Kishore et al. *Fixed Point Theory and Applications* (2018) 2018:21
https://doi.org/10.1186/s13663-018-0646-z

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Caristi type cyclic contraction and common fixed point theorems in bipolar metric spaces with applications

G.N.V. Kishore[1*], Ravi P. Agarwal[2,3], B. Srinuvasa Rao[1,4] and R.V.N. Srinivasa Rao[5*]

*Correspondence:
kishore.apr2@gmail.com;
rvnrepalle@gmail.com
[1]Department of Mathematics, SRKR Engineering College, Bhimavaram, India
[5]Department of Mathematics, College of Natural and Computational Sciences, Wollega University, Nekemte, Ethiopia
Full list of author information is available at the end of the article

Abstract

In this paper, we obtain the existence and uniqueness of the solution for three self mappings in a complete bipolar metric space under a new Caristi type contraction with an example. We also provide applications to homotopy theory and nonlinear integral equations.

MSC: 54H25; 47H10; 54E50

Keywords: Bipolar metric space; Covariant and contravariant map; Compatible mapping and common fixed point

1 Introduction

Fixed point theory plays a vital role in applications of many branches of mathematics. Finding fixed points of generalized contraction mappings has become the focus of fruitful research activity in fixed point theory. Recently, many investigators have published various papers on fixed point theory and applications in different ways. One of the recently popular topics in fixed point theory is addressing the existence of fixed points of contraction mappings in bipolar metric spaces, which can be considered as generalizations of the Banach contraction principle. In 2016, Mutlu and Gürdal [1] have introduced the concepts of bipolar metric space and they investigated certain basic fixed point and coupled fixed point theorems for covariant and contravariant maps under contractive conditions; see [1, 2].

Caristi's fixed point theorem [3] is a renowned extension of the Banach contraction principle [4]. The proof of Caristi's results has been generalized and extended in many directions [5–10].

The aim of this paper is to prove the common fixed point results in bipolar metric spaces by using a Caristi type cyclic contraction. Also, we give examples and applications to homotopy theory and integral equations.

2 Methods/experimental

Definition 2.1 ([1]) Let A and B be a two non-empty sets. Suppose that $d : A \times B \to [0,\infty)$ is a mapping satisfying the following properties:

(B_1) $d(a,b) = 0$ if and only if $a = b$ for all $(a,b) \in A \times B$,

(B_2) $d(a,b) = d(b,a)$, for all $a,b \in A \cap B$,

(B_3) $d(a_1,b_2) \leq d(a_1,b_1) + d(a_2,b_1) + d(a_2,b_2)$, for all $a_1,a_2 \in A$, $b_1,b_2 \in B$.

Then the mapping d is called a bipolar metric on the pair (A,B) and the triple (A,B,d) is called a bipolar-metric space.

Definition 2.2 ([1]) Assume (A_1,B_1) and (A_2,B_2) to be two pairs of sets.

The function $F : A_1 \cup B_1 \to A_2 \cup B_2$ is said to be a covariant map if $F(A_1) \subseteq A_2$ and $F(B_1) \subseteq B_2$ and we denote this as $F : (A_1,B_1) \rightrightarrows (A_2,B_2)$.

The mapping $F : A_1 \cup B_1 \to A_2 \cup B_2$ is said to be a contravariant map, if $F(A_1) \subseteq B_2$ and $F(B_1) \subseteq A_2$ and we denote this as $F : (A_1,B_1) \leftrightarrows (A_2,B_2)$.

In particular, if d_1 and d_2 are bipolar metrics in (A_1,B_1) and (A_2,B_2), respectively. Then sometimes we use the notations $F : (A_1,B_1,d_1) \rightrightarrows (A_2,B_2,d_2)$ and $F : (A_1,B_1,d_1) \leftrightarrows (A_2,B_2,d_2)$.

Definition 2.3 ([1]) Let (A,B,d) be a bipolar metric space. A point $\nu \in A \cup B$ is said to be a left point if $\nu \in A$, a right point if $\nu \in B$ and a central point if both hold.

Similarly, a sequence $\{a_n\}$ on the set A and a sequence $\{b_n\}$ on the set B are called a left and right sequence, respectively.

In a bipolar metric space, a sequence is the simple term for a left or right sequence.

A sequence $\{\nu_n\}$ is convergent to a point ν if and only if $\{\nu_n\}$ is a left sequence, ν is a right point and $\lim_{n\to\infty} d(\nu_n,\nu) = 0$; or $\{\nu_n\}$ is a right sequence, ν is a left point and $\lim_{n\to\infty} d(\nu,\nu_n) = 0$.

A bisequence $(\{a_n\},\{b_n\})$ on (A,B,d) is a sequence on the set $A \times B$. If the sequences $\{a_n\}$ and $\{b_n\}$ are convergent, then the bisequence $(\{a_n\},\{b_n\})$ is said to be convergent. $(\{a_n\},\{b_n\})$ is a Cauchy sequence, if $\lim_{n,m\to\infty} d(a_n,b_m) = 0$.

A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Definition 2.4 ([1]) Let (A_1,B_1,d_1) and (A_2,B_2,d_2) be two bipolar metric spaces.

(i) The mapping $F : (A_1,B_1,d_1) \rightrightarrows (A_2,B_2,d_2)$ is said to be left-continuous at a point $a_0 \in A_1$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that $d_1(a_0,b) < \delta$ implies that $d_2(F(a_0),F(b)) < \epsilon$ for all $b \in B_1$.

(ii) The mapping $F : (A_1,B_1,d_1) \rightrightarrows (A_2,B_2,d_2)$ is said to be right-continuous at a point $b_0 \in B_1$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that $d_1(a,b_0) < \delta$ implies that $d_2(F(a),F(b_0)) < \epsilon$ for all $a \in A_1$.

(iii) The mapping F is said to be continuous, if it is left-continuous at each point $a \in A_1$ and right-continuous at each point $b \in B_1$.

(iv) A contravariant mapping $F : (A_1,B_1,d_1) \leftrightarrows (A_2,B_2,d_2)$ is continuous if and only if it is continuous as a covariant map $F : (A_1,B_1,d_1) \rightrightarrows (A_2,B_2,d_2)$.

