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Common fixed point theorems for a finite family of multivalued mappings in an ordered Banach space

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Abstract

In this paper, we prove some common fixed point theorems for a finite family of multivalued and single-valued mappings operating on ordered Banach spaces. Our results extend and generalize many results in the literature on fixed point theory and lead to existence theorems for a system of integral inclusions.

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1 Introduction

A good part of the research publications on the fixed point theory was devoted to the existence of a common fixed point for pairs of single and multivalued functions in various types of spaces such as metric spaces, ordered spaces, and so on.

By using the measure of noncompactness, Dehage et al. [1] proved some common fixed point results for pairs of condensing mappings in ordered Banach spaces and they showed that these results have interesting applications in differential and integral equations. Several authors got generalizations of these results under a weaker hypothesis (see [2–6]). The main objective of the present paper is to generalize the results of [2, 3] by establishing some common fixed point results for a finite family of single and multivalued functions on ordered Banach spaces. Some examples will be given to support our results. As application, we will prove the existence solutions for a system of integral inclusions, which gives a generalization of results in [7].

2 Methods

Many authors studied the existence of a common fixed point for pairs of condensing mappings in an ordered Banach space with weak and strong topology.

The goal of this article is to generalize these results to the case of a finite family of multivalued condensing mapping. The main tool in this study is the notion of noncompactness measure on Banach spaces.

Our work is organized as follows: we discuss some concepts used in this paper, and we present our main results and their consequences. We give also some examples to validate our results. Then we apply the obtained results to solve a system of integral inclusions.

3 Results and discussion

Let us first give some preliminaries and notations. For a given real Banach space X, we denote by 2^X the space of all nonempty subsets of X. Recall that a multivalued function on X is a mapping from X into 2^X and that a point $x^* \in X$ is called a fixed point of T if $x^* \in T(x^*)$.

In the following, we denote by $T(A)$ the set $\bigcup_{x\in A} T(x)$ for every $A \in 2^X$, by

$$\Gamma(T) = \{(x,y) \in X \times X : y \in T(x)\}$$

the graph of T, and by X' be the topological dual space of X.

Definition 3.1 A sequence $\{x_n\}$ of X is weakly convergent to $x \in X$ if $\lim_n f(x_n) = f(x)$ for all $f \in X'$.

In this case, we denote $x_n \rightharpoonup x$.

Definition 3.2 Let X be an ordered Banach space, T is said to be monotone-closed if for each monotone sequence $\{x_n\}$ in X with $x_n \to x$ and for each sequence $\{y_n\}$ with $y_n \in T(x_n)$ such that $y_n \to y$, we have $y \in T(x)$.

Definition 3.3 T is said to be closed if for each sequence $\{(x_n, y_n)\}$ in $\Gamma(T)$ with $(x_n, y_n) \to (x, y)$ strongly in $X \times X$, we have $y \in T(x)$.

Definition 3.4 A nonempty closed subset P of X is called an order cone if

1. $P \neq \{0\}$;
2. For all $a, b \in \mathbb{R}^+$ and $x, y \in P$, we have $ax + by \in P$;
3. $P \cap (-P) = \{0\}$.

Given an order cone P on X, we can define a partial order "$\leq$" on X by

$$x \leq y \iff y - x \in P.$$

An order cone P is called normal if there is a real constant $N > 0$ such that, for all $x, y \in X$, we have

$$0 \leq x \leq y \quad \Rightarrow \quad \|x\| \leq N\|y\|.$$

The following lemma can be found in [3] and [8].

Lemma 3.5 *Let X be an ordered real Banach space with a normal order cone. Suppose that $\{x_n\}$ is a monotone sequence which contains a subsequence $\{x_{\sigma(n)}\}$ converging weakly to some $x \in X$. Then $\{x_n\}$ converges strongly to x.*

We can also define a partial order on 2^X by

$$A \leq B \iff x \leq y, \quad \forall (x,y) \in A \times B.$$

We define the measure of noncompactness on bounded subsets of X by

$$\psi(A) = \inf\left\{\delta > 0 : \exists A_1, \dots, A_n \subset X, \operatorname{diam}(A_i) \leq \delta \text{ and } A \subset \bigcup_{i=1}^{n} A_i\right\},$$

where $\operatorname{diam}(A)$ denotes the diameter of a subset A of X.

The following lemma describes some properties of the measure of noncompactness.

Lemma 3.6 *Let A and B be bounded sets of X, then*

1. $\psi(A) = 0$ *if and only if A is relatively compact*;
2. $A \subseteq B$ *implies* $\psi(A) \leq \psi(B)$;
3. $\psi(\overline{A}) = \psi(A)$;
4. $\psi(A \cup B) = \max\{\psi(A), \psi(B)\}$.

