}, x_{m_k-1}) \leq \varepsilon s^2. \tag{2.10}$$

Consequently,

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} \sigma_b(x_{n_k-1}, x_{m_k}) \leq \varepsilon s^2. \tag{2.11}$$

Also,

$$\sigma_b(x_{m_k-1}, x_{n_k}) \leq s\sigma_b(x_{m_k-1}, x_{n_k-1}) + s\sigma_b(x_{n_k-1}, x_{n_k}).$$

Then from (2.7), (2.8), and (2.1) we have

$$\limsup_{k\to\infty} \sigma_b(x_{m_k-1}, x_{n_k}) \leq s \limsup_{k\to\infty} \sigma_b(x_{m_k-1}, x_{n_k-1}) \leq \varepsilon s^2.$$

Consequently,

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} \sigma_b(x_{m_k-1}, x_{n_k}) \leq \varepsilon s^2. \tag{2.12}$$

This completes proof. □

3 Main results

We begin this section with the following definition.

Definition 3.1 Let (X, σ_b) be a b-metric-like space with parameter $s \geq 1$, let $\alpha : X \times X \to [0, \infty)$ be a function, and let $q \geq 1$ and $p \geq 2$ be arbitrary constants. A mapping $f : X \to X$ is α_{qs^p}-admissible if $\alpha(x,y) \geq qs^p$ implies $\alpha(fx,fy) \geq qs^p$ for all $x, y \in X$.

Remark 3.2

(i) Taking $q = 1$ in this definition, we obtain an α_{s^p}-admissible mapping defined in a b-metric-like space or in a b-metric space.

(ii) Note that, for $s = 1$, the definition reduces to an α_q-admissible mapping defined in a metric space or in a metric-like space.

(iii) For $s = 1$ and $q = 1$, the definition reduces to the definition of an α-admissible mapping in a metric space [8].

(iv) The class of α_{qs^p}-admissible mappings is strictly larger, and, more generally, because the constant $p \geq 2$, it is not restricted to some certain values.

We further consider the following properties.

Let (X, σ_b) be a complete b-metric-like space with parameter $s \geq 1$, and let $\alpha : X \times X \to [0, \infty)$ be a function. Then:

(H_{qs^p}) If $\{x_n\}$ is a sequence in X such that $x_n \to x \in X$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq qs^p$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq qs^p$ for all $k \in N$.

(U_{qs^p}) For all $x, y \in \mathrm{Fix}(f)$, we have $\alpha(x,y) \geq qs^p$, where $\mathrm{Fix}(f)$ denotes the set of fixed points of f.

Example 3.3 Let $X = (0, +\infty)$. Define $f : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ by $fx = \ln x$ for all $x \in X$, and let

$$\alpha(x,y) = \begin{cases} 2s^2, & x \neq y, \\ 0, & x = y \end{cases} \quad \text{for any } s \geq 1.$$

Then, f is α_{qs^p}-admissible.

Example 3.4 Let $X = (0, +\infty)$. Define $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $fx = 3x$ for all $x \in X$ and

$$\alpha(x,y) = \begin{cases} 2, & x \neq y, \\ 0, & x = y \end{cases} \quad \text{for all } x, y \in X.$$

Then f is α_{qs^p}-admissible.

Based on the definition of quasi-contraction from Ćirić, we introduce the following definition in the setting of a b-metric-like space.

Definition 3.5 Let (X, σ_b) be a complete b-metric-like space with parameter $s \geq 1$, and let $f : X \to X$ be a given mapping. We say that f is a generalized $\alpha_{qs^p} - \lambda$-quasi-contraction if f is an α_{qs^p}-admissible mapping such that

$$\alpha(x,y)\sigma_b(fx,fy) \leq \lambda \max \left\{ \begin{matrix} \sigma_b(x,y), \sigma_b(x,fx), \sigma_b(y,fy), \sigma_b(x,fy), \\ \sigma_b(y,fx), \sigma_b(x,x), \sigma_b(y,y) \end{matrix} \right\} \tag{3.1}$$

for all $x, y \in X$ and $\lambda \in [0, 1/2)$.

Remark 3.6 If we take $\alpha(x,y) = s^2$ ($p = 2$ and $q = 1$), then the definition reduces to the definition of an $s - \lambda$ quasi-contraction, and if we take $s = 1$, then the definition reduces to the λ-quasi-contraction in the setting of metric spaces.

Theorem 3.7 *Let (X, σ_b) be a complete b-metric-like space with parameter $s \geq 1$, let $f : X \to X$ be a self-mapping, and let $\alpha : X \times X \to R^+$ be a given function. Suppose that the following conditions are satisfied*:

(i) *f is an α_{qs^p}-admissible mapping*;
(ii) *f is an $\alpha_{qs^p} - \lambda$ contractive mapping*;
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq qs^p$*;
(iv) *either f is continuous, or property H_{qs^p} is satisfied.*

Then f has a fixed point. Moreover, f has a unique fixed point if property U_{qs^p} is satisfied.

Proof By hypothesis (iii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq qs^p$. We define the sequence $\{x_n\}$ in X by $x_n = fx_{n-1}$ for all $n \in N$. If $x_n = x_{n+1}$ for some $n \in N$, then $u = x_n$ is a fixed point for f. Consequently, we suppose that $x_n \neq x_{n+1}$ ($\sigma_b(x_n, x_{n+1}) > 0$) for all $n \in N$.