It follows from Definition 2.3 that a covariant (or a contravariant) mapping $F : (A_1, B_1,d_1) \rightrightarrows (A_2,B_2,d_2)$ is continuous if and only if $\{u_n\} \to \nu$ in (A_1,B_1,d_1) implies $\{F(u_n)\} \to F(\nu)$ in (A_2,B_2,d_2).

3 Results and discussions

In this section, we will prove some common fixed point theorems for three covariant mappings with some new Caristi type contractive conditions in bipolar metric spaces.

Definition 3.1 Let (A,B,d) be a bipolar metric space and $F,f:(A,B)\rightrightarrows(A,B)$ be covariant mappings. A pair $\{F,f\}$ is said to be compatible if and only if $\lim_{n\to\infty} d(Ffa_n,fFb_n) = \lim_{n\to\infty} d(fFa_n,Ffb_n) = 0$, whenever $(\{a_n\},\{b_n\})$ is a sequence in (A,B) such that $\lim_{n\to\infty} Fa_n = \lim_{n\to\infty} Fb_n = \lim_{n\to\infty} fa_n = \lim_{n\to\infty} fb_n = \kappa$ for some $\kappa \in A\cap B$.

Theorem 3.2 *Let (A,B,d) be a complete bipolar metric space. Suppose $F,f,g:(A,B)\rightrightarrows(A,B)$ is a covariant mappings satisfying*:

(3.2.1) $d(Fa,Fb) \le \psi(\alpha(fa))\alpha(fa) - \alpha(Fa) + \psi(\beta(gb))\beta(gb) - \beta(Fb)$ *for all $a\in A$ and $b\in B$, where $\alpha,\beta : A\cup B\to[0,\infty)$ are lower semi-continuous functions and $\psi:(-\infty,\infty)\to(0,1)$ be a continuous function.*

(3.2.2) $F(A\cup B)\subseteq g(A\cup B)$ *and* $F(A\cup B)\subseteq f(A\cup B)$.

(3.2.3) *Either (F,f) or (F,g) are compatible.*

(3.2.4) *Either f or g is continuous.*

Then the mappings $F,f,g:A\cup B\to A\cup B$ have a unique common fixed point.

Proof Let $a_0\in A$ and $b_0\in B$ and from (3.2.2) we construct the bisequence $(\{a_{2n}\},\{b_{2n}\})$, $(\{\omega_{2n}\},\{\xi_{2n}\})$ in (A,B) as

$$Fa_{2n} = ga_{2n+1} = \omega_{2n}, \qquad Fa_{2n+1} = fa_{2n+2} = \omega_{2n+1},$$
$$Fb_{2n} = gb_{2n+1} = \xi_{2n}, \qquad Fb_{2n+1} = fb_{2n+2} = \xi_{2n+1},$$

for $n = 0,1,2,\ldots$.

By using the condition (3.2.1), we have

$$\begin{aligned}
0 &\le d(\omega_{2n},\xi_{2n+1}) = d(Fa_{2n},Fb_{2n+1}) \\
&\le \psi\big(\alpha(fa_{2n})\big)\alpha(fa_{2n}) - \alpha(Fa_{2n}) \\
&\quad + \psi\big(\beta(gb_{2n+1})\big)\beta(gb_{2n+1}) - \beta(Fb_{2n+1}) \\
&\le \psi\big(\alpha(\omega_{2n-1})\big)\alpha(\omega_{2n-1}) - \alpha(\omega_{2n}) \\
&\quad + \psi\big(\beta(\xi_{2n})\big)\beta(\xi_{2n}) - \beta(\xi_{2n+1}) \\
&< \alpha(\omega_{2n-1}) - \alpha(\omega_{2n}) + \beta(\xi_{2n}) - \beta(\xi_{2n+1}).
\end{aligned} \tag{1}$$

It follows that

$$\alpha(\omega_{2n}) + \beta(\xi_{2n+1}) < \alpha(\omega_{2n-1}) + \beta(\xi_{2n}) \tag{2}$$

and

$$\begin{aligned}
d(\omega_{2n},\xi_{2n}) &= d(Fa_{2n},Fb_{2n}) \\
&\le \psi\big(\alpha(\omega_{2n-1})\big)\alpha(\omega_{2n-1}) - \alpha(\omega_{2n}) \\
&\quad + \psi\big(\beta(\xi_{2n-1})\big)\beta(\xi_{2n-1}) - \beta(\xi_{2n}) \\
&< \alpha(\omega_{2n-1}) - \alpha(\omega_{2n}) + \beta(\xi_{2n-1}) - \beta(\xi_{2n}).
\end{aligned} \tag{3}$$

Similarly, it follows that

$$\alpha(\omega_{2n}) + \beta(\xi_{2n}) < \alpha(\omega_{2n-1}) + \beta(\xi_{2n-1}). \tag{4}$$

Thus, from (2) and (4) one shows that the bisequences $(\{\alpha(\omega_{2n})\}, \{\beta(\xi_{2n})\})$ are non-increasing bisequences of non-negative real numbers. So they must converge to λ_1, λ_2 for $\lambda_1, \lambda_2 \geq 0$.

Suppose $\lambda_1 > 0$ or $\lambda_2 > 0$. Letting $n \to \infty$ in Eqs. (2) and (4), we get a contradiction.

Therefore,

$$\lim_{n\to\infty} \alpha(\omega_{2n}) = \lim_{n\to\infty} \beta(\xi_{2n}) = 0. \tag{5}$$

Now, from (1), we have

$$\begin{aligned}
\sum_{2n=1}^{2m} d(\omega_{2n}, \xi_{2n+1}) = d(\omega_1, \xi_2) + d(\omega_2, \xi_3) + \cdots + d(\omega_{2m}, \xi_{2m+1}) \\
&< \alpha(\omega_0) - \alpha(\omega_1) + \beta(\xi_1) - \beta(\xi_2) + \alpha(\omega_1) - \alpha(\omega_2) + \beta(\xi_2) - \beta(\xi_3) \\
&\quad + \cdots + \alpha(\omega_{2m-1}) - \alpha(\omega_{2m}) + \beta(\xi_{2m}) - \beta(\xi_{2m+1}) \\
&< \alpha(\omega_0) + \beta(\xi_1).
\end{aligned}$$

This shows $\sum_{2n=1}^{2m} d(\omega_{2n}, \xi_{2n+1})$ is a biconvergent series.

