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Fixed point theorems for a class of generalized weak cyclic compatible contractions

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Abstract

In this manuscript, we establish a coincidence point and a unique common fixed point theorem for (ψ,φ)-weak cyclic compatible contractions. We also present a fixed point theorem for a class of Λ-*weak cyclic compatible contractions* via altering distance functions. Our results extend and improve some well-known results in the literature. We provide examples to analyze and illustrate our main results.

MSC: 54H25; 47H10

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1 Introduction and preliminaries

The Banach contraction principle [1] is one of the most powerful and useful tools in modern analysis. Over time, this principle has been extended and improved in many ways and a variety of fixed point theorems have been obtained. In 2003, one of the more notable generalizations of the Banach contraction principle was introduced via cyclic contraction by Kirk et al. [2]. Following the publication of [2], many fixed point theorems for cyclic contractive mappings have been obtained. For more results on cyclic maps, we refer the reader to [3–9] and the references therein. Here, we present some essential definitions.

Definition 1.1 (see [2]) Let A, B be non-empty subsets of a set X and let $\mathcal{U} : A\cup B \to A\cup B$. $\mathcal{U}$ is called a cyclic map, if $\mathcal{U}(A) \subseteq B$ and $\mathcal{U}(B) \subseteq A$.

Throughout this manuscript, we assume that $\mathbb{R}^+ = [0,\infty)$, $\mathbb{N}$ = the set of all positive integers.

Definition 1.2 (see [10]) Let (X,d) be a complete metric space and let $\mathcal{U},\mathcal{V} : X \to X$ be self-mappings. Then $\mathcal{U}$ and $\mathcal{V}$ are said to be weakly compatible if $\mathcal{U}x = \mathcal{V}x$ implies $\mathcal{U}\mathcal{V}x = \mathcal{V}\mathcal{U}x$.

Definition 1.3 (see [11]) A function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance function if the following conditions are satisfied:

1. $\eta(0) = 0$;

2. η is monotonically nondecreasing;
3. η is continuous.

We will denote the set of all altering distance functions by Λ.

The concepts of (ψ,φ)-weak contractions, weakly compatible maps and altering distance functions is interesting, though brief. These concepts were widely used in the construction of existence theorems and many results, a number of applications have been obtained; see [12–17] for examples.

Below, we provide necessary definitions.

Definition 1.4 (see [6]) Let X be a non-empty set and $\{d_\alpha : \alpha \in (0,1]\}$ a family of the mapping d_α of $X \times X$ into $\mathbb{R}^+$. Then (X, d_α) is called a generating space of a b-quasi-metric family (abbreviated as G_{bq}-family), if it satisfies the following conditions, for any $x, y, z \in X$ and $s \geq 1$:

(a) $d_\alpha(x,y) = 0$ if and only if $x = y$.
(b) $d_\alpha(x,y) = d_\alpha(y,x)$.
(c) For any $\alpha \in (0,1]$ there exists $\beta \in (0,\alpha]$ such that $d_\alpha(x,z) \leq s[d_\beta(x,y) + d_\beta(y,z)]$.
(d) For any $x, y \in X, d_\alpha(x,y)$ is non-increasing and left continuous in α.

Definition 1.5 (see [6]) Let X be a non-empty set and $\{d_\alpha : \alpha \in (0,1]\}$ a family of the mapping d_α of $X \times X$ into $\mathbb{R}^+$. Then (X, d_α) is called a generating space of b-dislocated metric family (abbreviated as G_{bd}-family), if it satisfies the following conditions, for any $x, y, z \in X$ and $s \geq 1$:

(a) $d_\alpha(x,y) = 0$ implies $x = y$.
(b) $d_\alpha(x,y) = d_\alpha(y,x)$.
(c) For any $\alpha \in (0,1]$ there exists $\beta \in (0,\alpha]$ such that $d_\alpha(x,z) \leq s[d_\beta(x,y) + d_\beta(y,z)]$.
(d) For any $x, y \in X, d_\alpha(x,y)$ is non-increasing and left continuous in α.

The construction of topological concepts of the above spaces can be found in [6, 18]. Recently, Kumari and Panthi [7] introduced *cyclic compatible contractions* and established fixed point theorems in the generating space of a b-quasi-metric family (X, d_α).

Definition 1.6 (see [7]) Let A, B be non-empty subsets of a G_{bq}-family (X, d_α) and let $\mathcal{U}, \mathcal{V} : A \cup B \to A \cup B$ be cyclic mappings such that $\mathcal{U}(X) \subset \mathcal{V}(X)$. Then $\mathcal{U}, \mathcal{V}$ are said to be *cyclic compatible contraction*, if for some $x \in A$, there exists a $\gamma \in (0,1)$ such that

$$d_\alpha(\mathcal{U}^{2n}x, \mathcal{U}y) \leq \gamma d_\alpha(\mathcal{U}^{2n-1}x, \mathcal{V}y),$$

for all $n \in \mathbb{N}$ and $y \in B$.

In this paper, motivated and inspired by the above definitions, we investigate the weak *cyclic compatible contractions* via (ψ,φ)-weak contractions and altering distance functions.

2 Main results

We begin this section by introducing the following definition.

Definition 2.1 Let A and B be non-empty subsets of a G_{bd}-family (X, d_α). Suppose $\mathcal{U}, \mathcal{V} : A \cup B \to A \cup B$ are cyclic mappings with $\mathcal{U}(X) \subset \mathcal{V}(X)$. Then $\mathcal{U}$, $\mathcal{V}$ are called (ψ, φ)-*weak cyclic compatible contractions*, if for some $x \in A$

$$\psi\left(d_\alpha\left(\mathcal{U}^{2n}x, \mathcal{U}y\right)\right) \leq \psi\left(d_\alpha\left(\mathcal{U}^{2n-1}x, \mathcal{V}y\right)\right) - \varphi\left(d_\alpha\left(\mathcal{U}^{2n-1}x, \mathcal{V}y\right)\right), \tag{1}$$

where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$ and $n \in \mathbb{N}, y \in B$.