Since f is an α_{qs^p}-admissible mapping, we have

$$\alpha(x_0,x_1) = \alpha(x_0,fx_0) \geq qs^p, \qquad \alpha(fx_0,fx_1) = \alpha(x_1,x_2) \geq qs, \quad \text{and}$$
$$\alpha(fx_1,fx_2) = \alpha(x_2,x_3) \geq qs^p.$$

Hence, by induction we get

$$\alpha(x_n, x_{n+1}) \geq qs^p \quad \text{for all } n \in N.$$

By condition (3.1) we have:

$$\begin{aligned}
&qs^p \sigma_b(x_n, x_{n+1}) \\
&\quad = qs^p \sigma_b(fx_{n-1}, fx_n) \leq \alpha(x_{n-1}, x_n)\sigma_b(fx_{n-1}, fx_n) \\
&\quad \leq \lambda \max \left\{ \begin{matrix} \sigma_b(x_{n-1}, x_n), \sigma_b(x_{n-1}, fx_{n-1}), \sigma_b(x_n, fx_n), \sigma_b(x_{n-1}, fx_n), \\ \sigma_b(x_n, fx_{n-1}), \sigma_b(x_{n-1}, x_{n-1}), \sigma_b(x_n, x_n) \end{matrix} \right\} \\
&\quad = \lambda \max \left\{ \begin{matrix} \sigma_b(x_{n-1}, x_n), \sigma_b(x_{n-1}, x_n), \sigma_b(x_n, x_{n+1}), \sigma_b(x_{n-1}, x_{n+1}), \\ \sigma_b(x_n, x_n), \sigma_b(x_{n-1}, x_{n-1}), \sigma_b(x_n, x_n) \end{matrix} \right\} \\
&\quad \leq \lambda \max \left\{ \begin{matrix} \sigma_b(x_{n-1}, x_n), \sigma_b(x_{n-1}, x_n), \sigma_b(x_n, x_{n+1}), s[\sigma_b(x_{n-1}, x_n) + \sigma_b(x_n, x_{n+1})], \\ 2s\sigma_b(x_n, x_{n-1}), 2s\sigma_b(x_{n-1}, x_n), 2s\sigma_b(x_n, x_{n-1}) \end{matrix} \right\} \\
&\quad = \lambda \max \left\{ \begin{matrix} \sigma_b(x_{n-1}, x_n), \sigma_b(x_{n-1}, x_n), \sigma_b(x_n, x_{n+1}), s[\sigma_b(x_{n-1}, x_n) + \sigma_b(x_n, x_{n+1})], \\ 2s\sigma_b(x_n, x_{n-1}) \end{matrix} \right\}. \qquad (3.2)
\end{aligned}$$

If $\sigma_b(x_{n-1}, x_n) < \sigma_b(x_n, x_{n+1})$ for some $n \in N$, then from inequality (3.2) we have $\sigma_b(x_n, x_{n+1}) \leq 2\lambda/qs^{p-1}\sigma_b(x_n, x_{n+1})$, a contradiction since $2\lambda/qs^{p-1} < 1$.

Hence, for all $n \in N$, $\sigma_b(x_n, x_{n+1}) \leq \sigma_b(x_{n-1}, x_n)$, and also by inequality (3.2) we get

$$\sigma_b(x_n, x_{n+1}) \leq \frac{2\lambda}{qs^{p-1}}\sigma_b(x_{n-1}, x_n). \tag{3.3}$$

Similarly, by the contractive condition of theorem we have:

$$\sigma_b(x_{n-1}, x_n) \leq \frac{2\lambda}{qs^{p-1}}\sigma_b(x_{n-2}, x_{n-1}). \tag{3.4}$$

Generally, from (3.3) and (3.4) we have, for all n,

$$\sigma_b(x_n, x_{n+1}) \leq c\sigma_b(x_{n-1}, x_n) \leq \cdots \leq c^n\sigma_b(x_0, x_1), \tag{3.5}$$

where $0 \leq c = 2\lambda/qs^{p-1} < 1$. Taking limit as $n \to \infty$ in (3.5), we have

$$\sigma_b(x_n, x_{n+1}) \to 0. \tag{3.6}$$

Now we prove that $\{x_n\}$ is a Cauchy sequence. To do this, let $m, n > 0$ be such that $m > n$.

Using Definition 2.4(c), we have

$$\begin{aligned}\sigma_b(x_n,x_m) &\le s\bigl[\sigma_b(x_n,x_{n+1})+\sigma_b(x_{n+1},x_m)\bigr]\\ &\le s\sigma_b(x_n,x_{n+1})+s^2\sigma_b(x_{n+1},x_{n+2})+s^3\sigma_b(x_{n+2},x_{n+3})+\cdots\\ &\le sc^n\sigma_b(x_0,x_1)+s^2c^{n+1}\sigma_b(x_0,x_1)+s^3c^{n+2}\sigma_b(x_0,x_1)+\cdots\\ &= sc^n\sigma_b(x_0,x_1)\bigl[1+sc+(sc)^2+(sc)^3+\cdots\bigr]\\ &\le \frac{sc^n}{1-sc}\sigma_b(x_0,x_1).\end{aligned}$$

Taking the limit as $n,m\to\infty$, we have $\sigma_b(x_n,x_m)\to 0$, since $0\le cs = 2\lambda s/qs^{p-1} = 2\lambda/qs^{p-2}<1$. Therefore $\{x_n\}$ is a Cauchy sequence in the complete b-metric-like space (X,σ_b). Thus there is some $u\in X$ such that $\{x_n\}$ converges to u.

If f is a continuous mapping, then we get:

$$f(u)=f\Bigl(\lim_{n\to\infty}x_n\Bigr)=\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}(x_{n+1})=u.$$

Thus u is a fixed point of f.

On the other hand, if f is not a continuous function and property H_{qs^p} holds, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},u)\ge qs^p$ for all $k\in N$.