Similarly, we prove $\sum_{2n=1}^{2m} d(\omega_{2n}, \xi_{2n})$ is a biconvergent series. Hence it is convergent.

We use the property (B_3), for each $n, m \in N$ with $n < m$ and we use (1), (3). Then we have

$$\begin{aligned}
d(\omega_{2n}, \xi_{2m}) &\leq d(\omega_{2n}, \xi_{2n+1}) + d(\omega_{2n+1}, \xi_{2n+1}) + \cdots \\
&\quad + d(\omega_{2m-1}, \xi_{2m-1}) + d(\omega_{2m-1}, \xi_{2m}) \\
&< \alpha(\omega_{2n-1}) - \alpha(\omega_{2n}) + \beta(\xi_{2n}) - \beta(\xi_{2n+1}) \\
&\quad + \alpha(\omega_{2n}) - \alpha(\omega_{2n+1}) + \beta(\xi_{2n}) - \beta(\xi_{2n+1}) + \cdots \\
&\quad + \alpha(\omega_{2m-2}) - \alpha(\omega_{2m-1}) + \beta(\xi_{2m-2}) - \beta(\xi_{2m1}) \\
&\quad + \alpha(\omega_{2m-2}) - \alpha(\omega_{2m-1}) + \beta(\xi_{2m-1}) - \beta(\xi_{2m}) \\
&\to 0 \quad \text{as } n, m \to \infty.
\end{aligned}$$

Similarly, we can prove $d(\omega_{2m}, \xi_{2n}) \to 0$ as $n, m \to \infty$.

This shows (ω_{2n}, ξ_{2m}) is a Cauchy bisequence in (A, B).

Since (A, B, d) is complete, (ω_{2n}, ξ_{2m}) converges and thus it biconverges to a point $\kappa \in A \cap B$ such that

$$\lim_{n\to\infty} \omega_{2n+1} = \kappa = \lim_{n\to\infty} \xi_{2n+1}. \tag{6}$$

That is,

$$\lim_{n\to\infty} ga_{2n+1} = \lim_{n\to\infty} Fa_{2n} = \lim_{n\to\infty} gb_{2n+1} = \lim_{n\to\infty} Fb_{2n} = \kappa,$$

$$\lim_{n\to\infty} fa_{2n+2} = \lim_{n\to\infty} Fa_{2n+1} = \lim_{n\to\infty} fb_{2n+2} = \lim_{n\to\infty} Fb_{2n+1} = \kappa.$$

Since f is continuous, we have

$$\lim_{n\to\infty} f^2 a_{2n+2} = f\kappa, \qquad \lim_{n\to\infty} fFa_{2n+1} = f\kappa \quad \text{and}$$
$$\lim_{n\to\infty} f^2 b_{2n+2} = f\kappa, \qquad \lim_{n\to\infty} fFb_{2n+1} = f\kappa.$$

Since α, β are lower semi-continuous functions,

$$\lim_{n\to\infty} \alpha(\omega_{2n}) = \alpha(\kappa), \qquad \lim_{n\to\infty} \beta(\xi_{2n}) = \beta(\kappa).$$

From (5), we get $\alpha(\kappa) = \beta(\kappa) = 0$.

Since the pair $\{F,f\}$ is compatible, we have

$$\lim_{n\to\infty} d(Ffa_{2n+2}, fFb_{2n+1}) = \lim_{n\to\infty} d(fFa_{2n+1}, Ffb_{2n+2}) = 0.$$

Therefore

$$\lim_{n\to\infty} fFb_{2n+1} = \lim_{n\to\infty} Ffa_{2n+2} = f\kappa, \qquad \lim_{n\to\infty} Ffb_{2n+2} = \lim_{n\to\infty} fFa_{2n+1} = f\kappa.$$

Taking $a = fa_{2n+2}$ and $b = b_{2n+1}$ in (3.2.1), we get

$$\begin{aligned} d(Ffa_{2n+2}, Fb_{2n+1}) &\le \psi\bigl(\alpha(f^2 a_{2n+2})\bigr)\alpha\bigl(f^2 a_{2n+2}\bigr) - \alpha(Ffa_{2n+2}) \\ &\quad + \psi\bigl(\beta(gb_{2n+1})\bigr)\beta(gb_{2n+1}) - \beta(Fb_{2n+1}) \\ &\le \psi\bigl(\alpha(f^2 a_{2n+2})\bigr)\alpha\bigl(f^2 a_{2n+2}\bigr) - \alpha(Ffa_{2n+2}) \\ &\quad + \psi\bigl(\beta(\xi_{2n})\bigr)\beta(\xi_{2n}) - \beta(\xi_{2n+1}) \\ &< \alpha\bigl(f^2 a_{2n+2}\bigr) - \alpha(Ffa_{2n+2}) + \beta(\xi_{2n}) - \beta(\xi_{2n+1}). \end{aligned}$$

Letting $n \to \infty$, we see that $d(f\kappa,\kappa) < \alpha(f\kappa) - \alpha(f\kappa) + \beta(\kappa) - \beta(\kappa) = 0$ implies $d(f\kappa,\kappa) = 0$, that is, $f\kappa = \kappa$.

Similarly, by the continuity of g, we can prove that $g\kappa = \kappa$.