We state and prove our main results.

Theorem 2.2 *Let A and B be non-empty closed subsets of a complete G_{bd}-family (X, d_α) and let $\mathcal{U}, \mathcal{V} : A \cup B \to A \cup B$ be cyclic mappings with $\mathcal{U}(X) \subset \mathcal{V}(X)$ and $\mathcal{V}(X)$ closed in X. Suppose $\mathcal{U}$, $\mathcal{V}$ are (ψ, φ)*-weak cyclic compatible contractions, *then $\mathcal{U}$ and $\mathcal{V}$ have a point of coincidence and a unique common fixed point in $A \cap B$.*

Proof Let $x_0 \in X$ be fixed. As $\mathcal{U}(X) \subset \mathcal{V}(X)$, we may choose $x_1 \in X$ such that

$$\mathcal{U}x_0 = \mathcal{V}x_1.$$

Hence we can define the sequence $\{x_n\}$ in X by $\mathcal{U}^n x_0 = \mathcal{U}x_n = \mathcal{V}x_{n+1} = x_n$ for $n \in \mathbb{N} \cup \{0\}$.

Now consider

$$\begin{aligned}
\psi\left(d_\alpha(x_{2n}, x_{2n+1})\right) &= \psi\left(d_\alpha\left(\mathcal{U}^{2n}x_0, \mathcal{U}x_{2n+1}\right)\right) \\
&\leq \psi\left(d_\alpha\left(\mathcal{U}^{2n-1}x_0, \mathcal{V}x_{2n+1}\right)\right) - \varphi\left(d_\alpha\left(\mathcal{U}^{2n-1}x_0, \mathcal{V}x_{2n+1}\right)\right) \\
&= \psi\left(d_\alpha(x_{2n-1}, x_{2n})\right) - \varphi\left(d_\alpha(x_{2n-1}, x_{2n})\right) \\
&\leq \psi\left(d_\alpha(x_{2n-1}, x_{2n})\right).
\end{aligned} \tag{2}$$

This implies that

$$d_\alpha(x_{2n}, x_{2n+1}) \leq d_\alpha(x_{2n-1}, x_{2n}).$$

Similarly, we have

$$d_\alpha(x_{2n+1}, x_{2n+2}) \leq d_\alpha(x_{2n}, x_{2n+1}).$$

Inductively, we have

$$d_\alpha(x_n, x_{n+1}) \leq d_\alpha(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}.$$

Thus the sequence $\{d_\alpha(x_n, x_{n+1})\}$ is non-increasing and hence it is convergent. So, there exists $\kappa \geq 0$ such that $\lim_{n\to\infty} d_\alpha(x_n, x_{n+1}) = \kappa$.

From (2), we have

$$\psi\left(d_\alpha(x_{2n}, x_{2n+1})\right) \leq \psi\left(d_\alpha(x_{2n-1}, x_{2n})\right) - \varphi\left(d_\alpha(x_{2n-1}, x_{2n})\right). \tag{3}$$

Now taking the limit as $n \to \infty$, we get

$$\psi(\kappa) \le \psi(\kappa) - \varphi(\kappa).$$

This is a contradiction, unless $\kappa = 0$. Thus

$$\lim_{n\to\infty} d_\alpha(x_n, x_{n+1}) = 0. \tag{4}$$

For $n, m \in \mathbb{N}, m > n$, consider

$$\begin{aligned} d_\alpha\big(\mathcal{U}^n x_0, \mathcal{U}^m x_0\big) &= d_\alpha(x_n, x_m) \\ &\le s\big[d_\beta(x_n, x_{n+1}) + d_\beta(x_{n+1}, x_m)\big] \\ &\le s d_\beta(x_n, x_{n+1}) + s^2 d_\beta(x_{n+1}, x_{n+2}) + s^3 d_\beta(x_{n+2}, x_{n+3}) + \cdots. \end{aligned} \tag{5}$$

Letting $n, m \to \infty$, we get

$$\lim_{n,m\to\infty} d_\alpha\big(\mathcal{U}^n x_0, \mathcal{U}^m x_0\big) = 0.$$

Therefore $\{\mathcal{U}^n x_0\}$ is a Cauchy sequence. Since (X, d_α) is a complete G_{bd}-family, there exist sequences $\{\mathcal{U}^{2n} x_0\}$ in A and $\{\mathcal{U}^{2n-1} x_0\}$ in B such that $\lim_{n\to\infty} \mathcal{U}^{2n} x_0 \to u$ and $\lim_{n\to\infty} \mathcal{U}^{2n-1} x_0 \to u$. Since A and B are closed in X, $u \in A \cap B$. Since $\mathcal{V}(X)$ is closed in X, there exists z in X such that $\mathcal{V}z = u$. Now we shall prove that $\mathcal{U}z = u$. Consider

$$\begin{aligned} \psi\big(d_\alpha\big(\mathcal{U}^{2n} x_0, \mathcal{U}z\big)\big) &\le \psi\big(d_\alpha\big(\mathcal{U}^{2n-1} x_0, \mathcal{V}z\big)\big) - \varphi\big(d_\alpha\big(\mathcal{U}^{2n-1} x_0, \mathcal{V}z\big)\big) \\ &\le \psi\big(d_\alpha\big(\mathcal{U}^{2n-1} x_0, \mathcal{V}z\big)\big). \end{aligned} \tag{6}$$