Since $\alpha(x_{n_k},u)\ge qs^p$, applying condition (3.1) with $x=x_{n_k}$ and $y=u$, we obtain

$$\begin{aligned}qs^p\sigma_b(x_{n_k+1},fu) &= qs^p\sigma_b(fx_{n_k},fu)\le\alpha(x_{n_k},u)\sigma_b(fx_{n_k},fu)\\ &\le\lambda\max\left\{\begin{array}{c}\sigma_b(x_{n_k},u),\sigma_b(x_{n_k},fx_{n_k}),\sigma_b(u,fu),\sigma_b(x_{n_k},fu),\\ \sigma_b(u,fx_{n_k}),\sigma_b(x_{n_k},x_{n_k}),\sigma_b(u,u)\end{array}\right\}\\ &=\lambda\max\left\{\begin{array}{c}\sigma_b(x_{n_k},u),\sigma_b(x_{n_k},x_{n_k+1}),\sigma_b(u,fu),\\ \sigma_b(x_{n_k},fu),\sigma_b(u,x_{n_k+1}),\sigma_b(x_{n_k},x_{n_k}),\sigma_b(u,u)\end{array}\right\}.\end{aligned}\tag{3.7}$$

Taking the upper limit as $k\to\infty$ in (3.7) and using (3.6), and Lemmas 2.12 and 2.13, we have

$$qs^{p-1}\sigma_b(u,fu)=qs^p\frac{1}{s}\sigma_b(u,fu)\le 2\lambda s\sigma_b(u,fu).\tag{3.8}$$

From (3.8) we get $\sigma_b(u,fu)=0$, which implies that $fu=u$. Hence u is a fixed point of f.

Further, suppose that u and v are two fixed points of f, where $fu=u$ and $fv=v$ for some $u\neq v$. Since property U_{qs^p} is satisfied, we have $\alpha(u,v)\ge qs^p$. Hence, from (3.1) we have

$$\begin{aligned}qs^p\sigma_b(u,v) &= qs^p\sigma_b(fu,fv)\le\alpha(u,v)\sigma_b(fu,fv)\\ &\le\lambda\max\left\{\begin{array}{c}\sigma_b(u,v),\sigma_b(u,fu),\sigma_b(v,fv),\sigma_b(u,fv),\\ \sigma_b(v,fu),\sigma_b(u,u),\sigma_b(v,v)\end{array}\right\}\\ &=\lambda\max\left\{\begin{array}{c}\sigma_b(u,v),\sigma_b(u,u),\sigma_b(v,v),\sigma_b(u,v),\\ \sigma_b(v,u),\sigma_b(u,u),\sigma_b(v,v)\end{array}\right\}\\ &\le 2\lambda s\sigma_b(u,v).\end{aligned}\tag{3.9}$$

So $\sigma_b(u,v) = 0$, and since $0 \le c = 2\lambda/qs^{p-1} < 1$, we get $\sigma_b(u,v) = 0$. Hence the fixed point is unique. □

The following theorem is a version of the Hardy-Rogers result.

Theorem 3.8 *Let (X,σ_b) be a complete b-metric-like space with parameter $s \ge 1$, and let $f : X \to X$ be a given self-mapping. Suppose that there exists a function $\alpha : X \times X \to [0,\infty)$ such that*

$$\alpha(x,y)\sigma_b(fx,fy) \le \alpha_1\sigma_b(x,y) + \alpha_2\sigma_b(x,fx) + \alpha_3\sigma_b(y,fy) + \alpha_4\sigma_b(x,fy) + \alpha_5\sigma_b(y,fx),$$

for all $x,y \in X$ and the constants $a_i \ge 0$, $i = 1,\dots,5$, where $a_1 + a_2 + a_3 + a_4 + a_5 < 1/2$. Assume also that:

(i) *f is an α_{qs^p}-admissible mapping*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \ge qs^p$*;
(iii) *either f is continuous, or property H_{qs^p} is satisfied.*

Then f has a fixed point. Moreover, f has a unique fixed point if property U_{qs^p} is satisfied.

Proof This theorem can be considered as a corollary of Theorem 3.7, since, for all $x,y \in X$, we have

$$\begin{aligned}
&\alpha_1\sigma_b(x,y) + \alpha_2\sigma_b(x,fx) + \alpha_3\sigma_b(y,fy) + \alpha_4\sigma_b(x,fy) + \alpha_5\sigma_b(y,fx)\\
&\quad\le (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\max\bigl\{\sigma_b(x,y),\sigma_b(x,fx),\sigma_b(y,fy),\sigma_b(x,fy),\sigma_b(y,fx)\bigr\}\\
&\quad= k\max\bigl\{\sigma_b(x,y),\sigma_b(x,fx),\sigma_b(y,fy),\sigma_b(x,fy),\sigma_b(y,fx)\bigr\},
\end{aligned}$$

where $0 < k = a_1 + a_2 + a_3 + a_4 + a_5 < 1/2$. □

Corollary 3.9 *Let (X,σ_b) be complete b-metric-like space with parameter $s \ge 1$. If $f : X \to X$ is a self-mapping and there exist constants $a_i \ge 0$, $i = 1,\dots,5$, with $a_1 + a_2 + a_3 + a_4 + a_5 < 1/2$ such that*

$$qs^p\sigma_b(fx,fy) \le \alpha_1\sigma_b(x,y) + \alpha_2\sigma_b(x,fx) + \alpha_3\sigma_b(y,fy) + \alpha_4\sigma_b(x,fy) + \alpha_5\sigma_b(y,fx),$$

for all $x,y \in X$ and a constant $p \ge 2$, then f has a unique fixed point in X.

Proof In Theorem 3.8, take the function $\alpha(x,y) = qs^p$. □

Remark 3.10 Theorem 3.7 generalizes Theorem 18 in [7]. For $\alpha(x,y) = s^2$ and for all $x,y \in X$, Theorems 3.7 and 3.8 reduce to Theorems 3.2 and 3.13 of [19]. In Theorem 3.7 and Corollary 3.9, by choosing the constants a_i in certain manner, we obtain, as particular cases, certain classes of α_{qs^p}-types of Kannan, Chatterjea, Reich, and Zamfirescu contractions.