By using the condition (3.2.1) and (B_3), we obtain

$$\begin{aligned} d(F\kappa,\kappa) &\le d(F\kappa,\xi_{2n+1}) + d(\omega_{2n+1},\xi_{2n+1}) + d(\omega_{2n+1},\kappa) \\ &\le d(F\kappa, Fb_{2n+1}) + d(\omega_{2n+1},\xi_{2n+1}) + d(\omega_{2n+1},\kappa) \\ &\le \psi\bigl(\alpha(f\kappa)\bigr)\alpha(f\kappa) - \alpha(F\kappa) + \psi\bigl(\beta(gb_{2n+1})\bigr)\beta(gb_{2n+1}) - \beta(Fb_{2n+1}) \\ &\quad + d(\omega_{2n+1},\xi_{2n+1}) + d(\omega_{2n+1},\kappa) \\ &< \alpha(f\kappa) - \alpha(F\kappa) + \beta(gb_{2n+1}) - \beta(Fb_{2n+1}) \\ &\quad + d(\omega_{2n+1},\xi_{2n+1}) + d(\omega_{2n+1},\kappa) \\ &< \alpha(\kappa) - \alpha(F\kappa) + \beta(\xi_{2n}) - \beta(\xi_{2n+1}) \end{aligned}$$

$$\begin{aligned}&+ d(\omega_{2n+1}, \xi_{2n+1}) + d(\omega_{2n+1}, \kappa)\\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Thus $F\kappa = \kappa$. Hence $F\kappa = f\kappa = g\kappa = \kappa$.

Now we prove the uniqueness; we begin by taking ν to be another fixed point of covariant maps F, f and g. Then $F\nu = f\nu = g\nu = \nu$ implies $\nu \in A \cap B$ and we have

$$\begin{aligned} d(\kappa,\nu) = d(F\kappa,F\nu) &\le \psi\big(\alpha(f\kappa)\big)\alpha(f\kappa) - \alpha(F\kappa) + \psi\big(\beta(g\nu)\big)\beta(g\nu) - \beta(F\nu)\\ &\le \psi\big(\alpha(\kappa)\big)\alpha(\kappa) - \alpha(\kappa) + \psi\big(\beta(\nu)\big)\beta(\nu) - \beta(\nu)\\ &< \alpha(\kappa) - \alpha(\kappa) + \beta(\nu) - \beta(\nu) = 0. \end{aligned}$$

Thus $\kappa = \nu$. Hence κ is unique common fixed point of covariant mappings F, f and g. □

Corollary 1 *Let (A,B,d) be a complete bipolar metric space. Suppose $F,f : (A,B) \rightrightarrows (A,B)$ is a covariant mappings satisfying:*

(1.1) *$d(Fa,Fb) \le \psi(\alpha(fa))\alpha(fa) - \alpha(Fa,) + \psi(\beta(fb))\beta(fb) - \beta(Fb)$ for all $a \in A$ and $b \in B$, where $\alpha, \beta : A \cup B \to [0,\infty)$ are lower semi-continuous functions and $\psi : (-\infty,\infty) \to (0,1)$ is a continuous function.*

(1.2) *$F(A \cup B) \subseteq f(A \cup B)$.*

(1.3) *(F,f) are compatible.*

(1.4) *f is continuous.*

Then the mappings $F,f : A \cup B \to A \cup B$ have a unique common fixed point.

Proof Let us take $g = I_{A\cup B}$ (identity mapping on $A \cup B$), from Theorem 3.2 we see that F and f have a unique common fixed point. □

Example 3.3 Let $U_m(R)$ and $L_m(R)$ be the set of all $m \times m$ upper and lower triangular matrices over R. Define $d : U_m(R) \times L_m(R) \to [0,\infty)$ as

$$d(P,Q) = \sum_{i,j=1}^{m} |p_{ij} - q_{ij}|$$

for all $P = (p_{ij})_{m\times m} \in U_m(R)$ and $Q = (q_{ij})_{m\times m} \in L_m(R)$. Then obviously $(U_m(R), L_m(R), d)$ is a bipolar-metric space.

Define $F, f, g : U_m(R) \cup L_m(R) \to U_m(R) \cup L_m(R)$ as $F(P) = \frac{1}{8}(p_{ij})_{m\times m}$, and we have $f(P) = \frac{1}{2}(p_{ij})_{m\times m}$ and $g(P) = (p_{ij})_{m\times m}$ for all $P = (p_{ij})_{m\times m} \in U_m(R) \cup L_m(R)$.

Let $\alpha, \beta : U_m(R) \cup L_m(R) \to [0,\infty)$ be a lower semi-continuous mappings defined as $\alpha(P) = \sum_{i,j=1}^{m} |p_{ij}|$ and $\beta(P) = \frac{1}{2}\sum_{i,j=1}^{m} |p_{ij}|$ $\forall P = (p_{ij})_{m\times m} \in U_m(R) \cup L_m(R)$ and define

$$\psi : (-\infty,+\infty) \to (0,1) \quad \text{as } \psi(t) = \begin{cases} \frac{2}{3} & \text{if } t > 0,\\ 0 & \text{if } t < 0.\end{cases}$$

Obviously,

$$F\big(U_m(R) \cup L_m(R)\big) = f\big(U_m(R) \cup L_m(R)\big) = g\big(U_m(R) \cup L_m(R)\big) = O_{m\times m}.$$

Furthermore, we prove $\{F,f\}$ is compatible. Let (P_n, Q_n) be a bisequence in (A,B) such that, for some $\kappa \in A \cap B$, $\lim_{n\to\infty} d(fP_n,\kappa) = 0$, $\lim_{n\to\infty} d(\kappa,fQ_n) = 0$ and $\lim_{n\to\infty} d(FP_n,\kappa) = 0$, $\lim_{n\to\infty} d(\kappa,FQ_n) = 0$. Since F and f are continuous, we have

$$\begin{aligned}
\lim_{n\to\infty} d(fFP_n, FfQ_n) &= d\Bigl(\lim_{n\to\infty} fFP_n, \lim_{n\to\infty} FfQ_n\Bigr) = d(f\kappa, F\kappa) \\
&= d\left(\frac{1}{2}(\kappa_{ij})_{m\times m}, \frac{1}{8}(\kappa_{ij})_{m\times m}\right) \\
&= \sum_{i,j=1}^{m} \left|\frac{1}{2}\kappa_{ij} - \frac{1}{8}\kappa_{ij}\right| = \sum_{i,j=1}^{m} \frac{3}{8}|\kappa_{ij}|.
\end{aligned}$$

But $\sum_{i,j=1}^{m} \frac{3}{8}|\kappa_{ij}| = 0 \Leftrightarrow \kappa_{ij} = 0$. Similarly, we show $\lim_{n\to\infty} d(FfP_n, fFQ_n) = 0$. So the pair $\{F,f\}$ is compatible. Similarly, $\{F,g\}$ is also compatible.