By taking the limit as $n \to \infty$, we get $\psi(d_\alpha(u, \mathcal{U}z)) = 0$. Thus $d_\alpha(u, \mathcal{U}z) = 0$. Then $\mathcal{U}z = u$. Hence $\mathcal{U}z = \mathcal{V}z = u$. So u is a coincidence point of $\mathcal{U}$ and $\mathcal{V}$. From weak compatibility, we get

$$\mathcal{U}u = \mathcal{V}u. \tag{7}$$

Now we prove that $\mathcal{V}u = u = \mathcal{U}u$. We assume $u \neq \mathcal{U}u$, then

$$\begin{aligned} \psi\big(d_\alpha(u, \mathcal{U}u,)\big) &= \lim_{n\to\infty} \psi\big(d_\alpha\big(\mathcal{U}^{2n} x_0, \mathcal{U}u\big)\big) \\ &\le \lim_{n\to\infty}\big(\psi\big(d_\alpha\big(\mathcal{U}^{2n-1} x_0, \mathcal{V}u\big)\big) - \varphi\big(d_\alpha\big(\mathcal{U}^{2n-1} x_0, \mathcal{V}u\big)\big)\big) \\ &= \psi\big(d_\alpha(u, \mathcal{V}u) - \varphi\big(d_\alpha(u, \mathcal{V}u)\big) \\ &\le \psi\big(d_\alpha(u, \mathcal{U}u)\big). \end{aligned} \tag{8}$$

That is a contradiction. Therefore

$$u = \mathcal{U}u. \tag{9}$$

From (7) and (9), we have $\mathcal{U}u = \mathcal{V}u = u$. Therefore u is a common fixed point of $\mathcal{U}$ and $\mathcal{V}$. To prove uniqueness, suppose v is another fixed point of $\mathcal{U}$ and $\mathcal{V}$ such that $u \neq v$, then

$$\begin{aligned}\psi\bigl(d_\alpha(u,v)\bigr) &\leq \lim_{n\to\infty}\bigl(\psi\bigl(d_\alpha\bigl(\mathcal{U}^{2n}x_0, \mathcal{U}v\bigr)\bigr)\bigr) \\ &\leq \lim_{n\to\infty}\bigl(\psi\bigl(d_\alpha\bigl(\mathcal{U}^{2n-1}x_0, \mathcal{V}v\bigr)\bigr) - \varphi\bigl(d_\alpha\bigl(\mathcal{U}^{2n-1}x_0, \mathcal{V}v\bigr)\bigr)\bigr) \\ &= \psi\bigl(d_\alpha(u,v)\bigr) - \varphi\bigl(d_\alpha(u,v)\bigr) \\ &< \psi\bigl(d_\alpha(u,v)\bigr). \end{aligned} \tag{10}$$

This is a contradiction. Hence $u = v$. This completes our proof. □

Remark 2.3 We will obtain special cases of Theorem 2.2, if we

1. replace a G_{bq}-family (X, d_α) by a G_q-family, according to Definition 1.4 above, by putting $s = 1$;
2. replace a G_{bq}-family (X, d_α) by a b_d-metric space, and taking d instead of d_α;
3. replace a G_{bq}-family (X, d_α) by a complete dislocated metric space, by taking d instead of d_α and letting $s = 1$.

For more details of G_q-family, b_d-metric space and a complete dislocated metric space, we refer the reader to [6].

Example 2.4 Let $A = B = X = [0,1]$. Define $b_d : X \times X \to \mathbb{R}^+$ by $b_d(x,y) = (x+y)^2$. Then (X, b_d) is a b_d-metric space with $s = 2$, but not a dislocated metric space. Define

$$\mathcal{U}x = 0, \quad \text{if } 0 \leq x \leq 1$$

and

$$\mathcal{V}x = \begin{cases} 0, & \text{if } 0 \leq x < \frac{1}{2}, \\ \frac{1}{3}, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Clearly $\mathcal{U}(X) \subset \mathcal{V}(X)$. Define $\psi(t) = 2t$ and $\varphi(t) = t$.

Fix any $x \in [0,1]$, we have

$$\mathcal{U}x = \mathcal{U}^2x = \cdots = \mathcal{U}^n x = 0, \quad \forall n.$$

For any $y \in [0,1]$,

$$\mathcal{V}y = \begin{cases} 0, & \text{if } 0 \leq y < \frac{1}{2}, \\ \frac{1}{3}, & \text{if } \frac{1}{2} \leq y \leq 1. \end{cases}$$

Consider

Case (i). If $0 \leq y < \frac{1}{2}$, $\mathcal{V}y = 0$, we get

$$\begin{aligned}\psi\bigl(b_d\bigl(\mathcal{U}^{2n-1}x, \mathcal{V}y\bigr)\bigr) - \varphi\bigl(b_d\bigl(\mathcal{U}^{2n-1}x, \mathcal{V}y\bigr)\bigr) &= \psi\bigl(b_d(0,0)\bigr) - \varphi\bigl(b_d(0,0)\bigr) \\ &= 0 = \psi\bigl(b_d\bigl(\mathcal{U}^{2n}x, \mathcal{U}y\bigr)\bigr).\end{aligned}$$

Case (ii). If $\frac{1}{2} \leq y \leq 1$, $\mathcal{U}y = 0$, $\mathcal{V}y = \frac{1}{3}$, we get

$$\psi\big(b_d\big(\mathcal{U}^{2n}x, \mathcal{U}y\big)\big) = \psi\big(b_d(0,0)\big) = 0$$

and

$$\begin{aligned}\psi\big(b_d\big(\mathcal{U}^{2n-1}x, \mathcal{V}y\big)\big) - \varphi\big(b_d\big(\mathcal{U}^{2n-1}x, \mathcal{V}y\big)\big) &= \psi\left(b_d\left(0, \frac{1}{3}\right)\right) - \varphi\left(b_d\left(0, \frac{1}{3}\right)\right) \\ &= \psi\left(\frac{1}{9}\right) - \varphi\left(\frac{1}{9}\right) \\ &= \frac{2}{9} - \frac{1}{9} \\ &= \frac{1}{9}.\end{aligned}$$