For more properties of the measure of noncompactness, we refer to [9, 10].

Definition 3.7 A mapping $T : M \subseteq X \to X$ is said to be locally almost nonexpansive if, for each $x \in M$ and $\varepsilon > 0$, there exists a weak neighborhood U_x of x such that

$$\left\| T(u) - T(v) \right\| \leq \|u - v\| + \varepsilon \quad \text{for all } u, v \in U_x.$$

Now, we give our main results.

As extension of the results of [6], we define the notion of $(T_k)_{2\leq k\leq p}$-weakly isotone for a given integer $p \geq 2$ as follows.

Definition 3.8 Let M be a nonempty subset of an ordered Banach space X, and let $T_1, T_2, \dots, T_p : M \to 2^M$ be p multivalued mappings on M; we say that

1. T_1 is $(T_k)_{2\leq k\leq p}$-weakly isotone increasing if, for all $x \in M$, the relations

   $$x_1 \in T_1(x), \qquad x_2 \in T_2(x_1), \qquad \dots, \qquad x_p \in T_p(x_{p-1})$$

   imply

   $$T_1(x) \leq T_2(x_1) \leq \cdots \leq T_p(x_{p-1}) \leq T_1(x_p).$$

2. T_1 is $(T_k)_{2\leq k\leq p}$-weakly isotone decreasing if, for all $x \in M$, the relations

   $$x_1 \in T_1(x), \qquad x_2 \in T_2(x_1), \qquad \dots, \qquad x_p \in T_p(x_{p-1})$$

   imply

   $$T_1(x_p) \leq T_p(x_{p-1}) \leq T_{p-1}(x_{p-2}) \leq \cdots \leq T_1(x).$$

3. T_1 is $(T_k)_{2\leq k\leq p}$-weakly isotone if it is either $(T_k)_{2\leq k\leq p}$-weakly isotone increasing or weakly isotone decreasing.

The following definition will be used in the next of this paper.

Definition 3.9 ([11]) Let X be a Banach space, M be a nonempty subset of X, $T : M \to 2^M$ be a multivalued mapping on M, and let ψ be a measure of noncompactness on X. For $k \in [0,1]$, we have the following definitions:

1. T is called ψ-condensing if $T(M)$ is bounded and, for every nonempty bounded subset N of M with $\psi(N) > 0$, we have $\psi(T(N)) < \psi(N)$.
2. T is called $k - \psi$-contractive if $T(M)$ is bounded and, for every nonempty bounded subset N of M, we have $\psi(T(N)) \leq k\psi(N)$.

Our first main result is the following theorem.

Theorem 3.10 *Let X be an ordered Banach space, ψ be a measure of noncompactness on X and $p \geq 2$ be an integer. Let M be a nonempty closed subset of X and $T_1, T_2, \dots, T_p : M \to 2^M$ be p monotone-closed mappings satisfying*:

1. *$T_2, T_3, \dots, T_p$ are $1 - \psi$-contractive*;
2. *T_1 is ψ-condensing*;
3. *T_1 is $(T_k)_{2\leq k\leq p}$-weakly isotone.*

Then $T_1, T_2, \dots, T_p$ have a common fixed point.

Proof Assume that T_1 is $(T_k)_{2\leq k\leq p}$-weakly isotone increasing, and let $x \in M$. We define a sequence $\{x_n\}$ in M as follows:

$$x_0 = x, \qquad x_{pn+1} \in T_1(x_{pn}), \qquad x_{pn+2} \in T_2(x_{pn+1}), \qquad \dots, \qquad x_{pn+p} \in T_p(x_{pn+(p-1)})$$

for $n = 0, 1, 2, \dots$.

As we have

$$x_1 \in T_1(x_0), \qquad x_2 \in T_2(x_1), \qquad \dots, \qquad x_p \in T_p(x_{p-1}),$$

we obtain

$$T_1(x_0) \leq T_2(x_1) \leq \cdots \leq T_p(x_{p-1}) \leq T_1(x_p),$$

and then

$$x_1 \leq x_2 \leq \cdots \leq x_p \leq x_{p+1},$$

which gives

$$x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq \cdots.$$

Let $A_j = \{x_{np+j} : n \in \mathbb{N}\}$, $j = 0, 1, \dots, p-1$, and $A_p = A_0 \setminus \{x_0\}$, we have

$$A_j \subset T_j(A_{j-1}), \quad \text{for all } j \in \{1, 2, \dots, p\}.$$

The set

$$A = \left(\bigcup_{k=1}^{p} A_k\right) \cup \{x_0\}$$

is bounded; and since $T_2, T_3, \dots, T_p$ are $1 - \psi$-contractive, we have

$$\psi(A_k) \le \psi(T_k(A_{k-1}) \le \psi(A_{k-1}) \le \psi(T_{k-1}(A_{k-2}) \le \cdots \le \psi\bigl(T_1(A_0)\bigr)$$

for all $k = 1, 2, \dots, p$.