The notion of $\alpha - \psi$ contractive mappings is defined in a complete metric space in [8]. Thereafter, many authors provided various fixed point theorems for such a class of mappings. In the following definition, we extend and generalize the notions of $\alpha - \psi$ and $(\psi - \phi)$-contractive mappings in the context of larger spaces, such as b-metric-like spaces.

The aim of this section is to extend and generalize the main classical result and other existing results in the literature on b-metric and metric-like spaces.

Let (X,σ_b) be a b-metric-like space with parameter $s \geq 1$. For a self-mapping $f : X \to X$, we define $N(x,y)$ by

$$N(x,y) = \max\left\{\sigma_b(x,y), \sigma_b(x,fx), \sigma_b(y,fy), \frac{\sigma_b(x,fy) + \sigma_b(y,fx)}{4s}\right\} \tag{3.10}$$

for all $x, y \in X$.

The families Ψ, Φ with altering distance functions are defined as follows:

$$\psi : [0,\infty) \to [0,\infty) \quad \text{an increasing and continuous function;}$$

$$\phi : [0,\infty) \to [0,\infty) \quad \text{is continuous, and } \phi(t) < \psi(t) \text{ for all } t > 0.$$

Let $\mathbb{S}$ be the set of all mappings $\beta : [0,\infty) \to [0,1)$ satisfying the condition

$$\beta(t_n) \to 1 \quad \text{as } n \to \infty \text{ implies that } t_n \to 0 \text{ as } n \to \infty.$$

Definition 3.11 Let (X,σ_b) be a b-metric-like space with parameter $s \geq 1$, and let $f : X \to X$ be a self-mapping. Also, let $\alpha : X \times X \to [0,\infty)$ and $q \geq 1$, $p \geq 2$. We say that f is an $(\alpha_{qs^p} - \psi, \phi)$ generalized contractive mapping if there exist $\psi \in \Psi$, $\phi \in \Phi$ such that

$$\psi\bigl(\alpha(x,y)\sigma_b(fx,fy)\bigr) \leq \phi\bigl(N(x,y)\bigr) \tag{3.11}$$

for all $x, y \in X$ with $\alpha(x,y) \geq qs^p$, where $N(x,y)$ is defined by (3.10).

Remark 3.12

(i) Taking $q = 1$ in the definition, we obtain $\alpha_s - (\psi, \phi)$ admissible mappings defined in a b-metric-like space or in a b-metric space.

(ii) Note that, for $\alpha(x,y) = q$, the definition reduces to an α_q-admissible mapping defined in a metric space or in a metric-like space.

(iii) For $s = 1$ and $q = 1$, the definition reduces to the definition of an α-admissible mapping in a metric space.

(iv) The definition reduces to a (ψ, ϕ)-contractive mapping if we take $\alpha(x,y) = 1$.

(v) The definition reduces to an $\alpha_{qs^p} - \phi$ contractive mapping if we take $\psi(t) = t$.

(vi) The definition reduces to an $\alpha_{qs^p} - \lambda$ contractive mapping if we take $\psi(t) = t$ and $\phi(t) = \lambda t$ for $\lambda \in (0,1)$.

We now present the following theorem.

Theorem 3.13 *Let (X,σ_b) be a complete b-metric-like space with parameter $s \geq 1$, and let $f : X \to X$ be an $(\alpha_{qs^p} - \psi, \phi)$ generalized contractive mapping. Suppose that the following conditions are satisfied*:

(i) *f is an α_{qs^p}-admissible mapping*;

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq qs^p$*;

(iii) *either f is continuous, or property H_{qs^p} is satisfied.*

Then f has a fixed point $x \in X$. Moreover, f has a unique fixed point if property U_{qs^p} is satisfied.

Proof By assumption (ii) there exists a point $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq qs^p$. We construct a sequence $\{x_n\}$ in X by $x_n = f^n x_0 = f(x_{n-1})$ for all $n \in N$. If we suppose that $\sigma_b(x_n, x_{n+1}) = 0$ for some n, then $x_{n+1} = x_n$, and the proof is completed, since $u = x_n = x_{n+1} = f(x_n) = fu$. Consequently, throughout the proof, we assume that

$$\sigma_b(x_n, x_{n+1}) > 0 \quad \text{for all } n \in N. \tag{3.12}$$

Since f is an α_{qs^p}-admissible mapping, we observe that

$$\begin{aligned} &\alpha(x_0, x_1) = \alpha(x_0, fx_0) \geq qs^p, \qquad \alpha(fx_0, fx_1) = \alpha(x_1, x_2) \geq qs \quad \text{and} \\ &\alpha(fx_1, fx_2) = \alpha(x_2, x_3) \geq qs^p. \end{aligned}$$

In general, by induction we derive that

$$\alpha(x_n, x_{n+1}) \geq qs^p \quad \text{for all } n \in N. \tag{3.13}$$

By (3.13) and condition (3.11) we have:

$$\begin{aligned} \psi\big(\sigma_b(x_n, x_{n+1})\big) &\leq \psi\big(qs^p \sigma_b(x_n, x_{n+1})\big) = \psi\big(qs^p \sigma_b(fx_{n-1}, fx_n)\big) \\ &\leq \psi\big(\alpha(x_{n-1}, x_n)\sigma_b(fx_{n-1}, fx_n)\big) \\ &\leq \phi\big(N(x_{n-1}, x_n)\big) < \psi\big(N(x_{n-1}, x_n)\big), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} N(x_{n-1}, x_n) &= \max\left\{ \begin{matrix} \sigma_b(x_{n-1}, x_n), \sigma_b(x_{n-1}, fx_{n-1}), \sigma_b(x_n, fx_n), \\ \frac{\sigma_b(x_{n-1}, fx_n) + \sigma_b(x_n, fx_{n-1})}{4s} \end{matrix} \right\} \\ &= \max\left\{ \begin{matrix} \sigma_b(x_{n-1}, x_n), \sigma_b(x_{n-1}, x_n), \sigma_b(x_n, x_{n+1}), \\ \frac{\sigma_b(x_{n-1}, x_{n+1}) + \sigma_b(x_n, x_n)}{4s} \end{matrix} \right\} \\ &\leq \max\left\{ \begin{matrix} \sigma_b(x_{n-1}, x_n), \sigma_b(x_{n-1}, x_n), \sigma_b(x_n, x_{n+1}), \\ \frac{s[\sigma_b(x_{n-1}, x_n) + \sigma_b(x_n, x_{n+1})] + 2s\sigma_b(x_{n-1}, x_n)}{4s} \end{matrix} \right\}. \end{aligned} \tag{3.15}$$