From both cases, we obtain

$$\psi\big(b_d\big(\mathcal{U}^{2n}x, \mathcal{U}y\big)\big) \leq \psi\big(b_d\big(\mathcal{U}^{2n-1}x, \mathcal{V}y\big)\big) - \varphi\big(b_d\big(\mathcal{U}^{2n-1}x, \mathcal{V}y\big)\big).$$

Thus $\mathcal{U}$, $\mathcal{V}$ are (ψ, φ)-*weak cyclic compatible contractions*. All the conditions of Theorem 2.2 hold true, and $\mathcal{U}$ and $\mathcal{V}$ have a unique common fixed point. Here $u = 0$ is the unique common fixed point of $\mathcal{U}$ and $\mathcal{V}$.

Example 2.5 Let $A = B = X = [0, 20]$. Define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) = x + y$. Then (X, d) is a complete dislocated metric space.

Define

$$\mathcal{U}x = \begin{cases} 0, & \text{if } x \in \{0\} \cup (3, 20], \\ 2, & \text{if } 0 < x \leq 3, \end{cases}$$

and

$$\mathcal{V}x = \begin{cases} 0, & \text{if } x = 0, \\ x + 10, & \text{if } 0 < x \leq 3, \\ x - 2, & \text{if } 3 < x \leq 20. \end{cases}$$

Clearly, $\mathcal{U}(X) \subset \mathcal{V}(X)$. Define $\psi(t) = 2t$ and $\varphi(t) = t$.

Fix any $x \in (3, 20]$, we have

$$\mathcal{U}x = \mathcal{U}^2x = \cdots = \mathcal{U}^n x = 0, \quad \forall n.$$

For any $y \in [0, 20]$.

$$\mathcal{V}y = \begin{cases} 0, & \text{if } y = 0, \\ y + 10, & \text{if } 0 < y \leq 3, \\ y - 2, & \text{if } 3 < y \leq 20. \end{cases}$$

Consider

Case (i). If $y = 0$, $\mathcal{U}y = 0$, $\mathcal{V}y = 0$, we have

$$\psi\big(d\big(\mathcal{U}^{2n}x,\mathcal{U}y\big)\big) = \psi\big(d(0,0)\big) = 0$$

and

$$\begin{aligned}\psi\big(d\big(\mathcal{U}^{2n-1}x,\mathcal{V}y\big)\big) - \varphi\big(d\big(\mathcal{U}^{2n-1}x,\mathcal{V}y\big)\big) &= \psi\big(d(0,0)\big) - \varphi\big(d(0,0)\big)\\ &= 0.\end{aligned}$$

Hence

$$\psi\big(d\big(\mathcal{U}^{2n}x,\mathcal{U}y\big)\big) = \psi\big(d\big(\mathcal{U}^{2n-1}x,\mathcal{V}y\big)\big) - \varphi\big(d\big(\mathcal{U}^{2n-1}x,\mathcal{V}y\big)\big).$$

Case (ii). If $0 < y \leq 3$, $\mathcal{U}y = 2$ and $\mathcal{V}y = y + 10$. Then we have

$$\psi\big(d\big(\mathcal{U}^{2n}x,\mathcal{U}y\big)\big) = \psi\big(d(2,2)\big) = \psi(4) = 8$$

and

$$\begin{aligned}\psi\big(d\big(\mathcal{U}^{2n-1}x,\mathcal{V}y\big)\big) - \varphi\big(d\big(\mathcal{U}^{2n-1}x,\mathcal{V}y\big)\big) &= \psi\big(d(2,y+10)\big) - \varphi\big(d(2,y+10)\big)\\ &= \psi(y+12) - \varphi(y+12)\\ &= 2(y+12) - (y+12) = y+12.\end{aligned}$$

Therefore, for $0 < y \leq 3$,

$$\psi\big(d\big(\mathcal{U}^{2n}x,\mathcal{U}y\big)\big) < \psi\big(d\big(\mathcal{U}^{2n-1}x,\mathcal{V}y\big)\big) - \varphi\big(d\big(\mathcal{U}^{2n-1}x,\mathcal{V}y\big)\big).$$

Case (iii). If $3 < y \leq 20$, $\mathcal{U}y = 0$ and $\mathcal{V}y = y - 1$. Then we have

$$\psi\big(d\big(\mathcal{U}^{2n}x,\mathcal{U}y\big)\big) = \psi\big(d(0,0)\big) = 0$$

and

$$\begin{aligned}\psi\big(d\big(\mathcal{U}^{2n-1}x,\mathcal{V}y\big)\big) - \varphi\big(d\big(\mathcal{U}^{2n-1}x,\mathcal{V}y\big)\big) &= \psi\big(d(0,y-2)\big) - \varphi\big(d(0,y-2)\big)\\ &= \psi(y-2) - \varphi(y-2)\\ &= 2(y-2) - (y-2) = y-2.\end{aligned}$$

Hence

$$\psi\big(d\big(\mathcal{U}^{2n}x,\mathcal{U}y\big)\big) < \psi\big(d\big(\mathcal{U}^{2n-1}x,\mathcal{V}y\big)\big) - \varphi\big(d\big(\mathcal{U}^{2n-1}x,\mathcal{V}y\big)\big).$$

Therefore, from all cases, we have

$$\psi\big(d\big(\mathcal{U}^{2n}x,\mathcal{U}y\big)\big) \leq \psi\big(d\big(\mathcal{U}^{2n-1}x,\mathcal{V}y\big)\big) - \varphi\big(d\big(\mathcal{U}^{2n-1}x,\mathcal{V}y\big)\big).$$

Thus $\mathcal{U}$, $\mathcal{V}$ are (ψ,φ)-*weak cyclic compatible contractions*. All the conditions of Theorem 2.2 hold true, and $\mathcal{U}$ and $\mathcal{V}$ have a unique common fixed point. Here $u = 0$ is the unique common fixed point of $\mathcal{U}$ and $\mathcal{V}$.