Now for each $P,Q \in U_m(R) \cup L_m(R)$, we have

$$\begin{aligned}
d(FP,FQ) &= d\left(\frac{1}{8}(p_{ij})_{m\times m}, \frac{1}{8}(q_{ij})_{m\times m}\right) \\
&= \sum_{i,j=1}^{m} \left|\frac{1}{8}p_{ij} - \frac{1}{8}q_{ij}\right| \\
&= \frac{1}{8}\sum_{i,j=1}^{m} |p_{ij} - q_{ij}| \\
&\le \frac{1}{3}\sum_{i,j=1}^{m} |p_{ij}| - \frac{1}{8}\sum_{i,j=1}^{m} |p_{ij}| + \frac{1}{3}\sum_{i,j=1}^{m} |q_{ij}| - \frac{1}{16}\sum_{i,j=1}^{m} |q_{ij}| \\
&\le \frac{2}{3}\alpha\left(\frac{1}{2}(p_{ij})_{m\times m}\right) - \alpha\left(\frac{1}{8}(p_{ij})_{m\times m}\right) \\
&\quad + \frac{2}{3}\beta\bigl((q_{ij})_{m\times m}\bigr) - \beta\left(\frac{1}{8}(q_{ij})_{m\times m}\right) \\
&= \psi\bigl(\alpha(fP)\bigr)\alpha(fP) - \alpha(FP) + \psi\bigl(\beta(gQ)\bigr)\beta(gQ) - \beta(FQ).
\end{aligned}$$

Thus, F, f, g satisfy all the conditions of Theorem 3.2 and $O_{m\times m}$ is a unique common fixed point of F, f and g.

Theorem 3.4 *Let (A,B,d) be a complete bipolar metric space. Suppose $F,f,g : (A,B) \rightrightarrows (A,B)$ are covariant mappings satisfying*:

(3.4.1) *$d(Fa,Fb) \le \alpha(\psi(fa,gb))\psi(fa,gb) - \psi(Fa,Fb)$ for all $a \in A$ and $b \in B$, where $\psi : (A\times B) \cup (B \times A) \to [0,\infty)$ is a lower semi-continuous function and $\alpha : (-\infty,\infty) \to (0,1)$ is a continuous function*

(3.4.2) *$F(A\cup B) \subseteq g(A\cup B)$ and $F(A\cup B) \subseteq f(A\cup B)$.*

(3.4.3) *Either (F,f) or (F,g) are compatible.*

(3.4.4) *Either f or g is continuous.*

Then the mappings $F,f,g : A\cup B \to A\cup B$ have a unique common fixed point.

Corollary 2 *Let (A,B,d) be a complete bipolar metric space. Suppose $F,f : (A,B) \rightrightarrows (A,B)$ are covariant mappings satisfying*:

(2.1) $d(Fa,Fb) \le \alpha(\psi(fa,fb))\psi(fa,fb) - \psi(Fa,Fb)$ *for all* $a \in A$ *and* $b \in B$, *where* $\psi : (A \times B) \cup (B \times A) \to [0,\infty)$ *is a lower semi-continuous function and* $\alpha : (-\infty,\infty) \to (0,1)$ *is a continuous function*.

(2.2) $F(A \cup B) \subseteq f(A \cup B)$.

(2.3) (F,f) *is compatible*.

(2.4) f *is continuous*.

Then the mappings $F,f : A \cup B \to A \cup B$ *have a unique common fixed point*.

Corollary 3 *Let* (A,B,d) *be a complete bipolar metric space. Suppose* $F : (A,B) \rightrightarrows (A,B)$ *is a covariant mapping satisfying*:

(3.1) $d(Fa,Fb) \le \alpha(\psi(a,b))\psi(a,b) - \psi(Fa,Fb)$ *for all* $a \in A$ *and* $b \in B$, *where* $\psi : (A \times B) \cup (B \times A) \to [0,\infty)$ *is a lower semi-continuous function and* $\alpha : (-\infty,\infty) \to (0,1)$ *is a continuous function. Then the mapping* $F : A \cup B \to A \cup B$ *has a unique fixed point*.

3.1 Application to homotopy

In this section, we study the existence of a unique solution applied to homotopy theory.

Theorem 3.5 *Let* (A,B,d) *be complete bipolar metric space*, (U,V) *be an open subset of* (A,B) *and* $(\overline{U},\overline{V})$ *be a closed subset of* (A,B) *such that* $(U,V) \subseteq (\overline{U},\overline{V})$. *Suppose* $H : (\overline{U} \cup \overline{V}) \times [0,1] \to A \cup B$ *is an operator with the following conditions satisfied*:

(3.5.1) $x \neq H(x,\kappa)$ *for each* $x \in \partial U \cup \partial V$ *and* $\kappa \in [0,1]$ (*here* $\partial U \cup \partial V$ *is boundary of* $U \cup V$ *in* $A \cup B$)

(3.5.2) $d(H(x,\kappa),H(y,\kappa)) \le \alpha(\psi(x,y))\psi(x,y) - \psi(H(x,\kappa),H(y,\kappa))$ *for all* $x \in \overline{U}$, $y \in \overline{V}$ *and* $\kappa \in [0,1]$, *where* $\psi : (A \times B) \cup (B \times A) \to [0,\infty)$ *is a lower semi-continuous function and* $\alpha : (-\infty,+\infty) \to (0,1)$ *is a continuous function*.

(3.5.3) $\exists M \ge 0 \ni d(H(x,\chi),H(y,\zeta)) \le M|\chi - \zeta|$ *for every* $x \in \overline{U}$ *and* $y \in \overline{V}$ *and* $\chi,\zeta \in [0,1]$.

Then $H(\cdot,0)$ *has a fixed point* $\iff$ $H(\cdot,1)$ *has a fixed point*.