Assume that $\psi(A) \neq 0$, we have

$$\begin{aligned}
\psi(A) &= \max\bigl\{\psi(A_k) : k = 1, 2, \dots, p\bigr\} \\
&\le \max\bigl\{\psi\bigl(T_k(A_{k-1})\bigr) : k = 1, 2, \dots, p\bigr\} \\
&\le \max\bigl\{\psi(A_{k-1}) : k = 1, 2, \dots, p\bigr\} \\
&\le \max\bigl\{\psi\bigl(T_{k-1}(A_{k-2})\bigr) : k = 2, \dots, p\bigr\} \\
&\vdots \\
&\le \psi\bigl(T_1(A_0)\bigr).
\end{aligned}$$

As T_1 is ψ-condensing, we obtain

$$\psi(A) \le \psi\bigl(T_1(A_0)\bigr) \le \psi\bigl(T_1(A)\bigr) < \psi(A),$$

which is contraction. Thus $\psi(A) = 0$, and then A is relatively compact. Since $\{x_n\}$ is monotone increasing in A, it is convergent to some x^*. Since $x_{pn+1} \in T_1(x_{pn})$ and T_1 has a closed graph, we obtain that $x^* \in T_1(x^*)$. Similarly, we obtain $x^* \in T_k(x^*)$ for $k = 2, 3, \dots, p$; and consequently, x^* is a common fixed point for $T_1, T_2, T_3, \dots, T_p$.

The case T_1 is $(T_k)_{2\le k\le p}$-weakly isotone decreasing is similar, which ends the proof. □

Remark 3.11 For $p = 2$, we obtain [3] Theorem 3.10.

Let M be a nonempty subset of an ordered Banach space X, and let

$$T_1, T_2, \dots, T_p : M \to M$$

be p mappings. As in Definition 3.8, we can define that the notion of T_1 is $(T_k)_{2\le k\le p}$-weakly isotone. Note that in this case the set $T(x)$ becomes single $\{T(x)\}$ and $y \in \{T(x)\}$ becomes $y = T(x)$.

By Theorem 3.10, we obtain the following result.

Corollary 3.12 *Let X be an ordered Banach space and ψ be a measure of noncompactness on X. Let M be a nonempty closed subset of X, and $T_1, T_2, \dots, T_p : M \to M$ be p closed mappings satisfying*:

1. *$T_2, T_3, \dots, T_p$ are $1 - \psi$-contractive*;
2. *T_1 is ψ-condensing*;
3. *T_1 is $(T_k)_{2\le k\le p}$-weakly isotone.*

Then $T_1, T_2, \dots, T_p$ have a common fixed point.

Remark 3.13 For $p = 2$, this corollary is obtained in [2].

Using Definition 3.7, we have the following corollary.

Corollary 3.14 *Let M be a nonempty, closed subset of an ordered reflexive Banach space X, and ψ be a measure of noncompactness on X.*

Let $T_1, T_2, \dots, T_p : M \to M$ be p continuous mappings satisfying:

1. $T_2, T_3, \dots, T_p$ *are locally almost nonexpansive mappings*;
2. T_1 *is ψ-condensing*;
3. T_1 *is $(T_k)_{2 \le k \le p}$-weakly isotone.*

Then $T_1, T_2, \dots, T_p$ have a common fixed point.

Proof By [12], Lemma 1, p. 672, we have $T_2, T_3, \dots, T_p$ are $1 - \psi$-contractive; hence the proof of Theorem 3.10. □

Theorem 3.15 *Let X be an ordered Banach space, and ψ be a measure of noncompactness on X. Let M be a nonempty closed subset of X, and $T_1, T_2, \dots, T_p : M \to 2^M$ be p monotone-closed mappings ($p \ge 2$) satisfying*:

1. $T_1, T_2, T_3, \dots, T_p$ *are ψ-condensing*;
2. T_1 *is $(T_k)_{2 \le k \le p}$-weakly isotone.*

Then $T_1, T_2, \dots, T_p$ have a common fixed point.

Proof We use the same notations as in the proof of the previous theorem and assume that $\psi(A) \neq 0$; we have

$$A = \left(\bigcup_{k=1}^{p} T_k(A_{k-1}) \right) \cup \{x_0\},$$

then

$$\begin{aligned} \psi(A) &= \max\bigl\{\psi\bigl(T_k(A_{k-1})\bigr); k = 1, 2, \dots, p\bigr\} \\ &\le \max\bigl\{\psi\bigl(T_k(A)\bigr); k = 1, 2, \dots, p\bigr\} \\ &< \psi(A). \end{aligned}$$

Then $\psi(A) = 0$, which is contradiction. Therefore A is relatively compact, which ends the proof. □

Definition 3.16 Let M be a nonempty subset of an ordered Banach space X and $T_1, T_2, \dots, T_p : M \to 2^M$ be p mappings with $p \ge 2$.