If we take $\mathcal{V} = \mathcal{U}$ and $\mathcal{V} = I$ in Definition 1.6 and Definition 2.1, we obtain the following definition.

Definition 2.6 Let $\mathcal{U}$ be a cyclic mapping; then

(1) $\mathcal{U}$ is called a *cyclic idle contraction* $\Leftrightarrow d_\alpha(\mathcal{U}^{2n}x, \mathcal{U}y) \leq \gamma d_\alpha(\mathcal{U}^{2n-1}x, \mathcal{U}y)$;

(2) $\mathcal{U}$ is called a (ψ,φ)-*weak cyclic idle contraction* $\Leftrightarrow \psi(d_\alpha(\mathcal{U}^{2n}x, \mathcal{U}y)) \leq \psi(d_\alpha(\mathcal{U}^{2n-1}x, \mathcal{U}y)) - \varphi(d_\alpha(\mathcal{U}^{2n-1}x, \mathcal{U}y))$;

(3) $\mathcal{U}$ is called a (ψ,φ)-*weak cyclic orbital contraction* $\Leftrightarrow \psi(d_\alpha(\mathcal{U}^{2n}x, \mathcal{U}y)) \leq \psi(d_\alpha(\mathcal{U}^{2n-1}x, y)) - \varphi(d_\alpha(\mathcal{U}^{2n-1}x, y))$,

where $\psi,\varphi : [0,\infty) \to [0,\infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$.

Theorem 2.7 *Let A and B be non-empty closed subsets of a complete G_{bd}-family (X, d_α) and let $\mathcal{U} : A \cup B \to A \cup B$ be a $(\psi$-$\varphi)$-*weak cyclic idle contraction. *Assume that $\mathcal{U}(X)$ is closed in X. Then $\mathcal{U}$ has a unique fixed point in $A \cap B$.*

Proof Take $\mathcal{U} = \mathcal{V}$ in Theorem 2.2. □

Theorem 2.8 *Let A and B be non-empty closed subsets of a complete G_{bd}-family (X, d_α) and let $\mathcal{U} : A \cup B \to A \cup B$ be a $(\psi$-$\varphi)$-*weak cyclic orbital contraction. *Assume that $\mathcal{U}(X)$ is closed in X. Then $\mathcal{U}$ has a unique fixed point in $A \cap B$.*

Proof Take $\mathcal{V} = I$ in Theorem 2.2. □

Next we will state and prove a fixed point theorem via altering distance functions. We introduce the following definition.

Definition 2.9 Let A and B be non-empty closed subsets of the generating space of a b-quasi-metric family (X, d_α). Suppose $\mathcal{U}, \mathcal{V} : A \cup B \to A \cup B$ are cyclic mappings such that $\mathcal{U}(X) \subset \mathcal{V}(X)$. Then $\mathcal{U}$, $\mathcal{V}$ are said to be Λ-*cyclic compatible contractions,* if for some $x \in A$, there exists $\gamma \in (0,1)$ such that

$$\eta\bigl(d_\alpha\bigl(\mathcal{U}^{2n}x, \mathcal{U}y\bigr)\bigr) \leq \gamma\eta\bigl(d_\alpha\bigl(\mathcal{U}^{2n-1}x, \mathcal{V}y\bigr)\bigr), \tag{11}$$

for all $n \in \mathbb{N}, y \in B$ and $\eta \in \Lambda$.

Theorem 2.10 *Let A and B be non-empty closed subsets of the generating space of a complete b-quasi-metric family (X, d_α). Let $\mathcal{U}, \mathcal{V} : A \cup B \to A \cup B$ be cyclic mappings such that $\mathcal{U}(X) \subset \mathcal{V}(X)$. Suppose*

1. $\mathcal{U}$, $\mathcal{V}$ *are* Λ-cyclic compatible contractions,
2. $\mathcal{U}$, $\mathcal{V}$ *are weakly compatible,*
3. $\mathcal{V}(X)$ *is a closed subset of* X.

Then $\mathcal{U}$ and $\mathcal{V}$ have a point of coincidence in $A \cap B$. Moreover, $\mathcal{U}$ and $\mathcal{V}$ have a unique common fixed point in $A \cap B$.

Proof Take $x = x_0 \in A$. Since $\mathcal{U}(X) \subset \mathcal{V}(X)$, we may choose $x_1 \in X$ such that $\mathcal{U}x_0 = \mathcal{V}x_1$.