Proof Consider the sets

$$X = \{\chi \in [0,1] : x = H(x,\chi) \text{ for some } x \in U\},$$
$$Y = \{\zeta \in [0,1] : y = H(y,\zeta) \text{ for some } y \in V\}.$$

Since $H(\cdot,0)$ has a fixed point in $U \cup V$, we have $0 \in X \cap Y$. Thus, $X \cap Y$ is a non-empty set.

We will show $X \cap Y$ is both closed and open in $[0,1]$ and so, by the connectedness $X = Y = [0,1]$.

Let $(\{\chi_n\}_{n=1}^{\infty},\{\zeta_n\}_{n=1}^{\infty}) \subseteq (X,Y)$ with $(\chi_n,\zeta_n) \to (\chi,\zeta) \in [0,1]$ as $n \to \infty$.

We must show $\chi = \zeta \in X \cap Y$.

Since $(\chi_n,\zeta_n) \in (X,Y)$ for $n = 0,1,2,3,\ldots$, there exists a bisequence (x_n,y_n) with $x_{n+1} = H(x_n,\chi_n)$, $y_{n+1} = H(y_n,\zeta_n)$.

Consider

$$\begin{aligned} d(x_n,y_{n+1}) &= d\big(H(x_{n-1},\chi_{n-1}),H(y_n,\zeta_n)\big) \\ &\le \alpha\big(\psi(x_{n-1},y_n)\big)\psi(x_{n-1},y_n) - \psi\big(H(x_{n-1},\chi_{n-1}),H(y_n,\zeta_n)\big) \end{aligned}$$

$$\begin{aligned} &< \psi(x_{n-1},y_n) - \psi\big(H(x_{n-1},\chi_{n-1}),H(y_n,\zeta_n)\big) \\ &= \psi(x_{n-1},y_n) - \psi(x_n,y_{n+1}). \end{aligned} \tag{7}$$

It follows that

$$\psi(x_n,y_{n+1}) \le \alpha\big(\psi(x_{n-1},y_n)\big)\psi(x_{n-1},y_n) < \psi(x_{n-1},y_n). \tag{8}$$

Also, we have

$$\begin{aligned} d(x_n,y_n) &= d\big(H(x_{n-1},\chi_{n-1}),H(y_{n-1},\zeta_{n-1})\big) \\ &\le \alpha\big(\psi(x_{n-1},y_{n-1})\big)\psi(x_{n-1},y_{n-1}) - \psi\big(H(x_{n-1},\chi_{n-1}),H(y_{n-1},\zeta_{n-1})\big) \\ &< \psi(x_{n-1},y_{n-1}) - \psi\big(H(x_{n-1},\chi_{n-1}),H(y_{n-1},\zeta_{n-1})\big) \\ &= \psi(x_{n-1},y_{n-1}) - \psi(x_n,y_n); \end{aligned} \tag{9}$$

similarly, it follows

$$\psi(x_n,y_n) \le \alpha\big(\psi(x_{n-1},y_{n-1})\big)\psi(x_{n-1},y_{n-1}) < \psi(x_{n-1},y_{n-1}). \tag{10}$$

From (8) and (10) one shows the bisequence $(\{\psi(x_n,y_n)\}$ is a non-increasing bisequence of non-negative real numbers. So they must converge to $\lambda_1 \ge 0$.

Suppose $\lambda_1 > 0$. Letting $n \to \infty$ in Eqs. (8) and (10), we get a contradiction. Therefore,

$$\lim_{n\to\infty} \psi(x_n,y_n) = 0. \tag{11}$$

Now, from (7), we have

$$\begin{aligned} \sum_{n=1}^{m} d(x_n,y_{n+1}) &= d(x_1,y_2) + d(x_2,y_3) + \cdots + d(x_m,y_{m+1}) \\ &< \psi(x_0,y_1) - \psi(x_1,y_2) + \psi(x_1,y_2) - \psi(x_2,y_3) \\ &\quad + \cdots + \psi(x_{m-1},y_m) - \psi(x_m,y_{m+1}) \\ &< \psi(x_0,y_1). \end{aligned}$$

This shows $\sum_{n=1}^{m} d(x_n,y_{n+1})$ is a biconvergent series.

Similarly, we can also prove $\sum_{n=1}^{m} d(x_n,y_n)$ is a biconvergent series. Hence it is convergent.

Now for each $n, m \in N$, $n < m$, using the property (B_3) and from (7), (9), we have

$$\begin{aligned} d(x_n,y_m) &\le d(x_n,y_{n+1}) + d(x_{n+1},y_{n+1}) + \cdots \\ &\quad + d(x_{m-1},y_{m-1}) + d(x_{m-1},y_m) \\ &\le d\big(H(x_{n-1},\chi_{n-1}),H(y_n,\zeta_n)\big) \\ &\quad + d\big(H(x_n,\chi_n),H(y_n,\zeta_n)\big) + \cdots \\ &\quad + d\big(H(x_{m-2},\chi_{m-2}),H(y_{m-2},\zeta_{m-2})\big) \end{aligned}$$

$$
\begin{aligned}
&+ d\big(H(x_{m-2}, \chi_{m-2}), H(y_{m-1}, \zeta_{m-1})\big) \\
<\ & \psi(x_{n-1}, y_n) - \psi(x_n, y_{n+1}) + M|\chi_{n-1} - \zeta_{n-1}| + M|\chi_{m-2} - \zeta_{m-2}| \\
&+ \psi(x_{m-2}, y_{m-1}) - \psi(x_{m-1}, y_m) \\
\to\ & 0 \quad \text{as } n, m \to \infty.
\end{aligned}
$$

Similarly, we can also show $\lim_{n\to\infty} d(x_m, y_n) = 0$.