1. The p-uplet $(T_1, T_2, \dots, T_p)$ is called weakly isotone increasing if for all $x \in M$, we have

$$\begin{cases} T_p(x) \le T_1(y) & \text{for all } y \in T_p(x), \\ \forall i \in \{1, 2, \dots, p-1\}, \quad T_i(x) \le T_{i+1}(y) & \text{for all } y \in T_i(x). \end{cases}$$

2. We say that the p-uplet $(T_1, T_2, \dots, T_p)$ is weakly isotone decreasing if for all $x \in M$, we have

$$\begin{cases} T_1(y) \preceq T_p(x) \quad \text{for all } y \in T_p(x), \\ \forall i \in \{1, 2, \dots, p-1\}, \quad T_{i+1}(y) \preceq T_i(x) \quad \text{for all } y \in T_i(x). \end{cases}$$

3. The p-uplet $(T_1, T_2, \dots, T_p)$ is weakly isotone if it is either weakly isotone increasing or weakly isotone decreasing.

Remark 3.17 If $p = 2$ and T_1, T_2 are a single-valued mappings, then Definition 3.16 coincides with [6], Definition 2.2.

Remark 3.18 If $(T_1, T_2, \dots, T_p)$ is weakly isotone increasing (resp. decreasing), then T_1 is $(T_k)_{2\leq k\leq p}$-weakly isotone increasing (resp. decreasing).

Using this remark, the statement of Theorem 3.15 remains if we replace "T_1 is $(T_k)_{2\leq k\leq p}$-weakly isotone" with "$(T_1, T_2, \dots, T_p)$ is weakly isotone".

Let M be a nonempty subset of an ordered Banach space X. Motivated by [3], we introduce the following definition.

Definition 3.19 p maps $T_1, T_2, \dots, T_p$ with $p \geq 2$ are said to satisfy condition B_M if, for any monotone sequence $\{x_n\}$ of M and for any fixed $a \in M$, the condition

$$\{x_n\} \subset \{a\} \cup \left(\bigcup_{k=1}^{p} T_k\big(\{x_n\}\big) \right)$$

implies $\{x_n\}$ has a weakly convergent subsequence.

Theorem 3.20 *Let X be an ordered Banach space with a normal order cone. Let M be a nonempty closed subset of X and $T_1, T_2, \dots, T_p : M \to 2^M$ be p monotone-closed mappings with $p \geq 2$ satisfying*:

1. *$T_1, T_2, \dots, T_p$ satisfy condition B_M;*
2. *T_1 is $(T_k)_{2\leq k\leq p}$-weakly isotone.*

Then $T_1, T_2, \dots, T_p$ have a common fixed point.

Proof Assume that T_1 is $(T_k)_{2\leq k\leq p}$-weakly isotone increasing, and let $x \in M$ be fixed. We define a sequence $\{x_n\}$ in M as follows:

$$x_0 = x, \qquad x_{pn+1} \in T_1(x_{pn}), \qquad x_{pn+2} \in T_2(x_{pn+1}), \qquad \dots, \qquad x_{pn+p} \in T_p(x_{pn+(p-1)}),$$
$$n = 0, 1, 2, \dots.$$

Then as in the proof of Theorem 3.10, it follows that

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_n \preceq \cdots.$$

We have

$$\{x_n\} \subset \{x_0\} \cup \left(\bigcup_{k=1}^{p} T_k\big(\{x_n\}\big) \right).$$

Since $T_1, T_2, \dots, T_p$ satisfy condition B_M, there exist $x^* \in M$ and a subsequence $\{x_{\sigma(n)}\}$ of $\{x_n\}$ such that $x_{\sigma(n)} \rightharpoonup x^*$. Referring to Lemma 3.5, we get $\{x_n\}$ strongly converges to x^*. Now we have $x_{pn+1} \in T_1(x_{pn})$ for all $n \in \mathbb{N}$, and T_1 is monotone-closed, we obtain that $x^* \in T_1(x^*)$. A similar argument yields $x^* \in T_k(x^*)$, $k = 2, 3, \dots, p$, and consequently, x^* is a common fixed point for $T_1, T_2, T_3, \dots, T_p$.

The case when T_1 is $(T_k)_{2\leq k\leq p}$-weakly isotone decreasing is similar. □

Remark 3.21 For $p = 2$, Theorem 3.20 was proved in [3].