Hence we can define the sequence $\{x_n\}$ in X by $\mathcal{U}^n x_0 = \mathcal{U}x_n = \mathcal{V}x_{n+1} = x_{n+1}$ for $n \in \mathbb{N} \cup \{0\}$. Then $\{x_{2n}\} \in A$ and $\{x_{2n-1}\} \in B$. Thus we have

$$\begin{aligned}
\eta\bigl(d_\alpha(x_{2n}, x_{2n+1})\bigr) &= \eta\bigl(d_\alpha(\mathcal{V}x_{2n}, \mathcal{V}x_{2n+1})\bigr) \\
&= \eta\bigl(d_\alpha(\mathcal{U}x_{2n-1}, \mathcal{U}x_{2n})\bigr) \\
&= \eta\bigl(d_\alpha\bigl(\mathcal{U}^{2n}x_0, \mathcal{U}x_{2n-1}\bigr)\bigr) \\
&\le \gamma\eta\bigl(d_\alpha\bigl(\mathcal{U}^{2n-1}x_0, \mathcal{V}x_{2n-1}\bigr)\bigr) \\
&= \gamma\eta\bigl(d_\alpha(x_{2n}, x_{2n-1})\bigr) \\
&= \gamma\eta\bigl(d_\alpha(x_{2n-1}, x_{2n})\bigr).
\end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned}
\eta\bigl(d_\alpha(x_{2n+1}, x_{2n+2})\bigr) &= \eta\bigl(d_\alpha(\mathcal{V}x_{2n+1}, \mathcal{V}x_{2n+2})\bigr) \\
&= \eta\bigl(d_\alpha(\mathcal{U}x_{2n}, \mathcal{U}x_{2n+1})\bigr) \\
&= \eta\bigl(d_\alpha\bigl(\mathcal{U}^{2n+1}x_0, \mathcal{U}x_{2n}\bigr)\bigr) \\
&\le \gamma\eta\bigl(d_\alpha\bigl(\mathcal{U}^{2n}x_0, \mathcal{V}x_{2n}\bigr)\bigr) \\
&= \gamma\eta\bigl(d_\alpha(\mathcal{U}x_{2n}, x_{2n})\bigr) \\
&= \gamma\eta\bigl(d_\alpha(x_{2n+1}, x_{2n})\bigr) \\
&= \gamma\eta\bigl(d_\alpha(x_{2n}, x_{2n+1})\bigr).
\end{aligned} \tag{13}$$

In general, we have

$$\eta\bigl(d_\alpha(x_n, x_{n+1})\bigr) \le \gamma\eta\bigl(d_\alpha(x_{n-1}, x_n)\bigr).$$

Inductively, for each $n \in \mathbb{N}$, we get

$$\eta\bigl(d_\alpha(x_n, x_{n+1})\bigr) \le \gamma^n\eta\bigl(d_\alpha(x_0, x_1)\bigr).$$

Since $0 \le \gamma < 1$, letting $n \to \infty$, we get $\lim_{n\to\infty} \eta(d_\alpha(x_n, x_{n+1})) = 0$. From the definition of altering distance functions, we get

$$\lim_{n\to\infty} \bigl(d_\alpha(x_n, x_{n+1})\bigr) = 0. \tag{14}$$

Therefore

$$\lim_{n\to\infty} \bigl(d_\alpha(\mathcal{V}x_n, \mathcal{V}x_{n+1})\bigr) = 0,$$

or

$$\lim_{n\to\infty} \bigl(d_\alpha(\mathcal{U}x_{n-1}, \mathcal{U}x_n)\bigr) = 0.$$

Now we claim that $\{x_n\}$ is a Cauchy sequence. According to the definition of generating space of a b-quasi-metric family (X, d_α), and for $n, m \in \mathbb{N}, m > n$, we have

$$\begin{aligned} d_\alpha(x_n, x_m) &\le s\bigl[d_\beta(x_n, x_{n+1}) + d_\beta(x_{n+1}, x_m)\bigr] \\ &= sd_\beta(x_n, x_{n+1}) + sd_\beta(x_{n+1}, x_m) \\ &\le sd_\beta(x_n, x_{n+1}) + s^2 d_\beta(x_{n+1}, x_{n+2}) + s^3 d_\beta(x_{n+2}, x_{n+3}) + \cdots . \end{aligned} \tag{15}$$

Letting $n, m \to \infty$, and from (14), we have $\lim_{n,m\to\infty} d_\alpha(x_n, x_m) = 0$ for all $\alpha \in (0,1]$. Therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exist sequences $\{\mathcal{V}^{2n}x_0\}$ in A and $\{\mathcal{V}^{2n-1}x_0\}$ in B such that both sequences converge to some ω in X. Since A and B are closed in X, $\omega \in A \cap B$.

Since $\mathcal{V}(X)$ is closed in X, there exists z in X such that

$$\mathcal{V}z = \omega. \tag{16}$$

Consider

$$\eta\bigl(d_\alpha\bigl(\mathcal{U}^{2n}x_0, \mathcal{U}z\bigr)\bigr) \le \gamma\eta\bigl(d_\alpha\bigl(\mathcal{U}^{2n-1}x_0, \mathcal{V}z\bigr)\bigr).$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} \eta(d_\alpha(\mathcal{U}^{2n}x_0, \mathcal{U}z)) = 0$. From the definition of altering distance functions, we get

$$\lim_{n\to\infty} d_\alpha\bigl(\mathcal{U}^{2n}x_0, \mathcal{U}z\bigr) = 0.$$

This implies

$$d_\alpha(\omega, \mathcal{U}z) = 0.$$

Therefore

$$\omega = \mathcal{U}z. \tag{17}$$

From (16) and (17), it follows that $\mathcal{U}z = \mathcal{V}z = \omega$. Thus ω is a point of coincidence of $\mathcal{U}$ and $\mathcal{V}$. By weak compatibility, we get $\mathcal{V}\omega = \mathcal{U}\omega$. Now we prove $\mathcal{V}\omega = \omega$. Suppose $\mathcal{V}\omega \ne \omega$, then consider