If we assume that, for some $n \in N$,

$$\sigma_b(x_{n-1}, x_n) < \sigma_b(x_n, x_{n+1}),$$

then from inequality (3.15) we get

$$N(x_{n-1}, x_n) \leq \sigma_b(x_n, x_{n+1}). \tag{3.16}$$

Again, by (3.13) and condition (3.11) we have:

$$\begin{aligned}\psi\big(\sigma_b(x_n,x_{n+1})\big) &\le \psi\big(qs^p\sigma_b(x_n,x_{n+1})\big) = \psi\big(qs^p\sigma_b(fx_{n-1},fx_n)\big)\\ &\le \psi\big(\alpha(x_{n-1},x_n)\sigma_b(fx_{n-1},fx_n)\big)\\ &\le \phi\big(N(x_{n-1},x_n)\big) < \psi\big(N(x_{n-1},x_n)\big).\end{aligned} \tag{3.17}$$

By the property ψ inequality (3.17) implies that

$$\sigma_b(x_n,x_{n+1}) \le N(x_{n-1},x_n). \tag{3.18}$$

From (3.16) and (3.18) we have

$$N(x_{n-1},x_n) = \sigma_b(x_n,x_{n+1}). \tag{3.19}$$

From (3.17), using (3.19), we obtain

$$\begin{aligned}\psi\big(\sigma_b(x_n,x_{n+1})\big) &\le \psi\big(qs^p\sigma_b(x_n,x_{n+1})\big) = \psi\big(qs^p\sigma_b(fx_{n-1},fx_n)\big)\\ &\le \psi\big(\alpha(x_{n-1},x_n)\sigma_b(fx_{n-1},fx_n)\big)\\ &\le \phi\big(N(x_{n-1},x_n)\big) = \phi\big(\sigma_b(x_n,x_{n+1})\big)\\ &< \psi\big(\sigma_b(x_n,x_{n+1})\big),\end{aligned} \tag{3.20}$$

which gives a contradiction, since we have assumed that $\sigma_b(x_n,x_{n+1}) > 0$ and $\phi(t) < \psi(t)$ for all $t > 0$. Hence, for all $n \in N$, $\sigma_b(x_n,x_{n+1}) \le \sigma_b(x_{n-1},x_n)$, and the sequence $\{\sigma_b(x_n,x_{n+1})\}$ is decreasing and bounded below. Hence there exists $l \ge 0$ such that $\sigma_b(x_n,x_{n+1}) \to l$. Also,

$$\lim_{n\to\infty} \sigma_b(x_n,x_{n+1}) = \lim_{n\to\infty} N(x_{n-1},x_n) = l.$$

We shall prove that $l = 0$.

Consider

$$\begin{aligned}\psi\big(\sigma_b(x_n,x_{n+1})\big) &\le \psi\big(qs^p\sigma_b(x_n,x_{n+1})\big) = \psi\big(qs^p\sigma_b(fx_{n-1},fx_n)\big)\\ &\le \psi\big(\alpha(x_{n-1},x_n)\sigma_b(fx_{n-1},fx_n)\big)\\ &\le \phi\big(N(x_{n-1},x_n)\big) = \phi\big(\sigma_b(x_n,x_{n+1})\big).\end{aligned} \tag{3.21}$$

If we assume that $l > 0$, taking the limit in (3.21), we have

$$\psi(l) \le \phi(l),$$

which is a contradiction since $\psi(t) > \phi(t)$ for $t > 0$. Hence $l = 0$, and

$$\lim_{n\to\infty} \sigma_b(x_n,x_{n+1}) = \lim_{n\to\infty} N(x_{n-1},x_n) = 0. \tag{3.22}$$

Next, we shall show that $\{x_n\}$ is a Cauchy sequence in X. Suppose, on the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then by Lemma 2.14 there exist $\varepsilon > 0$ and two subsequences

$\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$, with $n_k > m_k > k$, such that

$$\begin{aligned}
&\sigma_b(x_{m_k}, x_{n_k}) \geq \varepsilon, \qquad \sigma_b(x_{m_k}, x_{n_k-1}) < \varepsilon, \\
&\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} \sigma_b(x_{m_k-1}, x_{n_k-1}) \leq \varepsilon s, \\
&\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} \sigma_b(x_{n_k-1}, x_{m_k}) \leq \varepsilon s^2, \quad \text{and} \\
&\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} \sigma_b(x_{m_k-1}, x_{n_k}) \leq \varepsilon s^2.
\end{aligned} \tag{3.23}$$