Example 3.22 Let $X = \mathbb{R}^2$, $P = \{x = (t,t) \in \mathbb{R}^2 : t \geq 0\}$ and

$$M = \big\{x \in X : \|x\|_\infty \leq 2\big\} \cup \big\{(t,0) \in X : t > 2\big\}.$$

$(X, \|\cdot\|_\infty)$ is an ordered Banach space with a normal order cone P, and M is a nonempty closed subset of X.

Let $T_1, T_2, T_3 : M \to 2^M$ be defined by, for all $x \in M$,

$$T_1(x) = \begin{cases} \{(\|x\|_\infty, \|x\|_\infty), (\frac{\|x\|_\infty+2}{2}, \frac{\|x\|_\infty+2}{2})\} & \text{if } \|x\|_\infty \leq 2, \\ \{(2,2)\} & \text{if } \|x\|_\infty > 2, \end{cases}$$

$$T_2(x) = \begin{cases} \{(2,2)\} & \text{if } \|x\|_\infty \leq 2, \\ \{(\frac{t}{t-1}, \frac{t}{t-1}) : t \in [2, \|x\|_\infty]\} & \text{if } \|x\|_\infty > 2 \end{cases}$$

and $T_3(x) = \{x\}$.

The multivalued mappings T_1, T_2, T_3 are closed, so monotone-closed, and satisfying the condition $\mathcal{B}_M$, and T_1 is (T_2, T_3)-weakly-isotone increasing. Besides, the mappings T_1, T_2, T_3 have the common fixed point $(2,2)$.

The following corollary can be obtained from Theorem 3.20.

Corollary 3.23 *Let X be an ordered Banach space with a normal order cone. Let M be a nonempty closed subset of X. Let $T_1, T_2, \dots, T_p : M \to M$ be p monotone-continuous mappings with $p \geq 2$ satisfying*:

1. *$T_1, T_2, \dots, T_p$ satisfy condition B_M*;
2. *T_1 is $(T_k)_{2\leq k\leq p}$-weakly isotone.*

Then $T_1, T_2, \dots, T_p$ have a common fixed point.

Let M be a nonempty subset of an ordered Banach space X, and let $a \in M$. Motivated by [6] we introduce the following condition.

Definition 3.24 p maps $T_1, T_2, \dots, T_p : M \to 2^M$ with $p \geq 2$ are said to satisfy the condition D_M if, for any countable set of M, the condition

$$A \subset \{a\} \cup \left(\bigcup_{k=1}^{p} T_k(A) \right)$$

implies $\overline{A}$ is weakly compact.

Theorem 3.25 *Let X be an ordered Banach space with a normal cone and M be a nonempty closed subset of X. Let $p \geq 2$ and let $T_1, T_2, \dots, T_p : M \to 2^M$ be p monotone-closed mappings satisfying*:

1. $T_1, T_2, \dots, T_p$ *satisfy condition* D_M;
2. T_1 *is* $(T_k)_{2 \leq k \leq p}$*-weakly isotone.*

Then $T_1, T_2, \dots, T_p$ have a common fixed point.

Proof Assume that T_1 is $(T_k)_{2 \leq k \leq p}$-weakly isotone increasing, and let $x \in M$ be fixed. We define a sequence $\{x_n\}$ in M as follows:

$$x_0 = x, \qquad x_{pn+1} \in T_1(x_{pn}), \qquad x_{pn+2} \in T_2(x_{pn+1}), \qquad \dots, \qquad x_{pn+p} \in T_p(x_{pn+(p-1)}), \quad n = 0, 1, 2, \dots.$$

Then, as in the proof of Theorem 3.10, it follows that

$$x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq \cdots.$$

Let $A = \{x_0, x_1, x_2, \dots\}$, A is countable and

$$A \subset \{x_0\} \cup \left(\bigcup_{k=1}^{p} T_k(A) \right),$$

or $T_1, T_2, \dots, T_p$ satisfy the condition D_M, so $\overline{A}$ is weakly compact. Then there exists a subsequence $\{x_{\sigma(n)}\} \subset A$ and $x^* \in \overline{A}$ such that

$$x_{\sigma(n)} \rightharpoonup x^*.$$

We refer to Lemma 3.5, we get $x_{\sigma(n)} \to x^*$. Since $x_{pn+1} \in T_1(x_{pn})$ for all $n \in \mathbb{N}$ and T_1 is monotone-closed, we obtain that $x^* \in T_1(x^*)$. A similar argument yields $x^* \in T_k(x^*)$, $k = 2, 3, \dots, p$, and consequently, x^* is a common fixed point for $T_1, T_2, T_3, \dots, T_p$. The case when T_1 is $(T_k)_{2 \leq k \leq p}$-weakly isotone decreasing is similar. □

Remark 3.26 For $p = 2$, Theorem 3.25 was proved in [3].

The following corollary is a single-valued version of Theorem 3.25.