Therefore, (x_n, y_n) is a Cauchy bisequence in (U, V). By completeness, there exist $\xi \in U$ and $\lambda \in V$ with

$$
\lim_{n\to\infty} x_n = \lambda, \qquad \lim_{n\to\infty} y_n = \xi. \tag{12}
$$

Now consider

$$
\begin{aligned}
d\big(H(\xi,\chi),\lambda\big) &\leq d\big(H(\xi,\chi), y_{n+1}\big) + d(x_{n+1}, y_{n+1}) + d(x_{n+1}, \lambda) \\
&\leq d\big(H(\xi,\chi), H(y_n, \zeta_n)\big) + d\big(H(x_n, \chi_n), H(y_n, \zeta_n)\big) + d(x_{n+1}, \lambda) \\
&\leq \alpha\big(\psi(\xi, y_n)\big)\psi(\xi, y_n) - \psi\big(H(\xi,\chi), H(y_n, \zeta_n)\big) \\
&\quad + M|\chi_n - \zeta_n| + d(x_{n+1}, \lambda) \\
&< \psi(\xi, y_n) - \psi\big(H(\xi,\chi), H(y_n, \zeta_n)\big) \\
&\quad + M|\chi_n - \zeta_n| + d(x_{n+1}, \lambda) \\
&< \psi(\xi, y_n) - \psi(\xi, y_{n+1}) \\
&\quad + M|\chi_n - \zeta_n| + d(x_{n+1}, \lambda) \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

It follows that $H(\xi,\chi) = \lambda$. Similarly, we get $H(\lambda,\zeta) = \xi$.

On the other hand from (12), we get

$$
d(\xi,\lambda) = d\Big(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n\Big) = \lim_{n\to\infty} d(x_n, y_n) = 0.
$$

Therefore, $\xi = \lambda$. Thus $\chi = \zeta \in X \cap Y$. Clearly $X \cap Y$ is closed in [0, 1].

Let $(\chi_0, \zeta_0) \in (X, Y)$. Then there exists a bisequence (x_0, y_0) such that

$$
x_0 = H(x_0, \chi_0), \qquad y_0 = H(y_0, \zeta_0).
$$

Since $U \cup V$ is open, there exists $r > 0$ such that $B_d(x_0, r) \subseteq U \cup V$ and $B_d(r, y_0) \subseteq U \cup V$.

Choose $\chi \in (\zeta_0 - \epsilon, \zeta_0 + \epsilon)$ and $\zeta \in (\chi_0 - \epsilon, \chi_0 + \epsilon)$ such that $|\chi - \zeta_0| \leq \frac{1}{M^n} < \frac{\epsilon}{2}$, $|\zeta - \chi_0| \leq \frac{1}{M^n} < \frac{\epsilon}{2}$ and $|\chi_0 - \zeta_0| \leq \frac{1}{M^n} < \frac{\epsilon}{2}$.

Then we have $y \in \overline{B_{X\cup Y}(x_0, r)} = \{y, y_0 \in V / d(x_0, y) \leq r + d(x_0, y_0)\}$ and $x \in \overline{B_{X\cup Y}(y_0, r)} = \{x, x_0 \in U / d(x, y_0) \leq r + d(x_0, y_0)\}$.

Also

$$
\begin{aligned}
d\big(H(x,\chi), y_0\big) &= d\big(H(x,\chi), H(y_0, \zeta_0)\big) \\
&\leq d\big(H(x,\chi), H(y, \zeta_0)\big) + d\big(H(x_0, \chi), H(y, \zeta_0)\big)
\end{aligned}
$$

$$\begin{aligned}
&+ d\big(H(x_0,\chi), H(y_0,\zeta_0)\big)\\
&\le 2M|\chi - \zeta_0| + d\big(H(x_0,\chi), H(y,\zeta_0)\big)\\
&\le 2M|\chi - \zeta_0| + \alpha\big(\psi(x_0,y)\big)\psi(x_0,y) - \psi\big(H(x_0,\chi), H(y,\zeta_0)\big)\\
&< \frac{2}{M^{n-1}} + \psi(x_0,y) - \psi\big(H(x_0,\chi), H(y,\zeta_0)\big).
\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}
d\big(H(x,\chi), y_0\big) &< \psi(x_0,y) - \psi\big(H(x_0,\chi), H(y,\zeta_0)\big)\\
&< \psi(x_0,y)\\
&\le d(x_0,y) \le r + d(x_0,y_0).
\end{aligned}$$

Similarly, we can also prove $d(x_0, H(y,\zeta)) \le d(x,y_0) \le r + d(x_0,y_0)$.

On the other hand

$$\begin{aligned}
d(x_0,y_0) &= d\big(H(x_0,\chi_0), H(y_0,\zeta_0)\big)\\
&\le M|\chi_0 - \zeta_0| \le \frac{1}{M^{n-1}} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

So $x_0 = y_0$. Thus, for each fixed ζ, $\zeta = \chi \in (\zeta_0 - \epsilon, \zeta_0 + \epsilon)$ and $H(\cdot,\chi) : \overline{B_{X\cup Y}(x_0,r)} \to \overline{B_{X\cup Y}(x_0,r)}$. Thus, we conclude $H(\cdot,\chi)$ has a fixed point in $\overline{U \cup V}$. But this must be in $U \cup V$.

Therefore, $H(\cdot,\chi)$ has a fixed point in $\overline{U} \cap \overline{V}$. But this must be in $U \cap V$.

Therefore, $\chi = \zeta \in X \cap Y$ for $\zeta \in (\zeta_0 - \epsilon, \zeta_0 + \epsilon)$. Hence $(\zeta_0 - \epsilon, \zeta_0 + \epsilon) \subseteq X \cap Y$. Clearly $X \cap Y$ is open in [0, 1].

To prove the reverse, we can use a similar process. □

3.2 Application to the existence of solutions of integral equations

In this section, we study the existence and unique solution to an integral equations as an application of Corollary 3.