From the definition of $N(x,y)$ we have

$$\begin{aligned}
N(x_{m_k-1}, x_{n_k-1}) &= \max\left\{\begin{array}{c}\sigma_b(x_{m_k-1}, x_{n_k-1}), \sigma_b(x_{m_k-1}, fx_{m_k-1}), \sigma_b(x_{n_k-1}, fx_{n_k-1}), \\ \frac{\sigma_b(x_{m_k-1}, fx_{n_k-1}) + \sigma_b(x_{n_k-1}, fx_{m_k-1})}{4s}\end{array}\right\} \\
&= \max\left\{\begin{array}{c}\sigma_b(x_{m_k-1}, x_{n_k-1}), \sigma_b(x_{m_k-1}, x_{m_k}), \sigma_b(x_{n_k-1}, x_{n_k}), \\ \frac{\sigma_b(x_{m_k-1}, x_{n_k}) + \sigma_b(x_{n_k-1}, x_{m_k})}{4s}\end{array}\right\}.
\end{aligned} \tag{3.24}$$

Taking the upper limit as $k \to \infty$ in (3.24) and using (3.22), (3.23), we get

$$\begin{aligned}
&\limsup_{k\to\infty} N(x_{m_k-1}, x_{n_k-1}) \\
&\quad = \limsup_{k\to\infty} \max\left\{\begin{array}{c}\sigma_b(x_{m_k-1}, x_{n_k-1}), \sigma_b(x_{m_k-1}, x_{m_k}), \sigma_b(x_{n_k-1}, x_{n_k}), \\ \frac{\sigma_b(x_{m_k-1}, x_{n_k}) + \sigma_b(x_{n_k-1}, x_{m_k})}{4s}\end{array}\right\} \\
&\quad \leq \max\left\{\varepsilon s, 0, 0, \frac{\varepsilon s}{2}\right\} \leq \varepsilon s.
\end{aligned} \tag{3.25}$$

Using the α_{qs^p}-weak contractive condition, we have

$$\begin{aligned}
\psi\left(qs^p \sigma_b(x_{m_k}, x_{n_k})\right) &\leq \psi\left(qs^p \sigma_b(fx_{m_k-1}, fx_{n_k-1})\right) \\
&\leq \psi\left(\alpha(x_{m_k-1}, x_{n_k-1}) \sigma_b(fx_{m_k-1}, fx_{n_k-1})\right) \\
&\leq \phi\left(N(x_{m_k-1}, x_{n_k-1})\right).
\end{aligned} \tag{3.26}$$

Taking the upper limit in (3.26), using (3.23) and (3.25), we obtain

$$\begin{aligned}
\psi(\varepsilon s) &\leq \psi\left(q\varepsilon s^{p-1}\right) = \psi\left(qs^p \frac{\varepsilon}{s}\right) \leq \psi\left(\limsup_{k\to\infty} \sigma_b(x_{m_k}, x_{n_k})\right) \\
&\leq \phi\left(\limsup_{k\to\infty}\left(N(x_{m_k-1}, x_{n_k-1})\right)\right) \leq \phi(\varepsilon s) \\
&< \psi(\varepsilon s),
\end{aligned}$$

which is a contradiction, since $\varepsilon > 0$. Therefore $\{x_n\}$ is a Cauchy sequence in the complete b-metric-like space (X, σ_b). Thus, there is some $u \in X$ such that $\{x_n\}$ converges to u. If f is a continuous mapping, we get:

$$f(u) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (x_{n+1}) = u,$$

and u is a fixed point of f.

If the self-map f is not continuous, then from (3.13) and condition H_{qs^p}, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},u) \geq qs^p$ for all $k \in N$. Since $\alpha(x_{n_k},u) \geq qs^p$, applying contractive condition (3.11), with $x = x_{n_k}$ and $y = u$, we obtain

$$\begin{aligned}\psi\big(qs^p\sigma_b(x_{n_k+1},fu)\big) &= \psi\big(qs^p\sigma_b(fx_{n_k},fu)\big)\\ &\leq \psi\big(\alpha(x_{n_k},u)\sigma_b(fx_{n_k},fu)\big)\\ &\leq \phi\big(N(x_{n_k},u)\big),\end{aligned} \tag{3.27}$$

where

$$\begin{aligned}N(x_{n_k},u) &= \max\left\{\begin{matrix}\sigma_b(x_{n_k},u),\sigma_b(x_{n_k},fx_{n_k}),\sigma_b(u,fu),\\ \frac{\sigma_b(x_{n_k},fu)+\sigma_b(u,fx_{n_k})}{4s}\end{matrix}\right\}\\ &= \max\left\{\begin{matrix}\sigma_b(x_{n_k},u),\sigma_b(x_{n_k},x_{n_k+1}),\sigma_b(u,fu),\\ \frac{\sigma_b(x_{n_k},fu)+\sigma_b(u,x_{n_k+1})}{4s}\end{matrix}\right\}.\end{aligned} \tag{3.28}$$

Taking the upper limit in (3.28) and using Lemma 2.13 and result (3.22), we obtain

$$\limsup_{n\to\infty} N(x_{n_k},u) \leq \max\left\{0,0,\sigma_b(u,fu),\frac{s\sigma_b(u,fu)}{4s}\right\} = \sigma_b(u,fu). \tag{3.29}$$

Taking the upper limit as $k \to \infty$ in (3.27) and using (3.29) and Lemma 2.13, we obtain

$$\begin{aligned}\psi\big(qs^{p-1}\sigma_b(u,fu)\big) = \psi\left(qs^p\frac{1}{s}\sigma_b(u,fu)\right) &\leq \psi\Big(qs^p\limsup_{k\to\infty}\sigma_b(x_{n_k},fu)\Big)\\ &\leq \phi\Big(\limsup_{k\to\infty} N(x_{n_k},u)\Big) < \psi\Big(\limsup_{k\to\infty} N(x_{n_k},u)\Big)\\ &\leq \psi\big(\sigma_b(u,fu)\big).\end{aligned} \tag{3.30}$$

From (3.30) we get $\sigma_b(u,fu) = 0$, which implies that $fu = u$. Hence u is a fixed point of f .