Corollary 3.27 *Let X be an ordered Banach space with a normal cone. Let M be a nonempty closed subset of X. Let $p \geq 2$ and let $T_1, T_2, \dots, T_p : M \to M$ be p closed mappings satisfying the condition D_M and T_1 is $(T_k)_{2 \leq k \leq p}$-weakly isotone. Then $T_1, T_2, \dots, T_p$ have a common fixed point.*

Example 3.28 Let $X = \mathbb{R}^2$, $P = \{x = (t,t) \in \mathbb{R}^2 : t \geq 0\}$ and

$$M = \bigl\{x \in X : \|x\|_\infty \leq 2\bigr\} \cup \bigl\{(t,t) \in X : t > 2\bigr\}.$$

$(X, \|\cdot\|_\infty)$ is an ordered Banach space with a normal order cone P, and M is a nonempty closed subset of X.

Let $T_1, T_2, T_3 : M \to 2^M$ be defined by, for all $x \in M$,

$$T_1(x) = \begin{cases} \{(\|x\|_\infty, \|x\|_\infty)\} & \text{if } \|x\|_\infty \leq 2, \\ \{(2,2)\} & \text{if } \|x\|_\infty > 2, \end{cases}$$

$$T_2(x) = \begin{cases} \{(\frac{\|x\|_\infty+2}{2}, \frac{\|x\|_\infty+2}{2})\} & \text{if } \|x\|_\infty \leq 2, \\ \{(\frac{3t}{t+1}, \frac{3t}{t+1}) : t \in [2, \|x\|_\infty]\} & \text{if } \|x\|_\infty > 2 \end{cases}$$

and

$$T_3(x) = \begin{cases} \{(2,2)\} & \text{if } \|x\|_\infty \leq 2, \\ \{(\frac{3t}{t+1}, \frac{3t}{t+1}) : t \in [2, \|x\|_\infty]\} & \text{if } \|x\|_\infty > 2. \end{cases}$$

The multivalued mappings T_1, T_2, T_3 are monotone-closed and satisfying the condition D_M, because the mappings T_1, T_2, T_3 are bounded, and T_1 is (T_2, T_3)-weakly-isotone. Besides, the mappings T_1, T_2, T_3 have the common fixed point $(2,2)$.

Remark 3.29 In Example 3.22 the mappings T_1, T_2, T_3 are not satisfying the condition D_M.

4 Application

Let $\mathbb{R}$ be the real line and $\mathbb{N}$ be the set of positive integers, and $p \in \mathbb{N}$ with $p \geq 2$. Let E be a Banach space with norm $\|\cdot\|_E$, and let $C(E)$ denote the class of all nonempty closed subsets of E. Given a closed and bounded interval $J = [0,1] \subset \mathbb{R}$, consider the system of nonlinear integral inclusions of the form

$$x(t) \in q(t) + \int_0^{\sigma(t)} K(t,s)F_i\bigl(s, x(s)\bigr)\, ds \quad \text{for all } i \in \{1, 2, \dots, p\}, \tag{1}$$

where $t \in [0,1]$, and $\sigma : [0,1] \to [0,1]$, $q : [0,1] \to E$, $K : [0,1] \times [0,1] \to \mathbb{R}$ are continuous, and $F_i : [0,1] \times E \to C(E)$ for all $i \in \{1, 2, \dots, p\}$.

By a common solution for the system of integral inclusions (1), we mean a continuous function $x : J \to E$ such that

$$x(t) = q(t) + \int_0^{\sigma(t)} K(t,s)v_i(s)\, ds \quad \text{for all } i \in \{1, 2, \dots, p\},$$

for some $v_i \in B(J, E)$ satisfying $v_i(t) \in F_i(t, x(t))$ for all $t \in J$, where $B(J, E)$ is the space of all E-valued Bochner integrable functions on J.

Let $C(J,E)$ denote the space of all continuous E-valued functions on J, define a norm $\|\cdot\|_C$ in $C(J,E)$ by

$$\|x\|_C = \sup\bigl\|x(t)\bigr\|_E.$$

Clearly, $C(J,E)$ is a Banach space with the norm $\|\cdot\|_C$. We introduce an order relation "$\leq$" in $C(J,E)$ with the help of the cone P_C in $C(J,E)$ defined by

$$P_C = \bigl\{x \in C(J,E) : x(t) \in P_E \text{ for all } t \in J\bigr\},$$

where P_E is a normal cone in E.

Definition 4.1 A multivalued map $F : J \to 2^E$ is said to be measurable if for any $y \in E$, the function $t \to d(y, F(t)) = \inf\{\|y - x\|_E : x \in F(t)\}$ is measurable.