Theorem 3.6 *Let us consider the integral equation*

$$\gamma(x) = f(x) + \int_{E_1\cup E_2} S\big(x,y,\gamma(y)\big)\,dy, \quad x \in E_1 \cup E_2,$$

where $E_1 \cup E_2$ is a Lebesgue measurable set. Suppose

(i) $S : (E_1^2 \cup E_2^2) \times [0,+\infty) \to [0,+\infty)$ *and* $f \in L^\infty(E_1) \cup L^\infty(E_2)$,

(ii) *there is a continuous function* $\tau : E_1^2 \cup E_2^2 \to [0,+\infty)$ *such that* $|S(x,y,\gamma(y)) - S(x,y,\beta(y))| \le \frac{1}{2}\tau(x,y)|\gamma(y) - \beta(y)|$, *for* $(x,y) \in E_1^2 \cup E_2^2$,

(iii) $\| \int_{E_1\cup E_2} \tau(x,y)\,dy\| \le 1$ *i.e* $\mathrm{Sup}_{x\in E_1\cup E_2} \int_{E_1\cup E_2} |\tau(x,y)|\,dy \le 1$.

Then the integral equation has a unique solution in $L^\infty(E_1) \cup L^\infty(E_2)$.

Proof Let $A = L^\infty(E_1)$ and $B = L^\infty(E_2)$ be two normed linear spaces, where E_1, E_2 are Lebesgue measurable sets and $m(E_1 \cup E_2) < \infty$.

Consider $d : A \times B \to [0, +\infty)$ to be defined by $d(f, g) = \|f - g\|_\infty$ for all $(f, g) \in A \times B$. Then (A, B, d) is a complete bipolar metric space.

Define the covariant mapping $F : L^\infty(E_1) \cup L^\infty(E_2) \to L^\infty(E_1) \cup L^\infty(E_2)$ by

$$F(\gamma(x)) = \int_{E_1 \cup E_2} S(x, y, \gamma(y))\,dy + f(x), \quad x \in E_1 \cup E_2.$$

Define $\psi : (A \times B) \cup (B \times A) \to [0, +\infty)$ by $\psi(\gamma(x), \beta(y)) = 2\|\gamma(x) - \beta(y)\|$ and define

$$\alpha : (-\infty, +\infty) \to (0, 1) \quad \text{as } \alpha(t) = \begin{cases} \frac{5}{6} & \text{if } t > 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Now, we have

$$\begin{aligned} d(F\gamma(x), F\beta(x)) &= \|F\gamma(x) - F\beta(x)\| \\ &= \left| \int_{E_1 \cup E_2} S(x, y, \gamma(y))\,dy + f(x) - \left(\int_{E_1 \cup E_2} S(x, y, \beta(y))\,dy + f(x) \right) \right| \\ &\leq \int_{E_1 \cup E_2} \left| S(x, y, \gamma(y)) - S(x, y, \beta(y)) \right| dy \\ &\leq \frac{1}{2} \int_{E_1 \cup E_2} \tau(x, y) \left| \gamma(y) - \beta(y) \right| dy \\ &\leq \frac{1}{2} \left\| \gamma(y) - \beta(y) \right\|_\infty \int_{E_1 \cup E_2} \tau(x, y)\,dy \\ &\leq \frac{1}{2} \|\gamma - \beta\|_\infty \| \operatorname*{Sup}_{x \in E_1 \cup E_2} \int_{E_1 \cup E_2} \left| \tau(x, y) \right| dy \\ &= \frac{5}{6} \times 2\|\gamma - \beta\| - 2\|F\gamma - F\beta\| \\ &= \alpha(\psi(\gamma, \beta))\psi(\gamma, \beta) - \psi(F\gamma, F\beta). \end{aligned}$$

It follows from Corollary 3 that F has a unique fixed point in $A \cup B$. □

4 Conclusions

In this paper, we obtain the existence and uniqueness of the solution for three self mappings in a complete bipolar metric space under a new Caristi type contraction with an example. Also, we provide some applications to homotopy theory and nonlinear integral equations by using fixed point theorems in bipolar metric spaces.

Acknowledgements

The authors are very thankful to the reviewers and editors for valuable comments, remarks and suggestions for improving the content of the paper.

Funding

For this work there was no funding.

Availability of data and materials

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interest.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, SRKR Engineering College, Bhimavaram, India. [2]Department of Mathematics, Texas A and M University-Kingsville, Kingsville, USA. [3]Florida Institute of Technology, Melbourne, USA. [4]Department of Mathematics, K L University, Guntur, India. [5]Department of Mathematics, College of Natural and Computational Sciences, Wollega University, Nekemte, Ethiopia.

Publisher's Note

Received: 12 January 2018 Accepted: 28 August 2018 Published online: 17 September 2018

References

1. Mutlu, A., Gürdal, U.: Bipolar metric spaces and some fixed point theorems. J. Nonlinear Sci. Appl. **9**(9), 5362–5373 (2016)
2. Mutlu, A., Özkan, K., Gürdal, U.: Coupled fixed point theorems on bipolar metric spaces. Eur. J. Pure Appl. Math. **10**(4), 655–667 (2017)
3. Caristi, J.: Fixed point theorems for mappings satisfying inwardness condition. Trans. Am. Math. Soc. **215**, 241–251 (1976). https://doi.org/10.1090/s0002-9947-1976-0394329-4
4. Banach, S.: Sur les operations dans les ensembles abstraits et leur applications aux equations integrales. Fundam. Math. **3**, 133–181 (1922)
5. Agarwal, R.P., Khamsi, M.A.: Extension of Caristi's fixed point theorem to vector valued metric space. Nonlinear Anal. TMA **74**, 141–145 (2011). https://doi.org/10.1016/j.na.2010.08.025
6. Khamsi, M.A.: Remarks on Caristi's fixed point theorem. Nonlinear Anal. TMA **71**, 227–231 (2009)
7. Khamsi, M.A., Kirk, W.A.: An Introduction to Metric Spaces and Fixed Point Theory. Pure and Applied Mathematics. Wiley-Interscience, New York (2001). https://doi.org/10.1002/9781118033074
8. Ekeland, I.: Sur les problems variationnels. C. R. Acad. Sci. Paris **275**, 1057–1059 (1972)
9. Ekeland, I.: On the variational principle. J. Math. Anal. Appl. **47**(2), 324–353 (1974)
10. Dur-e-Shehwar, Batul, S., Kamran, T., Ghiura, A.: Caristi's fixed point theorem on C^*-algebra valued metric spaces. J. Nonlinear Sci. Appl. **9**, 584–588 (2016)