$$\begin{aligned} \eta\bigl(d_\alpha(\mathcal{V}\omega, \omega)\bigr) &= \lim_{n\to\infty} \eta\bigl(d_\alpha\bigl(\mathcal{U}\omega, \mathcal{U}^{2n-1}x_0\bigr)\bigr) \\ &= \lim_{n\to\infty} \eta\bigl(d_\alpha\bigl(\mathcal{U}^{2n-1}x_0, \mathcal{U}\omega\bigr)\bigr) \\ &\le \gamma \lim_{n\to\infty} \eta\bigl(d_\alpha\bigl(\mathcal{U}^{2n-2}x_0, \mathcal{V}\omega\bigr)\bigr) \\ &= \gamma\eta\bigl(d_\alpha(\omega, \mathcal{V}\omega)\bigr) \\ &\le \eta\bigl(d_\alpha(\omega, \mathcal{V}\omega)\bigr), \end{aligned} \tag{18}$$

which is a contradiction. Hence $\mathcal{V}\omega = \omega$. Therefore $\mathcal{U}\omega = \mathcal{V}\omega = \omega$. To prove uniqueness, suppose that ω_1 and ω_2 are two common fixed points of $\mathcal{U}$ and $\mathcal{V}$. Assume that $\omega_1 \neq \omega_2$. Then $\mathcal{U}\omega_1 = \mathcal{V}\omega_1 = \omega_1$ and $\mathcal{U}\omega_2 = \mathcal{V}\omega_2 = \omega_2$.

Consider

$$\begin{aligned} \eta\big(d_\alpha(\omega_1,\omega_2)\big) &= \lim_{n\to\infty} \eta\big(d_\alpha\big(\mathcal{U}^{2n-1}x_0, \mathcal{U}\omega_2\big)\big) \\ &\le \gamma \lim_{n\to\infty} \eta\big(d_\alpha\big(\mathcal{U}^{2n-2}x_0, \mathcal{V}\omega_2\big)\big) \\ &= \gamma\eta\big(d_\alpha(\omega_1, \mathcal{V}\omega_2)\big) \\ &\le \eta\big(d_\alpha(\omega_1,\omega_2)\big), \end{aligned} \tag{19}$$

which is a contradiction. Hence $\omega_1 = \omega_2$ and our proof is finished. □

Remark 2.11 We will obtain special cases of Theorem 2.10 above, if we

1. replace G_{bq}-family (X,d_α) by G_q-family, according to Definition 1.4, by putting $s = 1$;
2. replace G_{bq}-family (X,d_α) by b-metric space and taking $d_\alpha = d$;
3. replace G_{bq}-family (X,d_α) by complete metric space and taking $d_\alpha = d$ and $s = 1$.

Example 2.12 Let $A = B = X = [0,1]$. Define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) = (x-y)^2$. This is a b-metric space with $s = 2$, not the usual metric space, since $d(0,1) \nleq d(0,\frac{1}{2}) + d(\frac{1}{2},1)$. Define

$$\mathcal{U}x = 0, \quad \text{if } 0 \le x \le 1$$

and

$$\mathcal{V}x = \begin{cases} 0, & \text{if } 0 \le x < \frac{1}{2}, \\ \frac{1}{5}, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Clearly $\mathcal{U}(X) \subset \mathcal{V}(X)$. Define $\eta(t) = t^2$. For any $x \in [0,1]$, we have

$$\mathcal{U}x = \mathcal{U}^2x = \cdots \mathcal{U}^n x = 0, \quad \text{for all } n.$$

For any $y \in [0,1]$, we have

$$\mathcal{V}y = \begin{cases} 0, & \text{if } 0 \le y < \frac{1}{2}, \\ \frac{1}{5}, & \text{if } \frac{1}{2} \le y \le 1. \end{cases}$$

Case (i). If $0 \le y < \frac{1}{2}$, $\mathcal{U}y = 0$, we have

$$\begin{aligned} \eta\big(d\big(\mathcal{U}^{2n}x, \mathcal{U}y\big)\big) &= \eta\big(d(0,0)\big) \\ &= 0 = \gamma\eta\big(d\big(\mathcal{U}^{2n-1}x, \mathcal{V}y\big)\big). \end{aligned}$$

Case (ii). If $\frac{1}{2} \le y \le 1$, $\mathcal{U}y = 0$, we have

$$\eta\big(d\big(\mathcal{U}^{2n}x, \mathcal{U}y\big)\big) = \eta\big(d(0,0)\big) = 0$$

and

$$\eta\big(d\big(\mathcal{U}^{2n-1}x, \mathcal{V}y\big)\big) = \eta\left(d\left(0, \frac{1}{5}\right)\right)$$
$$= \eta\left(\frac{1}{25}\right) = \frac{1}{625}.$$

So

$$\eta\big(d\big(\mathcal{U}^{2n}x, \mathcal{U}y\big)\big) \leq \gamma\eta\big(d\big(\mathcal{U}^{2n-1}x, \mathcal{V}y\big)\big), \quad \text{for } 0 \leq \gamma < 1.$$

Therefore, from both cases, we get

$$\eta\big(d\big(\mathcal{U}^{2n}x, \mathcal{U}y\big)\big) \leq \gamma\eta\big(d\big(\mathcal{U}^{2n-1}x, \mathcal{V}y\big)\big), \quad \text{for } 0 \leq \gamma < 1.$$

Thus $\mathcal{U}$, $\mathcal{V}$ are Λ*-cyclic compatible contractions.* All the conditions of Theorem 2.10 hold true and $\mathcal{U}$, $\mathcal{V}$ have a unique common fixed point. Here $\omega = 0$ is the unique common fixed point of $\mathcal{U}$ and $\mathcal{V}$.