Let $i \in \{1, 2, \dots, p\}$. Denote

$$\bigl\|F_i(t,x)\bigr\| = \bigl\{\|u\|_E : u \in F_i(t,x)\bigr\} \quad \text{and} \quad \bigl\|\bigl|F_i(t,x)\bigr|\bigr\| = \sup\bigl\{\|u\|_E : u \in F_i(t,x)\bigr\}.$$

Definition 4.2 A multivalued function $\beta : J \times E \to 2^E$ is called Carathéodory if

1. $t \to \beta(t,x)$ is measurable for each $x \in E$, and
2. $x \to \beta(t,x)$ is an upper semicontinuous almost everywhere for $t \in J$.

Definition 4.3 A Carathéodory multifunction $F_i(t,x)$ is called L^1-Carathéodory if, for every real number $r > 0$, there exists a function $h_{i,r} \in L^1(J,\mathbb{R})$ such

$$\bigl\|\bigl|F_i(t,x)\bigr|\bigr\| \leq h_{i,r} \quad \text{a.e. } t \in J$$

for all $x \in E$ with $\|x\|_E \leq r$.

Denote

$$S^1_{F_i}(x) = \bigl\{v \in B(J,E) : v(t) \in F_i\bigl(t, x(t)\bigr) \text{ a.e. } t \in J\bigr\}.$$

In the sequel, we also need the following lemmas from [13].

Lemma 4.4 *If* $\dim(E) < \infty$ *and* $F_i : J \times E \to 2^E$ *is* L^1*-Carathéodory, then* $S^1_{F_i}(x) \neq \emptyset$ *for each* $x \in C(J,E)$.

Lemma 4.5 *Let* E *be a Banach space,* F *be a Carathéodory multimap with* $S^1_{F_i} \neq \emptyset$*, and let* $\mathcal{L} : L^1(J,E) \to C(J,E)$ *be a continuous linear mapping. Then the operator* $\mathcal{L} \circ S^1_{F_i} : C(J,E) \to 2^{C(J,E)}$ *is a closed graph operator on* $C(J,E) \times C(J,E)$.

Now we introduce the following definition.

Definition 4.6 A multifunction $F_i(t,x)$ is said to be nondecreasing in x almost everywhere for $t \in J$ if, for any $x, y \in E$ with $x < y$, we have that $F_i(t,x) \leq F_i(t,y)$ for almost everywhere $t \in J$.

Now we have the following condition.

Condition 2 Integral inclusions of system (1) are said to satisfy Condition 2 if there exist $a, b \in C(J, E)$ such that

$$a(t) \leq q(t) + \int_0^{\sigma(t)} K(t,s)v_i(s)\,ds \leq b(t)$$

for all $v_i \in B(J, E)$ such that $v_i \in F_i(t, x(t))$ with $i \in \{1, 2, \dots, p\}$.

We refer to [7] and we consider the following set of hypotheses in the following:

(H_0) The function K is continuous and nonnegative on $J \times J$, with

$$M = \sup_{t,s \in J} K(t,s).$$

(H_1) The multivalued F_i is Carathéodry for all $i \in \{1, 2, \dots, p\}$.

(H_2) For any bounded set A of E, $\psi(F_i([0,1] \times A)) \leq \lambda\psi(A)$ for some reals $\lambda > 0$, with $i \in \{1, 2, \dots, p\}$.

(H_3) Multivalued functions F_i are nondecreasing in x almost everywhere for $t \in J$, with $i \in \{1, 2, \dots, p\}$.

(H_4) $S^1_{F_i}(x) \neq \emptyset$ for each $x \in C(J, E)$ and for all $i \in \{1, 2, \dots, p\}$.

(H_5) Integral inclusions of system (1) satisfy Condition 2.

(H_6) For each $i \in \{1, 2, \dots, p\}$, the function

$$t \to \left|\!\left|\!\left| F_i(t, a(t) \right|\!\right|\!\right|_E + \left|\!\left|\!\left| F_i(t, b(t) \right|\!\right|\!\right|_E$$

is Lebesgue integrable on J.

(H_7) $F_p(t, x) \leq F_1(t, y)$ for all $v \in S^1_{F_p}(x)$ and, for each $i \in \{1, 2, \dots, p-1\}$, we have $F_i(t, x) \leq F_{i+1}(t, y)$ for all $v \in S^1_{F_i}(x)$, where

$$y(t) = q(t) + \int_0^{\sigma(t)} K(t,s)v(s)\,ds.$$

The next theorem has been proved in [7] for $p = 2$.

Theorem 4.7 *Let $p \in \mathbb{N}$ with $p \geq 2$, and assume that hypotheses* (H_0)–(H_7) *hold. If $\lambda M < 1$, then the system of integral inclusions* (1) *has a common solution in* $C([0,1], E)$.