Suppose that u and v are two fixed points of f, where $fu = u$ and $fv = v$ are such that $u \neq v$. Then, by hypothesis U_{qs^p}, $\alpha(u,v) \geq qs^p$, and applying (3.11), we have

$$\begin{aligned}\psi\big(qs^p\sigma_b(u,u)\big) &= \psi\big(qs^p\sigma_b(fu,fu)\big) \leq \psi\big(\alpha(u,u)\sigma_b(fu,fu)\big)\\ &\leq \phi\big(N(u,u)\big) \leq \phi\big(\sigma_b(u,u)\big),\end{aligned} \tag{3.31}$$

where

$$N(u,u) = \max\left\{\sigma_b(u,u),\sigma_b(u,u),\sigma_b(u,u),\frac{\sigma_b(u,u)+\sigma_b(u,u)}{4s}\right\} = \sigma_b(u,u).$$

From inequality (3.31) it follows that $\sigma_b(u,u) = 0$ (also $\sigma_b(v,v) = 0$).

Again we have

$$\begin{aligned}\psi\big(qs^p\sigma_b(u,v)\big) &= \psi\big(qs^p\sigma_b(fu,fv)\big) \leq \psi\big(\alpha(u,v)\sigma_b(fu,fv)\big)\\ &\leq \phi\big(N(u,v)\big) \leq \phi\big(\sigma_b(u,v)\big),\end{aligned} \tag{3.32}$$

where $N(u,v) = \sigma_b(u,v)$.

Inequality (3.32) implies that $\sigma_b(u,v) = 0$. Therefore $u = v$, and the fixed point is unique. □

Remark 3.14 Our theorem extends Theorems 2.1, 2.2, and 2.7 of Aydi et al. [9].

By taking $\phi(t) = \psi(t) - \varphi(t)$, where $\varphi \in \Psi$, in Theorem 3.13 we obtain the following result.

Corollary 3.15 *Let (X,σ_b) be a complete b-metric-like space with parameter $s \geq 1$, let $f : X \to X$ be a self-mapping, and let $\alpha : X \times X \to [0,\infty)$. Suppose that the following conditions are satisfied:*

(i) *f is an α_{qs^p}-admissible mapping;*
(ii) *there exist functions $\psi, \varphi \in \Psi$ such that*

$$\psi\left(\alpha(x,y)\sigma_b(fx,fy)\right) \leq \psi\big(N(x,y)\big) - \varphi\big(N(x,y)\big);$$

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq qs^p$;*
(iv) *either f is continuous, or property H_{qs^p} is satisfied.*

Then f has a fixed point $x \in X$. Moreover, f has a unique fixed point if property U_{qs^p} is satisfied.

Remark 3.16 This corollary extends Theorems 3 and 4 of Roshan et al. [25].

By taking $\psi(t) = t$ and $\phi(t) = \beta(t)t$ where $\beta \in \mathbb{S}$ is as in Theorem 3.13, we obtain the following result.

Corollary 3.17 *Let (X,σ_b) be a complete b-metric-like space with parameter $s \geq 1$, let $f : X \to X$ be a self-mapping, and let $\alpha : X \times X \to [0,\infty)$. Suppose that the following conditions are satisfied:*

(i) *f is an α_{qs^p}-admissible mapping;*
(ii) *there exist functions $\psi, \varphi \in \Psi$ such that*

$$\psi\left(\alpha(x,y)\sigma_b(fx,fy)\right) \leq \beta\big(N(x,y)\big)\big(N(x,y)\big);$$

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq qs^p$;*
(iv) *either f is continuous, or property H_{qs^p} is satisfied.*

Then f has a fixed point $x \in X$. Moreover, f has a unique fixed point if property U_{qs^p} is satisfied.

If we take $\psi(t) = t$ in Theorem 3.13, then we get the following result.

Corollary 3.18 *Let (X,σ_b) be a complete b-metric-like space with parameter $s \geq 1$, let $f : X \to X$ be a self-mapping, and let $\alpha : X \times X \to [0,\infty)$. Suppose that the following conditions are satisfied:*

(i) *f is an α_{qs^p}-admissible mapping;*
(ii) *there exist functions $\varphi \in \Psi$ such that*

$$\alpha(x,y)\sigma_b(fx,fy) \leq \varphi\big(N(x,y)\big);$$

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0,fx_0) \geq qs^p$;
(iv) *either* f *is continuous, or property* H_{qs^p} *is satisfied.*

Then f *has a fixed point* $x \in X$. *If property* U_{qs^p} *is satisfied, then* f *has a unique fixed point.*

Remark 3.19 Corollary 3.18 generalizes and extends Theorem 2.7 of Samet et al. [8].

Corollary 3.20 *Let* (X,σ_b) *be a complete b-metric-like space with parameter* $s \geq 1$, *let* $f : X \to X$ *be a self-mapping, and let* $\alpha : X \times X \to [0,\infty)$. *Suppose that the following conditions are satisfied*:

(i) f *is an* α_{qs^p}*-admissible mapping*;
(ii) *there exists a function* $\varphi \in \Psi$ *such that*

$$\alpha(x,y)\sigma_b(fx,fy) \leq N(x,y) - \varphi\bigl(N(x,y)\bigr);$$

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0,fx_0) \geq qs^p$;
(iv) *either* f *is continuous, or property* H_{qs^p} *is satisfied.*

Then f *has a fixed point* $x \in X$. *Moreover,* f *has a unique fixed point if property* U_{qs^p} *is satisfied.*

Proof It follows from Corollary 3.15 by taking $\psi(t) = t$. □

Remark 3.21 Corollary 3.20 generalizes Theorem 2.7 of Harandi [5].

Corollary 3.22 *Let* (X,σ_b) *be a complete b-metric like space with parameter* $s \geq 1$, *and let* f,g *be two selfmaps of* X *with* $\psi \in \Psi$, $\varphi \in \Phi$ *satisfying the condition*

$$\psi\bigl(\alpha_{qs^p}\sigma_b(fx,fy)\bigr) \leq \lambda\psi\bigl(M(x,y)\bigr)$$

for all $x,y \in X$, *where* $M(x,y)$ *is defined in* (3.15), *and* $q > 1$. *Then* f *and* g *have a unique common fixed point in* X.