Example 2.13 Let $A = B = X = [0,1]$. Define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) = |x - y|$. So (X,d) is a complete metric space. Define

$$\mathcal{U}x = 0, \quad \text{if } 0 \leq x \leq 1$$

and

$$\mathcal{V}x = \begin{cases} 0, & \text{if } 0 \leq x < \frac{1}{2}, \\ \frac{1}{9}, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Clearly $\mathcal{U}(X) \subset \mathcal{V}(X)$. Define $\eta(t) = t$.

For any $x \in [0,1]$, we have

$$\mathcal{U}x = \mathcal{U}^2x = \cdots = \mathcal{U}^n x = 0, \quad \text{for all } n.$$

For any $y \in [0,1]$,

$$\mathcal{V}y = \begin{cases} 0, & \text{if } 0 \leq y < \frac{1}{2}, \\ \frac{1}{9}, & \text{if } \frac{1}{2} \leq y \leq 1. \end{cases}$$

Consider

Case (i). If $0 \leq y < \frac{1}{2}$, $\mathcal{U}y = 0$, $\mathcal{V}y = 0$. Then we have

$$\eta\big(d\big(\mathcal{U}^{2n}x, \mathcal{U}y\big)\big) = \eta\big(d(0,0)\big)$$
$$= 0 = \gamma\eta\big(d\big(\mathcal{U}^{2n-1}x, \mathcal{V}y\big)\big).$$

Case (ii). If $\frac{1}{2} \leq y \leq 1$, $\mathcal{U}y = 0$, $\mathcal{V}y = \frac{1}{9}$. Then we have

$$\eta\big(d\big(\mathcal{U}^{2n}x, \mathcal{U}y\big)\big) = \eta\big(d(0,0)\big) = 0$$

and

$$\eta\big(d\big(\mathcal{U}^{2n-1}x,\mathcal{V}y\big)\big)=\eta\left(d\left(0,\frac{1}{9}\right)\right)$$
$$=\frac{1}{9}.$$

Hence, from both cases we conclude that $\eta(d(\mathcal{U}^{2n}x,\mathcal{U}y))\leq\gamma\eta(d(\mathcal{U}^{2n-1}x,\mathcal{V}y))$, for all $0\leq\gamma<1$. Thus $\mathcal{U}$, $\mathcal{V}$ are Λ*-cyclic compatible contractions.* All the conditions of Theorem 2.10 hold true. Here $\omega=0$ is the unique common fixed point of $\mathcal{U}$ and $\mathcal{V}$.

In view of Definition 2.6, we introduce the following definition.

Definition 2.14 Let $\mathcal{U}$ be a cyclic mapping, then

(1) $\mathcal{U}$ is called Λ*-cyclic idle contraction.* $\Leftrightarrow\eta(d_\alpha(\mathcal{U}^{2n}x,\mathcal{U}y))\leq\gamma\eta(d_\alpha(\mathcal{U}^{2n-1}x,\mathcal{U}y))$.
(2) $\mathcal{U}$ is called Λ*-cyclic orbital contraction.* $\Leftrightarrow\eta(d_\alpha(\mathcal{U}^{2n}x,\mathcal{U}y))\leq\gamma\eta(d_\alpha(\mathcal{U}^{2n-1}x,y))$.

Theorem 2.15 *Let A and B be non-empty closed subsets of the generating space of a complete b-quasi-metric family* (X,d_α). *Suppose* $\mathcal{U}:A\cup B\to A\cup B$ *is a* Λ-cyclic idle contraction, *then* $\mathcal{U}$ *has a unique fixed point in* $A\cap B$.

Proof Take $\mathcal{U}=\mathcal{V}$ in Theorem 2.10. □

Theorem 2.16 *Let A and B be non-empty closed subsets of the generating space of a complete b-quasi-metric family* (X,d_α). *Suppose* $\mathcal{U}:A\cup B\to A\cup B$ *is a* Λ-cyclic orbital contraction, *then* $\mathcal{U}$ *has a unique fixed point in* $A\cap B$.

Proof Take $\mathcal{V}=I$ in Theorem 2.10. □

3 Discussion

In this manuscript, we establish fixed point theorems in generating space of a G_{bd}-family (X,d_α), and a b-quasi-metric family (X,d_α). One can see that Theorem 2.2 and Theorem 2.10 are generalizations of Theorem 2.2 obtained in [7] in the setting of a complete G_{bd}-family (X,d_α), and a complete b-quasi-metric family (X,d_α), respectively. Besides, we introduce other definition of weak cyclic contractions (see Definition 2.6), and obtain fixed point theorems for those contractions (Theorems 2.7 and 2.8). After Theorem 2.10, we also introduce other definition of Λ*-cyclic contractions* (Def. 2.14) and obtain fixed point theorems (Theorems 2.15 and 2.16). Our main results extend and improve some well-known results in the existing literature. However, in recent remarkable work of Lau and Zhang in [19, 20], the authors studied fixed point properties of semigroups of nonexpansive mappings, nonlinear mappings and amenability. In relation to Theorem 2.2 and Theorem 2.10, we pose the following open problem at the end.

Problem Can Theorem 2.2 and Theorem 2.10 be generalized to a family or a commuting or amenable semigroup of maps?

4 Conclusions

In this manuscript, we introduce a new class of generalized (ψ,φ)-weak cyclic compatible contractions (Definition 2.1) and prove a coincidence point and a unique common fixed point theorem for these new contractions in a complete G_{bd}-family (X,d_α). We also introduce a new class of Λ-*cyclic compatible contractions* via altering distance functions(Definition 2.9), and prove an existence theorem for this new class of generalized contractions in the generating space of a b-quasi-metric family (X,d_α). Our results extend and improve the results obtained in [7] and some well-known results in the literature. We provide examples to illustrate and support our results and we also pose a problem at the end.

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The authors declare that there is no conflict of interest regarding the publication of this article.

Authors' contributions
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