Proof Let $X = C(J, E)$ and consider the ordered interval $[a, b] \subset X$, which will be defined in view of (H_5). Define p mappings $T_1, T_2, \dots, T_p : [a, b] \to X$ by

$$T_i(x) = \left\{ u : u(t) = q(t) + \int_0^{\sigma(t)} K(t,s)v(s)\,ds, v \in S^1_{F_i}(x) \right\} \quad \text{for all } i \in \{1, 2, \dots, p\}.$$

Our strategy is to show that T_i satisfies all the conditions of Theorem 3.15 for all $i \in \{1, 2, \dots, p\}$.

First we show that the p-uplet $(T_1, T_2, \dots, T_p)$ is weakly isotone on $[a,b]$. Let $x, y \in [a,b]$, then with a similar reasoning as in [7] we get, for each $i \in \{1,2,\dots,p\}$,

$$\begin{aligned}
T_i(x) &= \left\{u_1 : u_1 = q(t) + \int_0^{\sigma(t)} K(t,s)v_1(s)\,ds, v_1 \in S^1_{F_i}(x)\right\} \\
&= \left\{u_1 : u_1 = q(t) + \int_0^{\sigma(t)} K(t,s)v_1(s)\,ds, v_1 \in \left\{v \in B(J,E) : v(t) \in F_i\bigl(t, x(t)\bigr)\right\}\right\} \\
&\leq \left\{u_2 : u_2 = q(t) + \int_0^{\sigma(t)} K(t,s)v_2(s)\,ds, v_2 \in \left\{v \in B(J,E) : v(t) \in F_{i+1}\bigl(t, y(t)\bigr)\right\}\right\} \\
&\leq \left\{\text{for all } y(t) = q(t) + \int_0^{\sigma(t)} K(t,s)v(s)\,ds \text{ and } v \in S^1_{F_i}(x)\right\} \\
&= \left\{u_2 : u_2 = q(t) + \int_0^{\sigma(t)} K(t,s)v_2(s)\,ds, v_2 \in S^1_{F_{i+1}}(y)\right\} \\
&\leq \left\{\text{for all } y(t) = q(t) + \int_0^{\sigma(t)} K(t,s)v(s)\,ds \text{ and } v \in S^1_{F_i}(x)\right\} \\
&= T_{i+1}(y), \quad \text{for all } y(t) = q(t) + \int_0^{\sigma(t)} K(t,s)v(s)\,ds \text{ and } v \in S^1_{F_i}(x) \\
&= T_{i+1}(y), \quad \text{for all } y \in T_i(x) = \left\{u : u = q(t) + \int_0^{\sigma(t)} K(t,s)v(s)\,ds, v \in S^1_{F_i}(x)\right\}.
\end{aligned}$$

Thus, $T_i(x) \leq T_{i+1}(y)$ for all $y \in T_i(x)$. Similarly, we have $T_p(x) \leq T_1(y)$ for all $y \in T_p(x)$. Hence, the p-uplet $(T_1, T_2, \dots, T_p)$ is weakly isotone increasing.

Next, let $x \in [a,b]$, by (H_5) we have $a \leq T_i(x) \leq b$ for all $i \in \{1,2,\dots,p\}$. Hence $T_1, T_2, \dots, T_p : [a,b] \to 2^{[a,b]}$.

The following results are justified in [7]. Let A be a nonempty subset in $[a,b]$.

1. The cone P_C is normal in $C(J,E)$.
2. For each $t \in J$ and $i \in \{1,2,\dots,p\}$, $\psi(T_i(A(t))) \leq \lambda M \psi(A)$.
3. For each $i \in \{1,2,\dots,p\}$, $T_i(A)$ is a uniformly bounded and equicontinuous set in $[a,b]$.

We obtain $\psi(T_i(A)) \leq \lambda M \psi(A)$, where $\lambda M < 1$. Hence, T_i is a ψ-condensing multivalued mapping on $[a,b]$ for all $i \in \{1,2,\dots,p\}$. In [7] and from Lemma 4.5, we can prove that T_i is a closed graph for all $i \in \{1,2,\dots,p\}$.

Thus, $T_1, T_2, \dots, T_p$ satisfy all the conditions of Theorem 3.15, and therefore an application of it yields that $T_1, T_2, \dots, T_p$ have a common fixed point in $[a,b]$. This further implies that system of integral inclusions (1) has a common solution on J. □

5 Conclusions

Dehage, BC proved in [14] some common fixed point theorems for pairs of weakly isotone condensing mappings in an ordered Banach space. These results will be generalized later by Hussain and Taoudi. The authors showed that these results can be obtained if we replace the strong topology by the weaker one, and they used these results to solve the existence problem for a system of integral inclusions.

In the present paper, we extend and generalize these results for a finite family of single and multivalued functions on an ordered Banach space. And we prove the existence of solutions for a system of integral inclusions.

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