Proof In Theorem 3.13, take $\varphi(t) = \lambda\psi(t)$ where $0 < \lambda < 1$. □

Corollary 3.23 *Let* (X,σ_b) *be a complete b-metric-like space with parameter* $s \geq 1$, *and let* $f : X \to X$ *be a self-mapping such that, for all* $x,y \in X$ *and any arbitrary coefficient* $p \geq 1$,

$$qs^p\sigma_b(fx,fy) \leq k\max\left\{\sigma_b(x,y),\sigma_b(x,fx),\sigma_b(y,fy),\frac{\sigma_b(x,fy)+\sigma_b(y,fx)}{4s}\right\},$$

where $k \in (0,1)$. *Then* f *has a unique fixed point.*

Proof It follows from Corollary 3.15 by taking $\alpha(x,y) = qs^p$, $\psi(t) = t$, and $\varphi(t) = (1-k)t$ for all $t \in [0,\infty)$ and $k \in (0,1)$. □

Remark 3.24 It is clear that we can derive several corresponding results by replacing the b-metric-like space with some other spaces such as a b-metric space, a metric space, a metric-like space, and a partial metric space. Conditions (3.1) and (3.12) are more general than the analogues in the previous literature, and theorems related to those conditions have a more general character because of the parameter s and arbitrary coefficients q, p.

3.1 Application

In this section, we will use Corollary 3.23 to show that there is a solution to the following integral equation:

$$x(t) = \int_0^T G\big(t,r,x(r)\big)\,dr. \tag{3.33}$$

Let $X = C([0,T])$ be the set of real continuous functions defined on $[0,T]$ for $T > 0$.

We endow X with

$$\sigma_b(x,y) = \max_{t\in[0,1]} \big(\big|x(t)\big| + \big|y(t)\big|\big)^m \quad \text{for all } x,y \in X.$$

It is evident that (X,σ_b) is a complete b-metric-like space with parameter $s = 2^{m-1}$ with $m > 1$.

Consider the mapping $f : X \to X$ defined by $fx(t) = \int_0^T G(t,r,x(r))\,dr$.

Theorem 3.25 *Consider equation* (3.33) *and suppose that*

(a) $G : [0,T] \times [0,T] \times R \to R^+ = [0,\infty)$ (*that is*, $G(t,r,x(r)) \geq 0$) *is continuous*;
(b) *there exists a continuous* $\gamma : [0,T] \times [0,T] \to R$;
(c) $\sup_{t\in[0,T]} \int_0^T \gamma(t,r)\,dr \leq 1$;
(d) *there exists a constant* $L \in (0,1)$ *such that, for all* $(t,r) \in [0,T]^2$ *and* $x,y \in R$,

$$\big|G\big(t,r,x(r)\big) + G\big(t,r,y(r)\big)\big| \leq \left(\frac{L}{s^3}\right)^{\frac{1}{m}} \gamma(t,r)\big(\big|x(r)\big| + \big|y(r)\big|\big).$$

Then the integral equation (3.33) *has a unique solution in* $x \in X$.

Proof For $x,y \in X$, from conditions (c) and (d), for all t, we have

$$\begin{aligned}
qs\sigma_b\big(fx(t),fy(t)\big) &= qs\big(\big|fx(t)\big| + \big|fy(t)\big|\big)^m \\
&= qs\left(\left|\int_0^T G\big(t,r,x(r)\big)\,dr\right| + \left|\int_0^T G\big(t,r,y(r)\big)\,dr\right|\right)^m \\
&\leq qs\left(\int_0^T \big|G\big(t,r,x(r)\big)\big|\,dr + \int_0^T \big|G\big(t,r,y(r)\big)\big|\,dr\right)^m \\
&\leq qs\left(\int_0^T \left(\frac{L}{s^3}\right)^{\frac{1}{m}} \gamma(t,r)\Big(\Big(\big(\big|x(r) + y(r)\big|\big)^m\Big)^{\frac{1}{m}}\Big)\,dr\right)^m \\
&\leq qs\left(\int_0^T \left(\frac{L}{s^3}\right)^{\frac{1}{m}} \gamma(t,r)\sigma_b^{\frac{1}{m}}\big(x(r),y(r)\big)\,dr\right)^m \\
&\leq qs \cdot \frac{L}{s^3}\sigma_b\big(x(r),y(r)\big)\left(\int_0^T \gamma(t,r)\,dr\right)^m \\
&= \frac{qL}{s^2}\sigma_b\big(x(r),y(r)\big)\left(\int_0^T \gamma(t,r)\,dr\right)^m \\
&\leq \frac{qL}{s^2}\sigma_b\big(x(r),y(r)\big),
\end{aligned}$$

which implies that

$$s\sigma_b\big(fx(t),fy(t)\big) \le \frac{L}{s^2}\sigma_b\big(x(r),y(r)\big)$$
$$\le k\max\left\{\sigma_b(x,y),\sigma_b(x,Tx),\sigma_b(y,Ty)\frac{\sigma_b(x,Ty)+\sigma_b(y,Tx)}{4s}\right\},$$

where $k = L/s^2 \in (0,1)$.

Therefore, all of the conditions of Corollary 3.23 are satisfied, and, as a result, the mapping f has a unique fixed point in X, which is a solution of the integral equation (3.33). □

4 Conclusions

In this paper, the class of α_{qs^p}-admissible mappings is introduced in a larger structure such as a b-metric-like space. Some fixed point results dealing with $(\alpha-\psi,\phi)$ contractions are obtained, and they cover and unify a huge number of published results in the related literature.

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